Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Connections between H_2 optimal filters and unknown input observers – Performance limitations of H_2 optimal filtering

Ali Saberi [1] Anton A. Stoorvogel [2] Peddapullaiah Sannuti [3]

***Abstract*— This paper establishes a connection between the unknown input observer problem (also known as the exact input-decoupling filtering problem) and the Kalman filtering problem. Such a connection leads to a better understanding of filtering. As a consequence of this, tools for the analysis as well as synthesis of filters can be developed. Moreover, such tools can be utilized to establish performance limitations of Kalman filtering as related to the structural properties of the given system.**

I. INTRODUCTION

In filtering theory, one of the well-known problems is the unknown input observer problem [1] where the asymptotic estimation error is required to be zero whatever might be the inputs to the system. Since, in that case, the asymptotic estimation error is exactly decoupled from the input, such a problem is also termed an *exact input-decoupling* or, for short, an *EID* filtering problem [2]. A problem related to EID filtering problem is the *almost input-decoupling (AID)* filtering problem. When the inputs are modeled as white noise, it relaxes the requirement by seeking a family of filters which can make the RMS norm of the estimation error signal arbitrarily small. It turns out that in this case one can, equivalently, make the H_2 norm of the transfer matrix from the input to the estimation error arbitrarily small. Hence such a problem can be called the H_2 AID filtering problem, and the family of filters that solves such a problem as an H_2 AID filters.

Another well-known and indeed much celebrated problem is the Kalman filtering problem which, when the inputs can be modeled by white noise, seeks to make the RMS norm of the estimation error signal as small as *possible* (which might not be arbitrarily close to zero). As before, it turns out that Kalman filtering tries to make the H_2 norm of the transfer matrix from the input to the estimation error as small as possible. Since in Kalman filtering, the input is optimally decoupled from the estimation error in H_2 norm sense, the Kalman filtering problem can also be termed the H_2 *optimal input-decoupling* (H_2 OID) filtering problem. Another concept highly tied to H_2 OID filtering is suboptimal input-decoupling (H_2 SOID) filtering where one seeks to find a family of filters which can make the H_2 norm of the transfer matrix from the input to the estimation error arbitrarily close to the optimal value.

The focus of this paper is establish a connection between H_2 OID filtering (H_2 SOID filtering) problem for a given system and the EID filtering (H_2 AID filtering) problem for an auxiliary system constructed from the data of the given system. Such a connection leads to a better understanding of filtering. As a consequence of this, tools for the analysis as well as synthesis of filters can be developed. Moreover, such tools can be utilized to establish performance limitations of Kalman filtering as related to the structural properties of the given system. All proofs are omitted in this conference version of the paper.

II. PRELIMINARIES AND PROBLEM STATEMENTS

Let us consider a continuous-time plant or system model as,

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \\ z = Ex + Fu, \end{cases} \tag{1}$$

where, $u \in \mathbb{R}^m$ is the input, $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^p$ is the measured output, and $z \in \mathbb{R}^q$ is the output signal to be estimated. Our interest lies in estimating the output signal z while using only the measured output y but not the input u. Let $\hat{z}$ be the estimate of z as given by a filter, and let e_z be the estimation error, $e_z = z - \hat{z}$ as depicted in Figure 1.

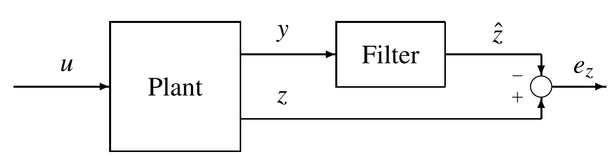

Fig. 1. General block diagram

We consider a general proper filter of the form,

$$\Sigma_f : \begin{cases} \dot{\xi} = L\xi + My \\ \hat{z} = N\xi + Py. \end{cases} \tag{2}$$

[1]School of Electrical Eng. and Comp. Science, Washington State University, Pullman, WA 99164-2752, U.S.A. E-mail: saberi@eecs.wsu.edu.

[2]Dept. of Mathematics and Computing Science, Eindhoven Univ. of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands. E-mail: A.A.Stoorvogel@tue.nl. Department of Electrical Engineering, Mathematics and Computer Science, Delft Univ. of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands, E-mail: a.a.stoorvogel@ewi.tudelft.nl

[3]Dept. of Electrical and Computer Engineering, Rutgers University, 94 Brett Road, Piscataway, NJ 08854-8058, U.S.A. E-mail: sannuti@ece.rutgers.edu

0-7803-8335-4/04/$17.00 ©2004 AACC

Whenever $P = 0$, the above filter is said to be a strictly proper filter. When the above filter is used as shown in Figure 1, the dynamics of the error e_z are described by

$$\Sigma^{ue} : \begin{cases} \dot{x} &= Ax + Bu, \\ \dot{\xi} &= MCx + L\xi + MDu \\ e_z &= (E - PC)x - N\xi + (F - PD)u. \end{cases} \qquad (3)$$

We define below what we mean by unbiased filters.

