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Adaptive Control of Dual-Rate Systems Based on Least Squares Methods

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Abstract— **This paper is motivated by a practical control problem that the output sampling rate is often limited. In particular, for a dual-rate system in which the output sampling period is an integer multiple of the input updating period, we use a polynomial transformation technique to obtain a frequency-domain model. Based on this model, we propose a self-tuning control algorithm by minimizing output tracking error criteria from directly the dual-rate input-output data, analyze convergence properties of the algorithm in detail in the stochastic framework, and show that the control algorithm can achieve virtually asymptotically optimal control, ensure the closed-loop systems to be globally convergent and stable, and the output tracking error at the output sampling instants has the property of minimum variance. The results from simulation are included.**

Keywords: Multirate systems, sampled-data systems, multirate modeling, self-tuning regulator, adaptive control, convergence properties, least squares.

I. INTRODUCCTION

In many industry applications, the outputs are sampled at slower rates than the control updating rates, mostly due to hardware limitation [1], [2], [3]. A typical example is the control of the bottom and top composition products of a distillation column by acting on the reflux and vapor flow rates; it is apparent that the control variables can be quickly manipulated, while infrequent and delayed composition measurements are obtained by gas chromatography [4]. In such cases where several sample rates co-exist, it is necessary to configure a control system to achieve a desired closed-loop system performance. This paper is concerned with a dual-rate case where the output samples are at a relative slow rate, whereas the control signal is updated faster.

More generally, the study of multirate systems goes back to the early 1950's. The first important work was performed by Kranc (1957) on the switch decomposition technique [5]. During the last decade, Al-Rahmani and Franklin studied multirate LQG/LQR optimal control [6], [7]; Chen and Qiu [8], Qiu and Chen [9], [10], and Sagfös *et al*. [11] studied $\mathcal{H}_\infty$ optimal control of multirate systems, considering the causality constraint. In the process control literature, Lee *et al*. [12], Scattolini and Schiavoni [13], Ling and Lim [14], and Sheng *et al*. [15] studied model based predictive control of multirate systems; Scattolini [4], Albertos *el al*. [16], and Zhang *et al*. [17] investigated adaptive control involving dual-rate/multirate systems.

One motivation for the work in this paper is inferential control. In this area, Lee and Morari developed a generalized inferential control scheme and discussed various optimal control problems for multirate/dual-rate systems [12]; Li *et al*. applied dual-rate modeling to Octane quality inferential control [3], [18]. However, most control algorithms reported in the area of multirate systems assume that the parameters of multirate models are known, which is usually not the case. Also, most theoretical results on parameter estimation based adaptive control assume that both the estimator and the controller are updated at the same rate, e.g., the well-known Åström and Wittenmark self-tuning regulator (1973) [19]. These results are not suitable for the dual-rate setting. For dual-rate sampled-data control systems, we expect that the control law is updated at a fast rate even if the output is sampled at a relative slow rate.

In the field of dual-rate sampled-data adaptive control, the algorithm presented by Kanniah *et al.* is based on a parameterized model with its AR coefficients corresponding to the fast sampling rate and the MA coefficients to the slow sampling rate [20]. Since the prediction and control are all based on the slow sampling rate, the desired fast-rate system performance may not be achieved. Also, Zhang *et al.* studied an indirect model reference multirate adaptive control [17]; Mitsuaki *et al.* presented a least squares based self-tuning control algorithm [21]. But these algorithms handle only noise-free systems. Scattolini presented a gradient-based adaptive control algorithm for multirate systems based on CARIMA models from lifted state-space models [4]. In multirate stochastic systems with noise, to our best knowledge, the control problems based on model identification have not been fully investigated, especially the self-tuning control and its convergence properties based on multirate data directly, which are the focus of this work.

The paper is organized as follows. In Section II, we simply introduce the adaptive control scheme of dual-rate systems. In Section III, using a polynomial transformation technique, we establish the mathematical model for dual-rate systems and a least squares based self-tuning control. We prove the global convergence of the control algorithm proposed in Sections IV and V. In Section VI we give an illustrative example demonstrating the effectiveness of the algorithm proposed in the paper. Finally, we offer some concluding remarks in Section VII.

II. Problem formulation

The focus of this paper is a class of multirate systems – the *dual-rate systems* – as depicted in Figure 1, where P_c is a continuous-time process; the input $u_c(t)$ to P_c is

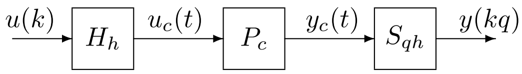

Fig. 1. The dual-rate system with noise

produced by a zero-order hold H_h with period h, processing a discrete-time signal $u(k)$; the output $y_c(t)$ of P_c is sampled by a sampler S_{qh} with period qh (q being a positive integer), yielding a discrete-time signal $y(kq)$ with period qh. The input-output data available are

- $\{u(k): \quad k = 0, 1, 2, \cdots\}$ at the fast rate, and
- $\{y(kq): \quad k = 0, 1, 2, \cdots\}$ at the slow rate.

