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Periodically Weighted Model-Matching Problems by LPTV Controllers Formulated in Dual Lifted Forms

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Abstract— **We propose two types of periodically weighted model-matching problem by linear periodically time-varying (LPTV) control for LTI plants. The causality constraint of LPTV controller is satisfied via a representation that we call** *dual lifted forms*. **We show the superiority of LPTV control to LTI control and demonstrate a design example.**

I. INTRODUCTION

Periodic digital control has been attracting many researchers for its ability for gain margin improvement [1], zero placement [2], simultaneous stabilization [1] etc. Motivated by these results, we focus on model-matching problem by periodic control because we can convert various control problems into model-matching form via Youla parameterization [3], [4].

Model-matching problems for LPTV controller have been investigated in [5]–[10]. Colaneri et al. [8] designed a state feedback LPTV controller so that the closed-loop transfer function is identical to a given LTI transfer function. Chapellat et al. [5] showed that time-varying controllers have not any advantage over periodic controllers. Tanaka et al. [7] dealt with the same problem in [5] and concluded that the cost of $\mathcal{H}^\infty$ control for LTI plant cannot be improved by any LPTV controller. This conclusion coincides with the preceding researches about fundamental performance limitation of LPTV control for an LTI plant [11], [12].

The main difficulty of LPTV controller design is the causality constraint in lifted system representation. Several approaches for this constraint can be found such as convex optimization [6], nest algebra [9] and LMI formulation [10]. In this paper, we show an LTI approximation approach completely different from them. Namely, we introduce two forms of causal LPTV systems as dual lifted forms and utilize them to restrict the Q-parameter in Youla parameterization. And our proposing problems are not for LPTV plants but for LTI plants. This is because one of our motivation is to reveal the fundamental difference between LPTV control and LTI control for LTI plants.

We use the following notation. The set of integers, the set of complex numbers and the set of real numbers are denoted by $\mathbb{Z}$, $\mathbb{C}$ and $\mathbb{R}$, respectively. The set of discrete-time sequences of n-th dimension is denoted by l^n. A time forward shift operator on l^n is denoted by q. The set of linear causal maps from l^m to l^n is denoted by $\mathbf{L}^{n\times m}$. The set $\mathbf{RL}^{n\times m}$ is a subset of $\mathbf{L}^{n\times m}$, which is represented by matrices whose elements are rational functions of z. The set $\mathbf{RH}_\infty$ is a subset of $\mathbf{RL}$, whose poles are in the open unit disk on $\mathbb{C}$. The symbol $e_i \in \mathbb{R}^N$ denotes the i-th unit vector: $(e_i)_j = \delta_{ij}$ where δ_{ij} is the Kronecker delta. We also use the symbol $E_i \triangleq e_i \otimes I_r$ where $\otimes$ is the matrix Kronecker product [4]. A linear discrete-time system $G \in \mathbf{L}^{n\times m}$ is called τ-periodic if G satisfies $G = q^\tau G q^{-\tau}$ where τ can take any non-negative integer value [1].

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II. PRELIMINARIES

Firstly, we define the lift operator $W : l^r \to l^{rN}$ introduced in [1], which is an isomorphism on l^r, and the inverse operator of W denoted by $W^{-1} : l^{rN} \to l^r$ as

$$W : \alpha(z) = \sum_{i=0}^{\infty} \alpha_i z^{-i} \to \begin{pmatrix} \alpha^{(0)}(z) \\ \vdots \\ \alpha^{(N-1)}(z) \end{pmatrix}, \quad (1)$$

$$W^{-1} : \begin{pmatrix} \alpha^{(0)}(z) \\ \vdots \\ \alpha^{(N-1)}(z) \end{pmatrix} \to \alpha(z) = \sum_{i=0}^{N-1} \alpha^{(i)}(z^N) z^{-i}, \quad (2)$$

where $\alpha^{(i)}(z) \triangleq \sum_{j=0}^{\infty} \alpha_{Nj+i} z^{-j}$ for $i = 0, \cdots, N-1$. In the following argument, we oftenly state that $\alpha \in l^r$ is in the original signal space and $W\alpha \in l^{rN}$ is in the lifted signal space.

We use the Youla parameterization [3], [4] of all stabilizing controllers, where the generalized plant $P(z) \in \mathbf{RL}$ and the stabilizing controller $\Omega \in \mathbf{RL}$ are

$$P(z) = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right), \quad (3)$$

$$\Omega = \Omega_1(z) - \Omega_2(z)Q(\Omega_3(z) + \Omega_4(z)Q)^{-1}, \quad (4)$$

$$[\ \Omega_1 \ \ \Omega_2\ ] = \left(\begin{array}{c|cc} A - B_2K + HC_2' & H & -B_2 - HD_{22} \\ \hline K & 0 & I \end{array}\right), \quad (5)$$

$$C_2' \triangleq C_2 - D_{22}K, \quad (6)$$

$$[\ \Omega_3 \ \ \Omega_4\ ] = \left(\begin{array}{c|cc} A - B_2K & -H & -B_2 \\ \hline C_2' & I & -D_{22} \end{array}\right). \quad (7)$$

We assume that the free parameter Q is an N-periodic stable LPTV system. Consequently, Ω is a stabilizing controller [13] and the closed-loop transfer function T has the following model-matching form where $T_1, T_2, T_3 \in \mathbf{RH}_\infty$:

$$T = T_1(z) - T_2(z)QT_3(z). \quad (8)$$

III. PROBLEM DEFINITION

Let us define an up-sampling operator U_i, a down-sampling operator D_i and a projection operator Pr^j on discrete time sequences as

$$U_i \triangleq W^{-1}E_{i+1}, \quad D_i \triangleq E_{i+1}^T W, \quad \mathrm{Pr}^j \triangleq U_j D_j. \tag{9}$$

It is easy to check Pr^j is a projection and has the eigen signal space $\mathcal{W}_j^r \triangleq \{w \in l^r \mid \mathrm{Pr}^j w = w\}$.