Definition II.1 *Consider a system Σ as in* (1). *We say a linear stable strictly proper (or proper) filter* (2) *is* ***unbiased*** *if, in the absence of the input u, the estimation error e_z decays asymptotically to zero for all possible finite initial values of the system* (1) *and the filter* (2).

Remark II.2 The above unbiasedness condition on the filter can be stated equivalently as follows: A filter is said to be unbiased if and only if, in the presence of an input u which is a white noise stochastic process, i.e. a zero mean wide sense stationary stochastic process with a unit power spectral density, and for all possible finite initial conditions on the plant and the filter, the mean of the estimation error e_z goes to zero asymptotically.

We have the following formal definition of the EID and Kalman filtering (e.g. see [3]) problems.

Problem II.3 Consider a system Σ as in (1), and the class of linear stable strictly proper (or proper) unbiased filters. Then, we have the following:

(i) The **exact input-decoupling (EID) filtering problem** consists of finding, whenever it exists, a filter such that the resulting transfer matrix G^{ue} from u to e_z equals zero.

(ii) The **Kalman filtering problem** consists of finding, whenever it exists, a filter which minimizes the RMS value of the estimation error, namely

$$\|e_z\|_{\mathrm{RMS}} = \lim_{T\to\infty} \mathbb{E}\frac{1}{T}\int_0^T \|e_z(t)\|^2 \, \mathrm{d}t,$$

where $\mathbb{E}$ denotes the expectation under the assumption that the input u is a white noise stochastic process, i.e. a zero mean wide sense stationary stochastic process with a unit power spectral density.

A filter that solves the EID filtering problem is said to be an EID filter. Similarly, a filter that solves the Kalman filtering problem is obviously said to be a Kalman filter.

Remark II.4 It is easy to see that the above EID filtering problem seeks to find a linear stable strictly proper unbiased (or proper) filter, whenever it exists, such that $e_z(t) \to 0$ as $t \to \infty$ for any input u and for all initial conditions for the system (1) and the filter (2). Similarly, it is well known that the Kalman filtering problem seeks to find a filter such that the resulting H_2 norm of G^{ue}, denoted here by $\|G^{ue}\|_2$, equals the infimum of $\|G^{ue}\|_2$ over the class of considered filters. Thus, the Kalman filtering problem can be called the H_2 *optimal input-decoupling* (H_2 *OID*) *filtering problem*, and the Kalman filter can be called the H_2 *OID filter*.

Definition II.5 *Consider a system Σ as in* (1) *where the input u is a zero mean wide sense stationary white noise with a unit power spectral density. The infimum of the RMS norm of the error signal e_z over the set of all linear stable strictly proper (or proper) unbiased filters, is called the* ***optimal input-decoupling (OID) filtering performance under white noise input*** *via linear stable strictly proper (or proper) unbiased filters, and is denoted by γ_{sp}^* (or γ_p^*).*

We can interpret γ_{sp}^* (or γ_p^*) as the infimum of the H_2 norm of the transfer function G^{ue} from u to e_z over the set of all linear strictly proper (or proper) unbiased filters. In other words, γ_{sp}^* (or γ_p^*) can be called the H_2 OID filtering performance via linear strictly proper (or proper) unbiased filters.

We define next the almost versions of EID and H_2 OID filtering problems.

Problem II.6 Consider a system Σ as in (1), and a family of linear stable strictly proper (or proper) unbiased filters parameterized in positive ε. Then,

(i) the ***H_2* almost input-decoupling (*H_2* AID) filtering problem** consists of finding, whenever it exists, a family of filters parameterized in positive ε such that $\|G_\varepsilon^{ue}\|_2 \to 0$ as $\varepsilon \to 0$.

(ii) the ***H_2* SOID filtering problem** is to find, whenever it exists, a family of strictly proper (or proper) filters parameterized in positive ε such that $\|G_\varepsilon^{ue}\|_2$ approaches γ_{sp}^* (or γ_p^*) as ε tends to zero.

A family of filters that solve the H_2 AID (or H_2 SOID) filtering problem is said to be a family of H_2 AID (or H_2 SOID) filters.

Most of the available literature on Kalman filtering deals with what is known as a *regular* filtering problem. If it is not a *regular* filtering problem, it is said to be a *singular* filtering problem. We have the following formal definition.

Definition II.7 *Consider a system Σ given by* (1) *with a white noise input. Then, a* ***regular H_2 OID filtering problem*** *refers to an H_2 OID filtering problem in which the matrix D is surjective, and the subsystem characterized by (A, B, C, D) has no invariant zeros on the imaginary axis.*

An H_2 OID filtering problem is said to be a ***singular H_2 OID filtering problem*** *if it is not a regular H_2 OID filtering problem.*

We will see that an H_2 OID filter always exists for the regular case. The regular case is the one which is always featured predominantly in many text books and hence creates the impression that an H_2 OID filter always exists even for the singular case. However, as will be seen subsequently, this is not the case!