Suppose that due to physical constraints, the intersample outputs, $y(kq + j)$, $j = 1, 2, \cdots, q - 1$, are not available, and thus we have missing output samples. Here, we refer to $\{u(k), y(kq)\}$ as the *dual-rate* measurement data.

The adaptive control scheme we propose is shown in Figure 2, where $y_r(k)$ denotes a deterministic reference input or desired output signal, $e(iq)$ a random noise with zero mean. For such a scheme to work, we can exploit an identification algorithm to produce the estimates $\hat{\theta}$ of the unknown system parameters based on the dual-rate data $\{u(k), y(kq)\}$, and compute the intersample (missing) outputs by using the estimated model and input $u(k)$. In order to feed back to the controller a fast rate signal $y_f(k)$, representing the output $y(k)$, we use the slow sampled output $y(iq)$ every q period, giving $y(0)$, $y(q)$, and $y(2q)$, etc., and use the estimated output $\hat{y}(iq + j)$ to fill in the missing samples in $y(k)$. In Figure 2, $y_f(k)$ connects to $y(iq)$ at times $k = iq$, and connects to $\hat{y}(iq + j)$ at $k = iq + j, j = 1, 2, \cdots, (q - 1)$. Thus the output of the switch is a fast rate signal given by $y_f(k)$. Due to the periodic switch, the fast rate signal $y_f(k)$ can be expressed as

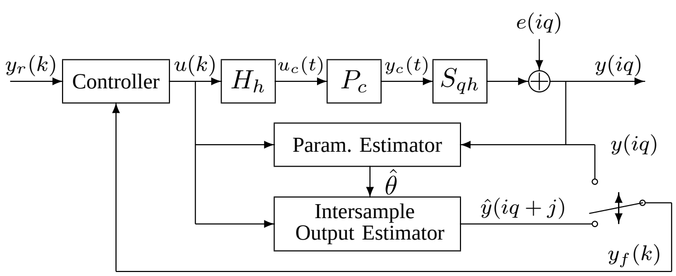

Fig. 2. The adaptive control scheme ($j = 1, 2, \cdots, q - 1$)

$$y_f(k) = \begin{cases} y(iq), & k = iq, \\ \hat{y}(iq + j), & k = iq + j, \ j = 1, 2, \cdots, (q - 1). \end{cases} \tag{1}$$

To summarize, the dual-rate adaptive control scheme uses a fast single-rate controller and a periodic switch. It is conceptually simple, easy to implement in digital computers, and practical for industry.

The objective of this paper is to propose an algorithm to estimate the intersample outputs $\{y(kq + j) : j = 1, 2, \cdots, (q - 1)\}$ based on the given dual-rate measurement data, design an adaptive controller so as to make the output $y(k)$ track a given desired output $y_r(k)$ by minimizing the tracking error criterion function

$$J[u(k)] = \mathrm{E}\{[y_f(k + d) - y_r(k + d)]^2 | \mathcal{F}_{k-1}\}, \tag{2}$$

and study the properties of the closed-loop system. Here, d represents the system delay, $\{\mathcal{F}_k\}$ is the σ algebra sequence generated by the observations up to and including time k.

III. Modeling and control algorithm of dual-rate systems

Setting the noise e to be zero in Figure 2, we assume that the open-loop transfer function from $u(k)$ to $y(k)$ takes the following real-rational form:

$$P_1(z) = \frac{z^{-d}b(z)}{a(z)}, \quad \text{or} \quad y(k) = \frac{z^{-d}b(z)}{a(z)}u(k) \tag{3}$$

with

$$\begin{aligned} a(z) &= 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}, \\ b(z) &= b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}. \end{aligned}$$

Here, d denotes the system delay and z^{-1} represents a unit backward shift operator at the fast rate: $z^{-1}u(k) = u(k-1)$.

This model in (3) is not suitable for dual-rate adaptive control because it would involve the unavailable outputs $\{y(kq + j) : \ j = 1, 2, \cdots, (q - 1)\}$. To obtain a model that we can use directly on the dual-rate data, by a polynomial transformation technique, $P_1(z)$ needs to be converted into a form so that the denominator is a polynomial in z^{-q} instead of z^{-1}.