Property 1: The projection operator Pr^i has the following properties:

1) $\forall\, i, j \in \mathbb{Z}[0, N-1], \quad \mathrm{Pr}^i\mathrm{Pr}^j = \delta_{ij}\mathrm{Pr}^i$.
2) $\mathrm{Pr}^0 + \mathrm{Pr}^1 + \cdots + \mathrm{Pr}^{N-1} = I$.

From the properties the signal space l^r can be decomposed by $\mathcal{W}_j^r$ as

$$l^r = \mathcal{W}_0^r \oplus \mathcal{W}_1^r \oplus \cdots \oplus \mathcal{W}_{N-1}^r. \tag{10}$$

The class $\mathcal{Q}^{\mathrm{p}}$ defined below represents causal, N-periodic and stable systems (See Section IV for the detail).

$$\mathcal{Q}^{\mathrm{p}} \triangleq \{Q \mid Q : \text{causal}, \; WQW^{-1} \in \mathbf{RH}_\infty\} \tag{11}$$

Then, we define the periodic model-matching problems with the projection operator Pr^i:

P1. Find Q^* such that

$$Q^* = \arg \inf_{Q\in\mathcal{Q}^{\mathrm{p}}} \left\| [T_1(z) - T_2(z)QT_3(z)]\mathrm{Pr}^i \right\|,$$

P2. Find Q^* such that

$$Q^* = \arg \inf_{Q\in\mathcal{Q}^{\mathrm{p}}} \left\| \mathrm{Pr}^i[T_1(z) - T_2(z)QT_3(z)] \right\|.$$

A possible practical control subject with regard to these problems is to reduce the effect by periodical impulsive disturbances caused by gas explosions in a cylinder or to move a hard disk head to desirable positions more precisely at every clock time when data arrives.

In the following sections, we show an LTI approximation approach to apply LTI design procedures to P1 and P2 problems.

IV. LIFTED SYSTEM REPRESENTATION

A. Lifting for LTI Systems

Similar to the equation (2), we can decompose an LTI transfer function $G(z) \in \mathbf{RL}^{n\times m}$ into LTI subsystems uniquely [1], [14] as

$$G(z) = \sum_{i=0}^{N-1} z^{-i} G^{(i)}(z^N), \tag{12}$$

where $G^{(i)}(z) \in \mathbf{RL}^{n\times m}$ for $i = 0, 1, \cdots, N-1$. By letting (A, B, C, D) be a state space realization of $G(z)$, each $G^{(i)}(z)$ is represented by

$$G^{(i)}(z) = \begin{cases} (a) \quad i = 0 \\ \quad D + C(zI - A^N)^{-1}A^{N-1}B \\ (b) \quad i = 1, \cdots, N-1 \\ \quad CA^{i-1}B + CA^i(zI - A^N)^{-1}A^{N-1}B \end{cases}. \tag{13}$$

Obviously, when $G(z)$ is stable, $G^{(i)}(z)$ is also stable for $i = 0, \cdots, N-1$. We introduce a pair of row and column operations for $\tilde{G}(z)$ as $\mathrm{cs} : \mathbf{RL}^{n\times m} \to \mathbf{RL}^{nN\times m}$ and $\mathrm{rs} : \mathbf{RL}^{n\times m} \to \mathbf{RL}^{n\times mN}$:

$$\begin{aligned} \mathrm{cs}\ G(z) &\triangleq \begin{pmatrix} G^{(0)}(z) \\ \vdots \\ G^{(N-1)}(z) \end{pmatrix}, \\ \mathrm{rs}\ G(z) &\triangleq \begin{pmatrix} G^{(N-1)}(z) & \cdots & G^{(0)}(z) \end{pmatrix}. \end{aligned}$$

Furthermore, let $\mathrm{ccs}\ G(z)$ and $\mathrm{rrs}\ G(z)$ denote

$$\begin{aligned} \mathrm{ccs}\ G(z) &\triangleq \big[\; G^{(0)}(z^N) \quad z^{-1}G^{(1)}(z^N) \quad \cdots \\ &\qquad \cdots \quad z^{-N+1}G^{(N-1)}(z^N) \;\big], \\ \mathrm{rrs}\ G(z) &\triangleq \left(\mathrm{ccs}\ G^T(z)\right)^T. \end{aligned}$$

where $G^{(i)}(z^N)$ is given by substituting z^N into $G^{(i)}(z)$ defined by (13). Note that $\mathrm{cs}\ G(z)$ and $\mathrm{rs}\ G(z)$ consist of matrix blocks because $G(z)$ is a matrix function and the operators cs and rs are one to one maps due to the uniqueness of $G^{(i)}(z)$ defined in (12) and (13). An existence condition of stable inverses of $\mathrm{rrs}\ G(z)$ and $\mathrm{ccs}\ G(z)$ can be derived as the following lemma. The operators ccs, rrs and their stable inverses are used in our main results.

Lemma 1: Let an LTI system $G(z)$ have a state space realization (A, B, C, D). And let us introduce two functions $M_N^{\mathrm{c}}(A, B, \xi)$ and $M_N^{\mathrm{o}}(A, C, \xi)$ with respect to two matrices A, B and a vector $\xi \in \mathbb{R}^{N-1}$ as

$$\begin{aligned} &M_N^{\mathrm{c}}(A, B, \xi) \\ &\quad \triangleq \left(A^{N-1} + \xi_1 I + \xi_2 A + \cdots + \xi_{N-1}A^{N-2}\right)B, \quad (14) \\ &M_N^{\mathrm{o}}(A, C, \xi) \\ &\quad \triangleq C\left(A^{N-1} + \xi_1 I + \xi_2 A + \cdots + \xi_{N-1}A^{N-2}\right). \quad (15) \end{aligned}$$

Then, when there exists ξ such that $M_N^{\mathrm{c}}(A, B, \xi) = 0$ or $M_N^{\mathrm{o}}(A, C, \xi) = 0$, the following two statements hold.