For a general discussion, let the roots of $a(z)$ be z_i to get

$$a(z) = \prod_{i=1}^{n}(1 - z_i z^{-1}).$$

Define

$$\phi_q(z) = \prod_{i=1}^{n}(1 + z_i z^{-1} + z_i^2 z^{-2} + \cdots + z_i^{q-1} z^{-q+1}).$$

Multiplying the numerator and denominator of $P_1(z)$ by $\phi_q(z)$ transforms the denominator into the desired form:

$$P_1(z) = \frac{z^{-d}b(z)\phi_q(z)}{a(z)\phi_q(z)} =: \frac{z^{-d}\beta(z)}{\alpha(z)}, \tag{4}$$

or

$$\alpha(z)y(k) = z^{-d}\beta(z)u(k) \tag{5}$$

with

$$\begin{aligned} \alpha(z) &= 1 + \alpha_1 z^{-q} + \alpha_2 z^{-2q} + \cdots + \alpha_n z^{-qn}, \\ \beta(z) &= \beta_0 + \beta_1 z^{-1} + \beta_2 z^{-2} + \cdots + \beta_{qn} z^{-qn}. \end{aligned}$$

Equation (4) is the frequency-domain model for the dual-rate system and it has the advantage that the denominator is a polynomial of z^{-q}; arising from here is a recursive equation using only slowly sampled outputs. The control algorithm we propose later for dual-rate systems will be based on this model which does not involve the unavailable intersample outputs.

Next, we derive an adaptive control algorithm based on the model discussed in (5). Define the parameter vector θ and information vector $\varphi(k)$ as ($N := qn + n + 1$)

$$\theta = [\alpha_1 \;\; \alpha_2 \;\; \cdots \;\; \alpha_n \;\; \beta_0 \;\; \beta_1 \;\; \cdots \;\; \beta_{qn}]^{\mathrm{T}} \in \mathbb{R}^N,$$

$$\varphi(k-d) = [-y(k-q) \;\; -y(k-2q) \;\; \cdots \;\; -y(k-qn)$$

$$u(k-d) \;\; u(k-d-1) \;\; \cdots \;\; u(k-d-qn)]^{\mathrm{T}} \in \mathbb{R}^N.$$

Notice that θ contains all parameters in the model in (4) to be estimated, and $\varphi(k-d)$ uses only available dual-rate data – if k is an integer multiple of q, then $\varphi(k-d)$ contains only the past measurement outputs (slow rate) and inputs (fast rate). Substituting the polynomials $\alpha(z)$ and $\beta(z)$ into (5) leads to the following regression equation:

$$y(k) = \varphi^{\mathrm{T}}(k-d)\theta, \tag{6}$$

or

$$y(k+d) = \varphi^{\mathrm{T}}(k)\theta.$$

Let $y_r(k)$ be a desired output signal, define the output tracking error

$$\xi(k+d) = y(k+d) - y_r(k+d).$$

If the control signal $u(k)$ is chosen according to the equation $y_r(k+d) = \varphi^{\mathrm{T}}(k)\theta$, then the tracking error $\xi(k+d)$ approaches zero finally. For stochastic systems, based on the model in (6), introducing a disturbance term $v(k)$, we have

$$y(k) = \varphi^{\mathrm{T}}(k-d)\theta + v(k), \tag{7}$$

where $\{v(k)\}$ is assumed to be a zero-mean random white noise sequence. Let $\hat{\theta}$ be the estimate of unknown parameter vector θ, then $\hat{y}(k+d) = \varphi^{\mathrm{T}}(k)\hat{\theta}$ is the best output prediction, which is computed by the intersample output estimator in Figure 2, then replacing θ by $\hat{\theta}$ and minimizing the criterion function in (2) yield the control law of the form:

$$y_r(k+d) = \varphi^{\mathrm{T}}(k)\hat{\theta}. \tag{8}$$

Replacing k in (7) by kq gives

$$y(kq) = \varphi^{\mathrm{T}}(kq-d)\theta + v(kq), \tag{9}$$

where

$$\begin{aligned}\varphi(kq-d) \;\; &= \;\; [-y(kq-q) \;\; \cdots \;\; -y(kq-qn) \\ &\qquad u(kq-d) \;\; \cdots \;\; u(kq-d-qn)]^{\mathrm{T}}.\end{aligned}$$

Let $\hat{\theta}(kq)$ be the estimate of θ at current time kq. We propose the self-tuning control algorithm for the dual-rate system in (9) as follows:

$$\begin{aligned}\hat{\theta}(kq) \;\; &= \;\; \hat{\theta}(kq-q) + P(kq)\varphi(kq-d) \\ &\qquad [y(kq) - \varphi^{\mathrm{T}}(kq-d)\hat{\theta}(kq-q)], \qquad (10)\\ \hat{\theta}(kq+j) \;\; &= \;\; \hat{\theta}(kq), \quad j = 0,\ 1,\ \cdots,\ q-1. \qquad (11)\\ P^{-1}(kq) \;\; &= \;\; P^{-1}(kq-q) + \varphi(kq-d)\varphi^{\mathrm{T}}(kq-d), \qquad (12)\\ \hat{\theta}(kq) \;\; &= \;\; [\hat{\alpha}_1(kq) \;\; \cdots \;\; \hat{\alpha}_n(kq) \;\; \hat{\beta}_0(kq) \;\; \cdots \;\; \hat{\beta}_{qn}(kq)]^{\mathrm{T}}. \qquad (13)\end{aligned}$$

Based on (8), the control law is given by

$$\varphi^{\mathrm{T}}(kq+j)\hat{\theta}(kq) = y_r(kq+d+j). \tag{14}$$

The control signal $u(kq+j),\ j = 0, 1, \cdots, (q-1)$, in (14) may be obtained from the following recursive equation

$$\begin{aligned}u(kq+j) \;\; &= \;\; \frac{1}{\hat{\beta}_0(kq)}[y_r(kq+d+j) \\ &\qquad + \sum_{i=1}^{n} \hat{\alpha}_i(kq) y(kq+d+j-iq) \\ &\qquad - \sum_{i=1}^{nq} \hat{\beta}_i(kq) u(kq+j-i)]. \qquad (15)\end{aligned}$$

Here, a difficulty arises in that on the interval $[kq,\ kq+q)$, except for $j = q-d$, the expression on the right-hand side of (15) contains the future missing outputs $y(kq+j_1)$ if $j_1 := d+j-iq > 0$, and the past missing outputs $y(kq-j_2)$ if $j_2 := -d-j+iq > 0$ and j_2 is not an integer multiple of q. In fact, only when $j = q-d$, the control term $u(kq+j)$ does not involve the missing outputs, and can be generated by

$$\begin{aligned}u(kq+q-d) \;\; &= \;\; \frac{1}{\hat{\beta}_0(kq)}[y_r(kq+q) + \\ &\qquad \sum_{i=1}^{n} \hat{\alpha}_i(kq) y(kq+q-iq) \\ &\qquad - \sum_{i=1}^{nq} \hat{\beta}_i(kq) u(kq+q-d-i)]. \qquad (16)\end{aligned}$$

So it is impossible to compute the control law by (15) and to realize the algorithm in (10)-(15). Our solution is based on the adaptive control scheme stated in Section 2: These unknown outputs $y(kq+j)$ in (15) are replaced by their estimates $\hat{y}(kq+j)$. Hence,

$$\begin{aligned}u(kq+j) \;\; &= \;\; \frac{1}{\hat{\beta}_0(kq)}[y_r(kq+d+j) + \\ &\qquad \sum_{i=1}^{n} \hat{\alpha}_i(kq) \hat{y}(kq+d+j-iq) \\ &\qquad - \sum_{i=1}^{nq} \hat{\beta}_i(kq) u(kq+j-i)], \qquad (17)\\ &\qquad j = 0,\ 1,\ \cdots,\ q-1;\ j \neq q-d.\end{aligned}$$

To initialize the control algorithm in (10)-(13), (16) and (17), we take $P(0) = p_0 I$ with p_0 normally a large positive number and I an identity matrix of appropriate dimension, and $\hat{\theta}(0) = \hat{\theta}_0$, some small real vector (e.g., $\hat{\theta}(0) = [10^{-6} \quad 10^{-6} \quad \cdots \quad 10^{-6}]^{\mathrm{T}}$). Notice that the parameter estimate $\hat{\theta}$ is updated every q (fast) samples, namely, at the slow rate; so is the covariance matrix P; in between the slow samples, we simply hold $\hat{\theta}$ unchanged. Thus, every time $\hat{\theta}$ is updated, we have q new input samples and one new output sample.

It is easy to see that by defining

$$\begin{aligned} L(kq) &:= P(kq)\varphi(kq-d) \\ &= \frac{P(kq-q)\varphi(kq-d)}{1+\varphi^{\mathrm{T}}(kq-d)P(kq-q)\varphi(kq-d)}, \end{aligned}$$

the covariance matrix P can be updated as follows:

$$P(kq) = [I - L(kq)\varphi^{\mathrm{T}}(kq-d)]P(kq-q).$$

IV. The output tracking performance

We assume that $\{v(k), \mathcal{F}_k\}$ is a martingale difference sequence defined on a probability space $\{\Omega, \mathcal{F}, \mathcal{P}\}$, where $\{\mathcal{F}_k\}$ is the σ algebra sequence generated by $\{v(k)\}$, i.e., $\mathcal{F}_k = \sigma(v(k), v(k-1), v(k-2), \cdots)$ [22]. We shall prove the main results of this paper by formulating a martingale process and by using stochastic process theory and the martingale convergence theorem (Lemma D.5.3 in [22]).