1) If there exists a constant matrix $D^\#$ such that $DD^\# = I$, then there exists $(\mathrm{ccs}\ G)^\#(z) \in \mathbf{RH}_\infty$ such that $\mathrm{ccs}\ G(z) \cdot (\mathrm{ccs}\ G)^\#(z) = I$.
2) If there exists a constant matrix $D^\#$ such that $D^\# D = I$, then there exists $(\mathrm{rrs}\ G)^\#(z) \in \mathbf{RH}_\infty$ such that $(\mathrm{rrs}\ G)^\#(z) \cdot \mathrm{rrs}\ G(z) = I$.

Proof: The proof of the first part is similar to that of the second part. So we only prove the second part here. By selecting $(\mathrm{rrs}\ G)^\#(z)$ as

$$\begin{aligned} (\mathrm{rrs}\ G)^\#(z) &\triangleq \big[\; D^\# \quad z^{-N+1}\xi_1 D^\# \quad \cdots \\ &\qquad \cdots \quad z^{-1}\xi_{N-1}D^\# \;\big], \end{aligned} \tag{16}$$

it is obvious that $(\mathrm{rrs}\ G)^\#(z) \in \mathbf{RH}_\infty$ and the following equations complete the proof.

$$\begin{aligned} &(\mathrm{rrs}\ G)^\#(z) \cdot \mathrm{rrs}\ G(z) \\ &\quad = D^\# D + D^\# M_N^{\mathrm{o}}(A, C, \xi)(z^N I - A^N)^{-1}B \\ &\quad = D^\# D + D^\# C(z^N - A^N)^{-1}M_N^{\mathrm{c}}(A, B, \xi). \end{aligned}$$

■

B. Dual Lifted Forms for N-Periodic Systems

We show that a causal LPTV system has a pair of dual representations in the lifted signal space. The dual lifted forms enable us to calculate the multiplication of two causal LPTV systems very simply and this is a key idea to solve problems P1 and P2.

The lifted forms of time shift operators have been shown in [15]. Let $\tilde{q}^{\tau} \triangleq Wq^{\tau}W^{-1}$ and $\tilde{q}^{-\tau} \triangleq Wq^{-\tau}W^{-1}$ for $\tau \in \mathbb{Z}[0, N]$. Then, $\tilde{q}^{\tau}$ and $\tilde{q}^{-\tau}$ can be represented as

$$\tilde{q}^{\tau} = \left(\begin{array}{c|c} 0 & I_{N-\tau} \\ \hline zI_{\tau} & 0 \end{array}\right) \otimes I_r, \tag{17}$$

$$\tilde{q}^{-\tau} = \left(\begin{array}{c|c} 0 & z^{-1}I_{\tau} \\ \hline I_{N-\tau} & 0 \end{array}\right) \otimes I_r. \tag{18}$$

By using the lifted time shift operators, we can give a parameterization of causal LPTV systems. In this parameterization, all free parameters are LTI and embedded via operators the rs and cs. Consequently, the causality constraint is satisfied naturally.

Theorem 1: For any linear causal N-periodic discrete-time system $G \in \mathbf{L}^{n\times m}$, $\tilde{G} = WGW^{-1}$ is an element of $\mathbf{RL}^{nN\times mN}$ and can be represented by the following pair of dual lifted forms:

F1. $\tilde{G}(z) = \left[\ \tilde{q}^{0}\mathrm{cs}\ G_0(z) \quad \cdots \quad \tilde{q}^{-N+1}\mathrm{cs}\ G_{N-1}(z)\ \right]$, where $G_i(z) \in \mathbf{RL}^{n\times m}$ for $i = 0, \cdots, N-1$.

F2. $\tilde{G}(z) = \begin{bmatrix} \mathrm{rs}\ G_{N-1}(z)\cdot\tilde{q}^{-N+1} \\ \mathrm{rs}\ G_{N-2}(z)\cdot\tilde{q}^{-N+2} \\ \vdots \\ \mathrm{rs}\ G_0(z)\cdot\tilde{q}^{0} \end{bmatrix}$, where $G_i(z) \in \mathbf{RL}^{n\times m}$ for $i = 0, \cdots, N-1$.

Proof: Because G is linear and N-periodic, G satisfies $G = q^N G q^{-N}$. From this and the equations $\tilde{q}^N = zI$ and $\tilde{q}^{-N} = z^{-1}I$, $\tilde{G} \triangleq WGW^{-1}$ satisfies $\tilde{G} = \tilde{q}^N\tilde{G}\tilde{q}^{-N} = q\tilde{G}q^{-1}$. This means that $\tilde{G}$ is an element of $\mathbf{RL}^{nN\times mN}$ and we can write $\tilde{G}(z)$ as a block matrix $[G_{ij}(z)]$ where $G_{ij}(z) \in \mathbf{RL}^{n\times m}$ and $i, j \in \mathbb{Z}[1, N]$. Furthermore, $\lim_{z\to\infty}\tilde{G}(z)$ must be block lower triangular because of causality [1], [13]. Therefore, the upper right blocks of $\tilde{G}$ can be represented as $z^{-1}\hat{G}_{ij}(z)$, where $j > i$ and $\hat{G}_{ij}(z) \in \mathbf{RL}^{n\times m}$. (For example, $\frac{1}{z-1} = z^{-1}\cdot\frac{z}{z-1}$.) In addition, because of the uniqueness of the maps cs and rs, there always exists $G_i(z) \in \mathbf{RL}^{n\times m}$ such that $\tilde{q}^{-i}\mathrm{cs}\ G_i(z)$ equals the $i+1$-th column block WGW^{-1} in the F1 case and rs $G_{N-1-i}(z)\cdot\tilde{q}^{-N+1+i}$ is equal to the $i+1$-th row block of WGW^{-1} in the F2 case. This completes the proof. ■

The relationship between an LPTV system represented in dual lifted forms and its implicit period is given as follows.

Corollary 1: Given a system $\tilde{G}$ in F1 or F2 form, then $G = W^{-1}\tilde{G}W$ is τ-periodic (i.e. $\tilde{G}(z) = \tilde{q}^{\tau}\tilde{G}(z)\tilde{q}^{-\tau}$) iff the following two conditions are satisfied.

1) $G_i(z) = G_{\tau+i}(z)$ is satisfied for $i = 0, \cdots, N-\tau-1$.
2) $G_i(z) = G_{N-\tau+i}(z)$ is satisfied for $i = 0, \cdots, \tau-1$.

In particular, G is LTI (i.e. 1-periodic) iff $G_0(z) = G_1(z) = \cdots = G_{N-1}(z) = G(z)$.

The two conditions in Corollary 1 always hold when $\tau = N$. Therefore, when given a system $\tilde{G} \in \mathbf{RL}^{nN\times mN}$ in F1 or F2 form, there always exists a linear causal N-periodic system $G \in \mathbf{L}^{n\times m}$ whose lifted form is $\tilde{G}$.

Let us define two classes of causal and stable LPTV systems for the Q parameter:

$$\mathcal{Q}^{\mathrm{c}} \triangleq \{Q \mid WQW^{-1} \text{ is (19) },\ \forall i\ Q_i^{\mathrm{c}}(z) \in \mathbf{RH}_\infty\},$$
$$\tilde{Q}(z) = \left[\ \tilde{q}^{0}\mathrm{cs}\ Q_0^{\mathrm{c}}(z) \quad \cdots \quad \tilde{q}^{-N+1}\mathrm{cs}\ Q_{N-1}^{\mathrm{c}}(z)\right], \tag{19}$$
$$\mathcal{Q}^{\mathrm{r}} \triangleq \{Q \mid WQW^{-1} \text{ is (20) },\ \forall i\ Q_i^{\mathrm{r}}(z) \in \mathbf{RH}_\infty\},$$
$$\tilde{Q}(z) = \begin{bmatrix} \mathrm{rs}\ Q_{N-1}^{\mathrm{r}}(z)\cdot\tilde{q}^{-N+1} \\ \vdots \\ \mathrm{rs}\ Q_0^{\mathrm{r}}(z)\cdot\tilde{q}^{0} \end{bmatrix}. \tag{20}$$

The two classes $\mathcal{Q}^{\mathrm{c}}$ and $\mathcal{Q}^{\mathrm{r}}$ are equivalent to $\mathcal{Q}^{\mathrm{p}}$ although their representations are different.

The well-posedness condition of the LPTV controller Ω expressed by (4)–(7) with a causal N-periodic Q parameter is equivalent to the existence of a causal inverse map of $\Omega_3 + \Omega_4 Q$ in $\mathbf{L}^{m\times m}$ and the following theorem states the necessary and sufficient condition for it.

Theorem 2: For causal N-periodic Q, $\Omega_3 + \Omega_4 Q$ has a causal inverse in $\mathbf{L}^{m\times m}$ iff the following equation is satisfied for $i = 0, \cdots, N-1$.

$$\lim_{z\to\infty}\det(I_m - D_{22}Q_i^{(0)}(z)) \neq 0 \tag{21}$$

where $Q_0(z), \cdots, Q_{N-1}(z)$ are parameters of F1 form or F2 form.

Proof: By lifting (4) from both sides we get $\tilde{\Omega} = \tilde{\Omega}_1 - \tilde{\Omega}_2\tilde{Q}(\tilde{\Omega}_3 + \tilde{\Omega}_4\tilde{Q})^{-1}$. Because Q is causal N-periodic, we can represent $\tilde{Q}$ in F1 form:

$$\tilde{Q} = \left[\ \tilde{q}^{0}\mathrm{cs}\ Q_0(z) \quad \cdots \quad \tilde{q}^{-N+1}\mathrm{cs}\ Q_{N-1}(z)\ \right].$$

Therefore, $\tilde{\Omega}_3 + \tilde{\Omega}_4\tilde{Q}$ is a rational matrix function of z in the lifted signal space and has an inverse iff its constant term is non-singular. Let us define D_∞ as $D_\infty \triangleq \lim_{z\to\infty}(\tilde{\Omega}_3 + \tilde{\Omega}_4\tilde{Q}) = I_N \otimes I_m - I_N \otimes D_{22}\cdot\lim_{z\to\infty}\tilde{Q}(z)$. Note that D_∞ is block lower triangular because Ω_3, Ω_4, Q are causal transfer functions. Hence, D_∞ is non-singular iff its diagonal blocks are non-singular. Moreover, D_∞^{-1} is also block lower triangular if it exists. Consequently, the condition:

$$\lim_{z\to\infty}\det(\Omega_3^{(0)}(z) + \Omega_4^{(0)}(z)Q_i^{(0)}(z)) \neq 0\ \ \forall\, i \in \mathbb{Z}[0, N-1]$$

is necessary and sufficient for the existence of a causal N-periodic inverse of $\Omega_3 + \Omega_4 Q$. ■

From this theorem, we know that only the constant part of Q is relevant to the well-posedness of the LPTV controller. Therefore, any LPTV controller parameterized as (4) can always be made well-posed by adding an appropriate nonzero constant ϵ to the free parameter Q.

V. MAIN RESULTS

The following theorem shows that the problems P1 and P2 can be expressed by corresponding LTI transfer functions.

Theorem 3: The following two statements hold.