Let us introduce some notation first. Let X be a square matrix; the symbols $\lambda_{\max}[X]$ and $\lambda_{\min}[X]$ represent the maximum and minimum eigenvalues of X, respectively; $\lambda_i[X]$ represent the ith eigenvalue of X. For $g(k) \geq 0$, we write $f(k) = O(g(k))$ if there exists a constant $\delta_m > 0$ such that $|f(k)| \leq \delta_m g(k)$.

Define

$$r(kq) = \mathrm{tr}[P^{-1}(kq)].$$

It follows easily that

$$r(kq) = \frac{N}{p_0} + \sum_{i=1}^{k} \|\varphi(iq-d)\|^2.$$

From here we get

$$\begin{aligned} r(kq) &= r(kq-q) + \|\varphi(kq-d)\|^2, \\ r(kq) &= \lambda_1[P^{-1}(t)] + \lambda_2[P^{-1}(t)] + \cdots \\ &\quad + \lambda_N[P^{-1}(t)] \\ &\leq N\lambda_{\max}[P^{-1}(kq)], \\ |P^{-1}(kq)| &= \lambda_1[P^{-1}(t)]\lambda_2[P^{-1}(t)]\cdots\lambda_N[P^{-1}(t)] \\ &\leq \lambda_{\max}^N[P^{-1}(kq)] \\ &\leq r^N(kq), \end{aligned}$$

and

$$\ln|P^{-1}(kq)| \leq N\ln r(kq), \tag{18}$$

or

$$\ln|P^{-1}(kq)| = O(\ln r(kq)).$$

In order to study the output tracking performance of the algorithm, the following lemma is required.

Lemma 1: The following inequality holds:

$$\sum_{i=1}^{\infty} \frac{\varphi^{\mathrm{T}}(iq-d)P(iq)\varphi(iq-d)}{\{\ln|P^{-1}(iq)|\}^c} < \infty, \ \text{a.s.}, \quad \text{for any } c > 1,$$

where $|\cdot|$ is the matrix determinant. (If $|P^{-1}(0)|$ is too small, we then begin the summation at some $i = i_0 > 1$ instead of $i = 1$.)

Theorem 1: For the dual-rate system in (9) and the adaptive control algorithm in (10)-(13), (16) and (17), assume that the noise sequence $\{v(k)\}$ satisfies the following conditions [22]:

$$\begin{aligned} &(A1) \qquad \mathrm{E}[v(k)|\mathcal{F}_{k-1}] = 0, \quad \text{a.s.}; \\ &(A2) \qquad \mathrm{E}[v^2(k)|\mathcal{F}_{k-1}] = \sigma_v^2(k) \leq \sigma_v^2 < \infty, \quad \text{a.s.}; \\ &(A3) \qquad \limsup_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} v^2(i) \leq \sigma_v^2 < \infty, \ \text{a.s.} \end{aligned}$$

That is, $\{v(k)\}$ is an independent random noise sequence with zero mean and bounded variance. The system delay $d \leq q$ is known, and the control law is given by (16) and (17). Then the adaptive control algorithm proposed ensures that the output tracking error at the output sampling instants has the property of minimum variance, i.e.,

1) $$\lim_{k\to\infty} \frac{1}{[\ln r(kq)]^c} \sum_{i=1}^{k} [y_r(iq) - y(iq) + v(iq)]^2 < \infty,$$ a.s., *for any* $c > 1$.

2) $$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} [y_r(iq) - y(iq) + v(iq)]^2 = 0, \quad \text{a.s.}$$

3) $$\limsup_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \mathrm{E}\{[y_f(iq) - y_r(iq)]^2|\mathcal{F}_{iq-1}\} \leq \sigma_v^2,$$ a.s.

Since single-rate systems belong to a special class of dual-rate systems with $q = 1$, the results in Theorems 1 still hold for single-rate systems.

V. The missing output estimation

From (1) and (9), we have

$$\begin{aligned} y_f(kq) &= y(kq) = \varphi^{\mathrm{T}}(kq-d)\theta + v(kq), & (19) \\ y_f(kq+j) &= \hat{y}(kq+j), \quad j = 1, 2, \cdots, q-1. & (20) \end{aligned}$$

From Figure 2 and (5), all missing output estimates $y_f(kq + j)$ can be obtained by

$$\hat{y}(kq+j) = \frac{z^{-d}\hat{\beta}(kq,z)}{\hat{\alpha}(kq,z)} u(kq+j), \tag{21}$$

where

$$
\begin{aligned}
\hat{\alpha}(kq,z) &= 1+\hat{\alpha}_1(kq)z^{-q}+\cdots+\hat{\alpha}_n(kq)z^{-qn},\\
\hat{\beta}(kq,z) &= \hat{\beta}_0(kq)+\hat{\beta}_1(kq)z^{-1}+\cdots+\hat{\beta}_{qn}(kq)z^{-qn},\\
\hat{\theta}(kq) &= [\ \hat{\alpha}_1(kq)\quad \hat{\alpha}_2(kq)\quad \cdots\quad \hat{\alpha}_n(kq)\\
&\qquad \hat{\beta}_0(kq)\quad \hat{\beta}_1(kq)\quad \cdots\quad \hat{\beta}_{qn}(kq)\ ].
\end{aligned}
$$

The output estimates $\hat{y}(kq+j)$ can also be computed from the recursive equation:

$$
\hat{y}(kq+j)+\sum_{i=1}^{n}\hat{\alpha}_i(kq)\hat{y}(kq+j-iq)
$$

$$
=\sum_{i=0}^{nq}\hat{\beta}_i(kq)u(kq-d+j-i),\quad j=0,\ 1,\ \cdots,\ q-1. \tag{22}
$$

Or

$$
\hat{y}(kq+j)=\hat{\varphi}^{\mathrm{T}}(kq+j)\hat{\theta}(kq),\quad j=1,2,\cdots,q-1,
$$

where

$$
\begin{aligned}
\hat{\varphi}(kq+j) &= [-\hat{y}(kq-q+j)\ \ -\hat{y}(kq-2q+j)\ \cdots\\
&\quad -\hat{y}(kq-qn+j)\ \ u(kq-d+j)\\
&\quad u(kq-d+j-1)\ \cdots\ u(kq-d+j-qn)]^{\mathrm{T}}.
\end{aligned}
$$

Comparing (17) with (22), we find that the missing intersample output estimates $\hat{y}(kq+j)$, $j=1,2,\cdots,\ q-1$, equal the desired outputs $y_r(kq+j)$, so we have

$$
\begin{aligned}
y_r(kq+j) &= \hat{y}(kq+j)=\hat{\varphi}^{\mathrm{T}}(kq+j)\hat{\theta}(kq), \qquad (23)\\
\hat{\varphi}(kq+j) &= [-y_r(kq-q+j)\ \ -y_r(kq-2q+j)\ \cdots\\
&\quad -y_r(kq-qn+j)\ \ u(kq-d+j)\\
&\quad u(kq-d+j-1)\ \cdots\ u(kq-d+j-qn)]^{\mathrm{T}}.
\end{aligned}
$$

It is easy to understand that the unknown intersample outputs $y(kq+j)$ are replaced by the desired outputs $y_r(kq+j)$ in that our goal is to make $y(k)$ track $y_r(k)$. Hence, combining (16) with (23) generates the control signal sequence $\{u(kq+j),\ j=0,1,\cdots,q-1\}$ based on the parameter estimates $\hat{\theta}(kq)$ obtained. Thus, the following theorem is easily established.

Theorem 2: For the dual-rate system in (9) and the adaptive control algorithm in (10)-(13), (16) and (17), assume the conditions of Theorem 1 hold, the open-loop system $(b(z)/a(z))$ is minimum phase, and the reference input $y_r(k)$ is bounded, i.e.,

$$
(A4)\qquad |y_r(k)|\le \delta_r<\infty.
$$

Then the adaptive control algorithm proposed ensures the closed-loop system to be globally convergent and stable with probability 1; in mathematical terms:

- The input and output variables are uniformly bounded, i.e.,

$$
\limsup_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}[u^2(i)+y^2(i)+y_f^2(i)]<\infty,\quad \text{a.s.}
$$

- Since $v(k)$ is an unpredicted white noise, the average tracking error approaches zero, i.e.,

$$
\lim_{k\to\infty}\frac{1}{[\ln k]^c}\sum_{i=1}^{k}[y_f(i)-y_r(i)-v(i)]^2=0,\quad \text{a.s.,}
$$

for any $c>1$. Or

$$
\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}[y_f(i)-y_r(i)-v(i)]^2=0,\quad \text{a.s.}
$$

In order to avoid generating $u(k)$ with too large magnitudes, for a given small positive ε, if $|\hat{\beta}_0(kq)|<\varepsilon$, we take $\hat{\beta}_0(kq)=\mathrm{sgn}[\hat{\beta}_0(kq)]\varepsilon$, where the sign function is defined by

$$
\mathrm{sgn}(x)=\begin{cases}1, & x\ge 0,\\ 0, & x<0.\end{cases}
$$

VI. Example

Assume that the discrete system model with period $h=2$ s takes the following form

$$
P_1(z)=\frac{z^{-d}b(z)}{a(z)}=\frac{4.12z^{-1}+3.09z^{-2}}{1-1.60z^{-1}+0.80z^{-2}},\quad d=1.
$$