1) Let $Q \in \mathcal{Q}^{\mathrm{c}}$, then

$$T_1(z) - T_2(z)QT_3(z) = \sum_{i=0}^{N-1} T^{\mathrm{I(i)}}(z)\mathrm{Pr}^i, \quad (22)$$

$$T^{\mathrm{I(i)}}(z) \triangleq T_1(z) - T_2(z)Q^{\mathrm{I(i)}}(z)\cdot \mathrm{rrs}\ T_3(z),$$

$$Q^{\mathrm{I(i)}}(z) \triangleq \begin{cases} (a) \quad i = 1, \cdots, N-1 \\ \left[\ Q_i^{\mathrm{c}}(z) \quad \cdots \quad Q_{N-1}^{\mathrm{c}}(z), \right. \\ \left. \qquad Q_0^{\mathrm{c}}(z) \quad \cdots \quad Q_{i-1}^{\mathrm{c}}(z)\ \right] \\ (b) \quad i = 0 \\ \left[\ Q_0^{\mathrm{c}}(z) \quad \cdots \quad Q_{N-1}^{\mathrm{c}}(z)\ \right] \end{cases}.$$

2) Let $Q \in \mathcal{Q}^{\mathrm{r}}$, then

$$T_1(z) - T_2(z)QT_3(z) = \sum_{i=0}^{N-1} \mathrm{Pr}^i T^{\mathrm{O(i)}}(z), \quad (23)$$

$$T^{\mathrm{O(i)}}(z) \triangleq T_1(z) - \mathrm{ccs}\ T_2(z)\cdot Q^{\mathrm{O(i)}}(z)T_3(z),$$

$$Q^{\mathrm{O(i)}}(z) \triangleq$$

$$\begin{cases} (a) \quad i = 0, \cdots, N-2 \\ \left[\ Q_{N-1-i}^{\mathrm{r}}(z)^T \quad \cdots \quad Q_{N-1}^{\mathrm{r}}(z)^T, \right. \\ \left. \qquad Q_0^{\mathrm{r}}(z)^T \quad \cdots \quad Q_{N-2-i}^{\mathrm{r}}(z)^T\ \right]^T \\ (b) \quad i = N-1 \\ \left[\ Q_0^{\mathrm{r}}(z)^T \quad \cdots \quad Q_{N-1}^{\mathrm{r}}(z)^T\ \right]^T \end{cases}.$$

Note that $T^{\mathrm{I(i)}}(z)$ and $T^{\mathrm{O(i)}}(z)$ contain all free parameters of $\mathcal{Q}^{\mathrm{c}}$ and $\mathcal{Q}^{\mathrm{r}}$. Hence, the statements of the theorem are not conservative at all. The proof is based on dual lifted forms.

Proof: From linearity of T and Property 1, $T = T\mathrm{Pr}^0 + \cdots + T\mathrm{Pr}^{N-1} = \mathrm{Pr}^0 T + \cdots + \mathrm{Pr}^{N-1}T$ hold. We show that $T\mathrm{Pr}^i = T^{\mathrm{I(i)}}(z)\mathrm{Pr}^i$ and $\mathrm{Pr}^i T = \mathrm{Pr}^i T^{\mathrm{I(i)}}(z)$. Because $T_3(z)$ is LTI, $\tilde{T}_3(z)$ can be represented in F2 form:

$$\tilde{T}_3(z) = \begin{bmatrix} \mathrm{rs}\ T_3(z)\cdot \tilde{q}^{-N+1} \\ \vdots \\ \mathrm{rs}\ T_3(z)\cdot \tilde{q}^0 \end{bmatrix}. \quad (24)$$

Therefore, from $T\mathrm{Pr}^i = T_1\mathrm{Pr}^i - T_2W^{-1}\tilde{Q}\tilde{T}_3E_{i+1}D_i$,

$$\begin{aligned} &W^{-1}\tilde{Q}\tilde{T}_3E_{i+1} \\ &= W^{-1}\left(\sum_{j=0}^{N-1} \tilde{q}^{-j}\mathrm{cs}\ Q_j^{\mathrm{c}}\cdot \mathrm{rs}\ T_3\cdot \tilde{q}^{j-N+1}\right)E_{i+1} \\ &= W^{-1}\sum_{j=0}^{N-1} \tilde{Q}_j^{\mathrm{c}}E_{j+1}E_{j+1}^T\tilde{T}_3E_{i+1} \\ &= W^{-1}\left(\sum_{j=0}^{i-1} \tilde{Q}_j^{\mathrm{c}}E_{j+1}z^{-1}T_3^{(N-i+j)}(z) \right. \\ &\quad \left. + \sum_{j=i}^{N-1} \tilde{Q}_j^{\mathrm{c}}E_{j+1}T_3^{(j-i)}(z)\right) \\ &= \left(\sum_{j=0}^{i-1} Q_j^{\mathrm{c}}(z)V_{N-i+j}^{\mathrm{c}}(z) + \sum_{j=i}^{N-1} Q_j^{\mathrm{c}}(z)V_{j-i}^{\mathrm{c}}(z)\right)U_i \end{aligned}$$

where $V_j^{\mathrm{c}}(z) \triangleq z^{-j}T_3^{(j)}(z^N) \quad (j = 0, \cdots, N-1)$. Consequently, we get the first part of the theorem.