Take $qh=4$ s and $qh=8$ s, i.e., $q=2$ and $q=4$, respectively. Use $\{v(k)\}$ as a white noise sequence with zero mean and variance $\sigma_v^2=1.00^2$. We apply the adaptive control algorithm in Section 3 to this system, and the results with different q are shown in Figures 3 and 4, where $y(k)$ represents the system output, $y_r(k)$ denotes the desired output, and

$$
y_r(400i+j)=(-1)^i 2,\ \ i=0,1,2,\cdots;\ \ j=1,2,\cdots,400.
$$

Figure 5 is the simulated results in terms of the Åström-Wittenmark self-tuning regulator (STR).

From Figures 3-5, it is clear that our control law is superior to that of the Åström-Wittenmark self-tuning regulator, but when q is too large, the output tracking performance degrades.

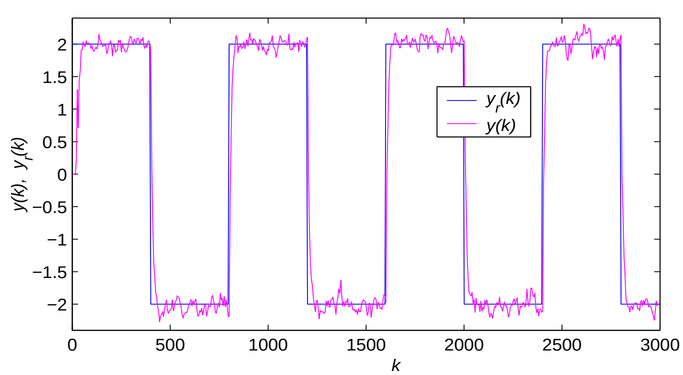

Fig. 3. $y_r(k)$ and $y(k)$ versus k $(q=2)$

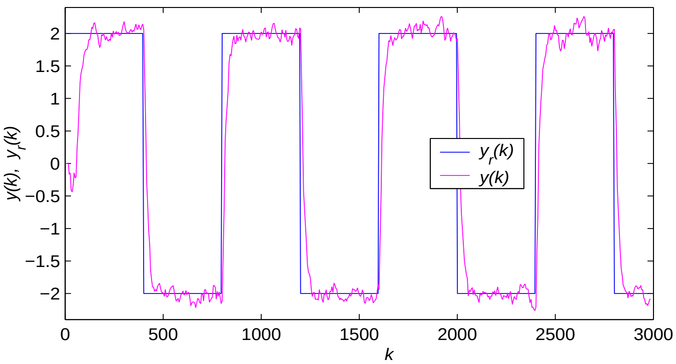

Fig. 4. $y_r(k)$ and $y(k)$ versus k ($q = 4$)

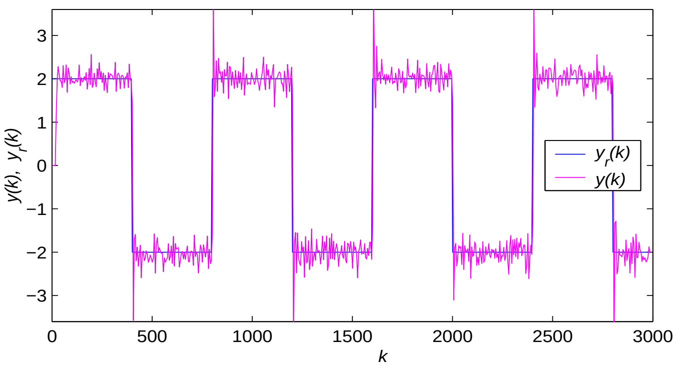

Fig. 5. $y_r(k)$ and $y(k)$ versus k ($q = 1$) of the Åström-Wittenmark STR

VII. Conclusions

A least squares based control algorithm for dual-rate systems is presented; the algorithm uses only slow-rate output measurement data and generates a relative fast-rate control signal. Convergence performance of the proposed algorithm is analyzed in detail in the stochastic framework. The algorithm achieves the desired control performance under certain conditions. The algorithm is also applied to a simulated system successfully, and the simulated results verify the theoretical findings. The control method for the case $d > q$ is currently being studied in the stochastic framework.

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IX. References

[1] M. Ohshima, I. Hashimoto, M.Takeda, T.Yoneyama and F.Goto, "Multirate multivariable model predictive control and its application to a semi-commercial polymerization reactor," *Proc. of the 1992 ACC*, vol. 2, pp. 1576-1581, 1992.

[2] R.D.Gudi, S.L.Shah and M.R.Gray, "Multirate state and parameter estimation in an antibiotic fermentation with delayed measurements," *Biotechnology and Bioengineering*, vol. 44, pp. 1271-1278, 1994.

[3] D. Li, S.L. Shah, T. Chen and K.Z. Qi, "Application of dual-rate modeling to CCR octane quality inferential control," *IEEE Trans. on Control Systems Technology*, vol. 11, no. 1, pp. 43-51, 2003.

[4] R. Scattolini, "Self-tuning control of systems with infrequent and delayed output sampling," *IEE Proceedings, Part D, Control Theory and Applications*, vol. 135, no. 4, pp. 213-221, 1988.

[5] G.M. Kranc, "Input-output analysis of multirate feedback systems," *IRE Trans. Automat. Contr.*, vol. 3, pp. 21-28, 1957.

[6] H.M. Al-Rahmani and G.F. Franklin, "A new optimal multirate control of linear periodic and time-invariant systems," *IEEE Trans. Automat. Contr.*, vol. 35, no. 4, pp. 406-415, 1990.

[7] H.M. Al-Rahmani and G.F. Franklin, "Multirate control: A new approach," *Automatica*, vol. 28, no. 1, pp. 35-44, 1992.

[8] T. Chen and L. Qiu, "$\mathcal{H}_\infty$ design of general multirate sampled-data control systems," *Automatica*, vol. 30, no. 7, pp. 1139-1152, 1994.

[9] L. Qiu and T. Chen, "$\mathcal{H}_2$-optimal design of multirate sampled-data systems," *IEEE Trans. Automat. Contr.*, vol. 39, no. 12, pp. 2506 - 2511, 1994.

[10] L. Qiu and T. Chen, "Multirate sampled-data systems: all $\mathcal{H}_\infty$ suboptimal controllers and the minimum entropy controller," *IEEE Trans. Automat. Contr.*, vol. 44, no. 3, pp. 537-550, 1999.

[11] M.F. Sagförs, H.T. Toivonen and B. Lennartson, "State-space solution to the periodic multirate $\mathcal{H}_\infty$ control problem: a lifting approach," *IEEE Trans. Automat. Contr.*, vol. 45, no. 45, pp. 2345-2350, 2000.

[12] J.H. Lee, M.S. Gelormino and M. Morari, "Model predictive control of multirate sampling-data systems: A state-space approach," *Int. J. Control*, vol. 55, pp. 153-191, 1992.

[13] R. Scattolini and N. Schiavoni, "A multirate model-based predictive controller," *IEEE Trans. Automat. Contr.*, vol. 40, no. 6, pp. 1093-1097, 1995.

[14] K.V. Ling and K.W. Lim, "A state-space GPC with extensions to multirate control," *Automatica*, vol. 32, no. 7, pp. 1067-1071, 1996.

[15] J. Sheng, T. Chen and S.L. Shah, "Generalized predictive control for non-uniformly sampled systems," *J. Process Control*, vol. 12, no. 8, pp. 875-885, 2002.

[16] P. Albertos, J. Salt and J. Tormero, "Dual-rate adaptive control," *Automatica*, vol. 32, no. 7, pp. 1027-1030, 1996.

[17] C. Zhang, R.H. Middleton and R.J. Evans, "An algorithm for multirate sampling adaptive control," *IEEE Trans. Automat. Contr.*, vol. 34, no. 7, pp. 792-795, 1989.

[18] D. Li, S.L. Shah and T. Chen, "Analysis of dual-rate inferential control systems," *Automatica*, vol. 38, no. 6, pp. 1053-1059, 2002.

[19] K. J. Åström and B. Wittenmak, "On self-tuning regulators," *Automatica*, vol. 9, no. 2, pp. 185-199, 1973.

[20] J. Kanniah, O.P. Malik, and G.S. Hope, "Self-tuning regulator based on dual-rate sampling," *IEEE Trans. Automat. Contr.*, vol. 29, no. 8, pp. 755-759, 1984.

[21] I. Mitsuaki, K. Masaki and N. Hiroaki, "Ripple-suppressed multirate adaptive control," *15th IFAC World Congress*, Barcelona, Spain, 2002.

[22] G.C. Goodwin and K.S. Sin, *Adaptive Filtering Prediction and Control*. Englewood Cliffs, NJ: Prentice-hall, 1984.

[23] T. Chen and B.A. Francis, *Optimal Sampled-data Control Systems*. London: Springer, 1995.

[24] D. Li, S.L. Shah and T. Chen, "Identification of fast-rate models from multirate data," *Int. J. Control*, vol. 74, no. 7, pp. 680-689, 2001.

[25] A.K. Tangirala, D. Li, R.S. Patwardhan, S.L. Shah and T. Chen, "Ripple-free conditions for lifted multirate control systems," *Automatica*, vol. 37, no. 10, pp. 1637-1645, 2001.