The proof of the second part is similar to that of the first part. At first $\tilde{T}_2(z)$ can be represented in F1 form:

$$\tilde{T}_2(z) = \left[\ \tilde{q}^0\mathrm{cs}\ T_2(z) \quad \cdots \quad \tilde{q}^{-N+1}\mathrm{cs}\ T_2(z)\ \right]. \quad (25)$$

The equation $\mathrm{Pr}^i T = \mathrm{Pr}^i T_1 - U_iE_{i+1}^T\tilde{T}_2\tilde{Q}WT_3$ and

$$\begin{aligned} &E_{i+1}^T\tilde{T}_2\tilde{Q}W \\ &= E_{i+1}^T\left(\sum_{j=0}^{N-1} \tilde{q}^{-j}\mathrm{cs}\ T_2\cdot \mathrm{rs}\ Q_{N-1-j}^{\mathrm{r}}\cdot \tilde{q}^{j-N+1}\right)W \\ &= D_i\left(\sum_{j=0}^{i} V_{i-j}^{\mathrm{r}}(z)Q_{N-1-j}^{\mathrm{r}}(z) \right. \\ &\quad \left. + \sum_{j=i+1}^{N-1} V_{N-j+i}^{\mathrm{r}}(z)Q_{N-1-j}^{\mathrm{r}}(z)\right) \end{aligned}$$

where $V_j^{\mathrm{r}}(z) \triangleq z^{-j}T_2^{(j)}(z^N)\ (j = 0, \cdots, N-1)$ complete the proof. ∎

From Theorem 3, we can formulate LTI model-matching problems corresponding to P1 and P2 as P1' and P2':

P1'. Find $Q^{\mathrm{I(i)}*} \in \mathbf{RH}_\infty$ such that

$$Q^{\mathrm{I(i)}*} = \arg \inf_{Q^{\mathrm{I(i)}} \in \mathbf{RH}_\infty} \left\|T^{\mathrm{I(i)}}(z)\right\|, \quad (26)$$

P2'. Find $Q^{\mathrm{O(i)}*} \in \mathbf{RH}_\infty$ such that

$$Q^{\mathrm{O(i)}*} = \arg \inf_{Q^{\mathrm{O(i)}} \in \mathbf{RH}_\infty} \left\|T^{\mathrm{O(i)}}(z)\right\|. \quad (27)$$

Because (22) and (23) are satisfied for any norm definition, we can apply various design methods such as $\mathcal{H}^\infty$ control [16] and l_1 optimal control [17] to solve P1' and P2'. And we can estimate upper and lower bounds about P1 and P2 with P1' and P2'.

Proposition 1: For any LTI transfer matrix $T = T(z) \in \mathbf{RL}$,

$$\|T\| \geq \left\|T\mathrm{Pr}^0\right\| = \cdots = \left\|T\mathrm{Pr}^{N-1}\right\| \geq \frac{\|T\|}{N}, \quad (28)$$

$$\|T\| \geq \left\|\mathrm{Pr}^0 T\right\| = \cdots = \left\|\mathrm{Pr}^{N-1}T\right\| \geq \frac{\|T\|}{N}. \quad (29)$$

Proof: By representing $\tilde{T}(z)$ in F1 form, we get $T(z)\mathrm{Pr}^j = W^{-1}\tilde{q}^j\mathrm{cs}\ T\cdot D_j = q^j(T(z)\mathrm{Pr}^0)U_0D_j$ for $j = 0, \cdots, N-1$, where the third equality is satisfied from the equation $\mathrm{cs}\ T(z) = WT(z)U_0$ and the definition of Pr^j. Because q and U_j are norm preserving operators, $\left\|T(z)\mathrm{Pr}^j\right\| = \left\|T(z)\mathrm{Pr}^0U_0D_j\right\|$ and $\|U_jD_0w\| = \|D_0w\| = \left\|\mathrm{Pr}^0w\right\|$ for $w \in l_2$. In addition, from $U_0D_j\ \cdot$

$U_j D_0 = \mathrm{Pr}^0$ and $\mathrm{Pr}^0 \cdot \mathrm{Pr}^0 = \mathrm{Pr}^0$,

$$\begin{aligned}\sup_{\substack{w\in l_2\\ w\neq 0}} \frac{\|T\mathrm{Pr}^0 U_0 D_j w\|}{\|w\|} &= \sup_{\substack{w\in l_2\\ w\neq 0}} \frac{\|T\mathrm{Pr}^0 U_0 D_j U_j D_0 w\|}{\|U_j D_0 w\|} \\ &= \sup_{\substack{w\in l_2\\ w\neq 0}} \frac{\|T\mathrm{Pr}^0 w\|}{\|\mathrm{Pr}^0 w\|}.\end{aligned}$$

From this, $\left\|T(z)\mathrm{Pr}^j\right\| = \left\|T(z)\mathrm{Pr}^0\right\|$ for $j = 0, \cdots, N-1$.

We proceed to the proof of (29). Firstly, $\left\|\mathrm{Pr}^j T\right\| = \|D_j T(z)\|$ is satisfied because of the norm preserving property of U_j. In addition, $D_j T(z) = \mathrm{rs}\ T \cdot \tilde{q}^{j-N+1} W = \mathrm{rs}\ T \cdot W q^{j-N+1}$ hold by representing $\tilde{T}(z)$ in F2 form. From this and the equation $\mathrm{rs}\ T(z) = D_{N-1} T(z) W^{-1}$, the equation $D_j T(z) = D_{N-1} T(z) q^{j-N+1}$ is satisfied . Therefore, $\left\|\mathrm{Pr}^j T(z)\right\| = \left\|\mathrm{Pr}^{N-1} T(z)\right\|$ for $j = 0, \cdots, N-1$.

In addition, $\left\|\mathrm{Pr}^i T(z)\right\| \leq \|T(z)\|$ and $\left\|T(z)\mathrm{Pr}^i\right\| \leq \|T(z)\|$ are satisfied because Pr^i is a projection operator. The first and the second inequalities of (28) and (29) are satisfied from the triangular inequality and the two equations $\|T(z)\| = \left\|\sum_{j=0}^{N-1} T(z)\mathrm{Pr}^j\right\|$ and $\|T(z)\| = \left\|\sum_{j=0}^{N-1} \mathrm{Pr}^j T(z)\right\|$. ■

We can simplify the problems P1' and P2' in some cases.

Theorem 4: The following two statements hold.

1) If there exists $\xi \in \mathbb{R}^{N-1}$ such that $M_N^{\mathrm{c}}(A + HC_2, B_1 + HD_{21}, \xi) = 0$ or $M_N^{\mathrm{o}}(A + HC_2, C_2, \xi) = 0$ is satisfied, in addition, there exists a constant matrix $D_{21}^{\#}$ such that $D_{21}^{\#} D_{21} = I$, then

$$\begin{aligned}&\inf_{Q^{\mathrm{I(i)}}(z)\in\mathbf{RH}_\infty} \left\|T^{\mathrm{I(i)}}(z)\right\| \\ &\quad= \inf_{Q^{\mathrm{I}'}(z)\in\mathbf{RH}_\infty} \left\|T_1(z) - T_2(z) Q^{\mathrm{I}'}(z)\right\|. \quad (30)\end{aligned}$$

2) If there exists $\xi \in \mathbb{R}^{N-1}$ such that $M_N^{\mathrm{c}}(A - B_2 K, B_2, \xi) = 0$ or $M_N^{\mathrm{o}}(A - B_2 K, C_1 - D_{12} K, \xi) = 0$ is satisfied, in addition, there exists a constant matrix $D_{12}^{\#}$ such that $D_{12} D_{12}^{\#} = I$, then

$$\begin{aligned}&\inf_{Q^{\mathrm{O(i)}}(z)\in\mathbf{RH}_\infty} \left\|T^{\mathrm{O(i)}}(z)\right\| \\ &\quad= \inf_{Q^{\mathrm{O}'}(z)\in\mathbf{RH}_\infty} \left\|T_1(z) - Q^{\mathrm{O}'}(z) T_3(z)\right\|. \quad (31)\end{aligned}$$

Proof: We can confirm these equations by substituting $Q^{\mathrm{I(i)}}(z) = Q^{\mathrm{I}'}(z)(\mathrm{rrs}\ T_3)^{\#}(z)$ into $T^{\mathrm{I(i)}}(z)$ and substituting $Q^{\mathrm{O(i)}}(z) = (\mathrm{ccs}\ T_2)^{\#}(z) Q^{\mathrm{O}'}(z)$ into $T^{\mathrm{O(i)}}(z)$ from Lemma 1. ■

The fact that $\mathrm{ccs}\ T_2(z)$ or $\mathrm{rrs}\ T_3(z)$ can be removed from P1' and P2' indicates that unstable zeros of $T_2(z)$ or $T_3(z)$ have no effect on resulting control performances. This enables LPTV controllers be superior to LTI controllers with regard to our model-matching problems.

Remark 1: The conditions of Lemma 1 except for the existence of $D^{\#}$ are always satisfied in two cases: (1) when N is larger than the size of A; (2) when N is equal to the size of A and there exist uncontrollable modes of the pair (A, B) or unobservable modes of the pair (A, C). In the first case, we can achieve both $M_N^{\mathrm{c}}(A, B, \xi) = 0$ and $M_N^{\mathrm{o}}(A, C, \xi) = 0$ by selecting $\xi_1, \cdots, \xi_{N-1}$ as the coefficients of the characteristic polynomial of A from Cayley-Hamilton theorem (See [4]). In the second case, there exists ξ such that $M_N^{\mathrm{c}}(A, B, \xi) = 0$ or $M_N^{\mathrm{c}}(A, B, \xi) = 0$ because the controllability matrix or the observability matrix is not full rank. Conversely, it is sufficient to take $N = \mathrm{size}(A) + 1$ to construct stable inverses of the operators rrs and ccs.

Remark 2: When N is a fixed number less than the size of A matrix and $V_i^{\mathrm{r}}(z) = T_2(z)$ or $V_i^{\mathrm{c}}(z) = T_3(z)$ is satisfied for some i where $V_i^{\mathrm{r}}(z)$ and $V_i^{\mathrm{c}}(z)$ are defined in the proof of Theorem 3, minimizing $\left\|T^{\mathrm{I(i)}}(z)\right\|$ subject to $Q^{\mathrm{I(i)}}(z) \in \mathbf{RH}_\infty$ or minimizing $\left\|T^{\mathrm{O(i)}}(z)\right\|$ subject to $Q^{\mathrm{O(i)}}(z) \in \mathbf{RH}_\infty$ is equivalent to minimizing $\|T\|$ subject to $Q(z) \in \mathbf{RH}_\infty$. For example, when $T_2(z) = \frac{z}{z^2-2}$ and $N = 2$, the equation $V_1^{\mathrm{r}}(z) = T_2(z)$ is satisfied. Therefore, when the number N is fixed, our proposed LPTV controller is effective in the case that $V_i^{\mathrm{r}}(z) \neq T_2(z)$ or $V_i^{\mathrm{c}} \neq T_3(z)$ is satisfied for $i = 0, \cdots, N-1$.

VI. ILLUSTRATIVE EXAMPLE

We show an illustrative example in which the controller designed by the method based on P2' achieves arbitrary small induced norm performance. We consider 2-periodic LPTV controller for the P2 problem with regard to l_2 induced norm and compare the resulting controller with LTI controllers. We deal with the closed-loop illustrated in Fig. 1 where the plant P is $P(z) = \frac{z-2}{z-3}$.

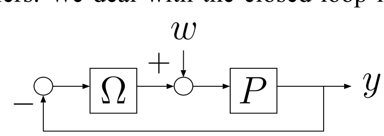

Fig. 1. A closed-loop system of the example problem

Because $P(z)$ has an unstable pole at 3 and an unstable zero at 2, trivial controllers $\Omega = 0$ and $\Omega = \infty$ do not stabilize the closed-loop. Selecting a feedback gain K and an observer gain H in (4)–(7) as $K = H = -2.5$, $\Omega_1, \cdots, \Omega_4$ is given by

$$\begin{bmatrix} \Omega_1(z) & \Omega_2(z) \\ \Omega_3(z) & \Omega_4(z) \end{bmatrix} = \begin{bmatrix} \frac{6.25}{z-4.25} & \frac{z-0.5}{z-4.25} \\ \frac{z-4.25}{z-0.5} & \frac{z-2}{z-0.5} \end{bmatrix}. \quad (32)$$

Then, we can represent the closed-loop transfer function T ($y = Tw$) as $T = T_1(z) + 4T_2(z) Q T_3(z)$, where

$$T_1(z) = -2\frac{z-2}{1-2z}\left(1 - \frac{3.75}{z-0.5}\right), \quad (33)$$

$$T_2(z) = T_3(z) = \frac{z-2}{1-2z}. \quad (34)$$

We select Q from $\mathcal{Q}^{\mathrm{r}}|_{N=2}$ which is a class of causal 2-periodic stable functions, then there exists $T^{\mathrm{O(1)}}(z)$ which satisfies $\mathrm{Pr}^1 T = \mathrm{Pr}^1 T^{\mathrm{O(1)}}(z)$ from Theorem 3:

$$T^{\mathrm{O(1)}}(z) = T_1(z) + 4\ \mathrm{ccs}\ T_2(z) \cdot Q^{\mathrm{O(1)}}(z) T_3(z). \quad (35)$$

By following Theorem 4, we set

$$Q^{\mathrm{O(1)}} = \frac{1}{4}\ (\mathrm{ccs}\ T_2)^{\#}(z) Q^{\mathrm{O}'}(z) = \begin{bmatrix} -0.5 \\ 0.25 z^{-1} \end{bmatrix} Q^{\mathrm{O}'}(z),$$

where $Q^{\mathrm{O}'}(z)$ is a new parameter in $\mathbf{RH}_\infty$. This leads to

$$T^{\mathrm{O}(1)}(z) = \frac{z-2}{1-2z}\left\{Q^{\mathrm{O}'}(z) - \left(2 - \frac{7.5}{z-0.5}\right)\right\}. \quad (36)$$

Therefore, we can design a suboptimal $Q^{\mathrm{O}'}(z)$ as

$$Q^{\mathrm{O}'}(z) \triangleq 2 + \epsilon - \frac{7.5}{z-0.5}, \quad (37)$$

where ϵ is a constant that guarantees the well-posedness of the LPTV controller Ω (See Theorem 2). Note that $\left\|T^{\mathrm{O}(1)}(z)\right\|_\infty = \epsilon$ because $\frac{z-2}{1-2z}$ is an inner function. However, the well-posedness condition is violated for $\epsilon = 0$ and we set $\epsilon = 0.01$ for the following arguments. Substituting (37) into $Q^{\mathrm{O}(1)}(z)$ and using the definition of $Q^{\mathrm{O}(1)}$ and (20) yield $\tilde{Q}$. Furthermore, substituting $\tilde{Q}$ into $\tilde{\Omega}$, we get

$$\tilde{\Omega} = \left[\begin{array}{cc|cc} 3.943 & -4.075 & -3.282 & -983.3 \\ -5.498 & 5.682 & 4.577 & -1312 \\ \hline -1.806 & 1.846 & 0 & 0 \\ -3.003 & 3.104 & 2.5 & -201 \end{array}\right]. \quad (38)$$

Then, $\tilde{T} = (I + \tilde{P}\tilde{\Omega})^{-1}\tilde{P}$ is an element of $\mathbf{RH}_\infty^{2\times 2}$ and we readily calculate $\mathcal{H}^\infty$ norm $\left\|\mathrm{Pr}^0 T\right\|_\infty = 28.0$ and $\left\|\mathrm{Pr}^1 T\right\|_\infty = 0.01$. Hence, as we can see in Fig. 2, the controller drastically refine $\left\|\mathrm{Pr}^1 T\right\|_\infty$ by ignoring $\left\|\mathrm{Pr}^0 T\right\|_\infty$. On the other hand, the optimal l_2 induced norm by LTI controllers is $\|T\|_\infty = 3.00$. Therefore, any LTI controllers cannot achieve $\left\|\mathrm{Pr}^1 T\right\|_\infty$ less than 1.50 from Proposition 1. This suggests that the resulting LPTV controller is superior to LTI controllers for our design objective P2 and contrasts with performance limitation results [11], [12] .

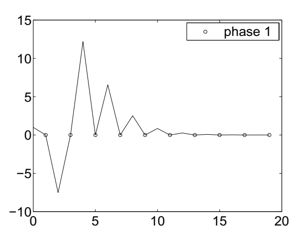

Fig. 2. An impulse response of the closed-loop transfer function T

VII. CONCLUSIONS AND FUTURE WORKS

We formulated two types of periodically weighted model-matching problem, P1 and P2, in which input signals or output signals of a closed-loop transfer function are projected onto the signal space $\mathcal{W}_i$ where any time sequence can take non-zero value at a specified phase in each period.

The Q parameter in Youla parameterization was selected so that the resulting LPTV controller is causal, N-periodic and well-posed. We firstly formulated causal LPTV discrete-time systems as the dual lifted forms F1 and F2, then we factorized the closed-loop transfer function into N sub LTI transfer functions in model-matching form. Consequently, the problems P1 and P2 with a class of LPTV controllers can be solved approximately by LTI model-matching problems P1' and P2' in which new stable LTI free parameters $Q^{\mathrm{I(i)}}(z)$ and $Q^{\mathrm{O(i)}}(z)$ are to be designed. These relaxed problems P1' and P2' give upper and lower bounds of achievable performances for the problems P1 and P2. In particular, we can remove the effect by unstable zeros of $T_2(z)$ or $T_3(z)$ to solve P1' and P2' under conditions in Theorem 4, which is easily satisfied for any period N larger than the size of A from Cayley-Hamilton theorem. However, the conditions are conservative and we will try to overcome this conservativeness. In addition, it is necessary to clarify the relationship between our periodic performance criteria and previously revealed advantages, such as arbitrary zero placement [2] or gain margin improvement [1] etc.

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