We recall next the definitions of classical invariant subspaces of geometric control theory that are needed in stating the solvability conditions of the above problems.

Definition II.8 *Consider a linear system Σ characterized by a quadruple (A, B, C, D). Then,*

(i) *The stabilizable weakly unobservable subspace, denoted by $\mathcal{V}^g(A, B, C, D)$, is defined as the maximal subspace of $\mathbb{R}^n$ which is $(A + BF)$-invariant and contained in $\ker(C + DF)$ such that the eigenvalues of $(A + BF)|\mathcal{V}^g$ are contained in $\mathbb{C}_g \subseteq \mathbb{C}$ for some F.*

(ii) *The detectable strongly controllable subspace, denoted by $\mathcal{S}^g(A, B, C, D)$, is defined as the minimal subspace of $\mathbb{R}^n$ which is $(A + KC)$ invariant and contains $\operatorname{im}(B + KD)$ such that the eigenvalues of the map which is induced by $(A + KC)$ on the factor space $\mathbb{R}^n/\mathcal{S}^g$ are contained in $\mathbb{C}_g \subseteq \mathbb{C}$ for some K.*

It is of interest to have $\mathbb{C}_g$ representing different sets in the complex plane, namely the entire complex plane $\mathbb{C}$, the open left-half complex plane $\mathbb{C}^-$, the imaginary axis $\mathbb{C}^0$, the closed left-half plane $\mathbb{C}^- \cup \mathbb{C}^0$, or the open right-half complex plane $\mathbb{C}^+$. Whenever $\mathbb{C}_g$ represents respectively the sets $\mathbb{C}$, $\mathbb{C}^-$, $\mathbb{C}^0$, $\mathbb{C}^- \cup \mathbb{C}^0$, and $\mathbb{C}^+$, the superscript g in $\mathcal{V}^g$ and $\mathcal{S}^g$ is replaced by a superscript $*$, $-$, 0, -0, and $+$.

III. Connection between H_2 OID (H_2 SOID) and EID (H_2 AID) Filtering Problems

In this section, we develop fundamental results enabling us to connect the H_2 OID (SOID) filtering problems for a given system to that of EID (H_2 AID) filtering problems for an auxiliary system. This provide us a *road-map* for a variety of filtering issues. In this paper, we deal with the following:

(i) computing the H_2 OID filtering performance via linear stable strictly proper (or proper) unbiased filters namely γ_{sp}^* (or γ_p^*),

(ii) determining the performance limitations of H_2 OID filtering due to the structural properties of a given system, and

(iii) developing the existence and uniqueness conditions for H_2 OID (SOID) filters.

To start with, certain preliminaries are necessary. Let us consider a linear matrix inequality (CLMI),

$$G(Q) := \begin{pmatrix} AQ + QA' + BB' & QC' + BD' \\ CQ + DB' & DD' \end{pmatrix} \geq 0. \quad (4)$$

In the expression for $G(Q)$, the matrix Q is unknown, while the matrix quadruple (A, B, C, D) corresponds to the data of the given system Σ as in (1). We are interested in a semi-stabilizing or stabilizing solution Q of the above CLMI, i.e. a matrix Q which satisfies the CLMI and additionally satisfies:

$$\operatorname{rank}\begin{pmatrix} A - s_0 I & AQ + QA' + BB' & QC' + BD' \\ C & CQ + DB' & DD' \end{pmatrix} = n + \operatorname{normrank} C(sI - A)^{-1}B + D$$

for all $s_0 \in \mathbb{C}^+$ (semi-stabilizing) or for all $s_0 \in \mathbb{C}^+ \cup \mathbb{C}^0$ (stabilizing). In view of the pair (C, A) being detectable, one can determine a unique semi-stabilizing solution Q of the CLMI (4) (see [4]). Such a solution Q is positive semi-definite, rank minimizing, and is the largest among all symmetric solutions.

Remark III.1 Whenever the matrix D is surjective, one can equivalently determine Q by solving for the unique semi-stabilizing solution Q of an H_2 algebraic Riccati equation given by

$$QA' + AQ + BB' - (QC' + BD')(DD')^{-1}(CQ + DB') = 0.$$

For a regular H_2 OID filtering problem, the unique semi-stabilizing solution of the above Riccati equation is indeed *stabilizing* rather than merely *semi-stabilizing*. Thus, for a regular H_2 OID filtering problem, one can obtain the matrix Q equivalently by solving for the unique stabilizing solution of the above H_2 Riccati equation rather than solving the CLMI (4).

Next, once the matrix Q is determined, we can define matrices $B_Q \in \mathbb{R}^{n\times\rho}$ and $D_Q \in \mathbb{R}^{p\times\rho}$ with $\rho = \operatorname{rank} G(Q)$ such that

$$G(Q) = \begin{pmatrix} B_Q \\ D_Q \end{pmatrix} \begin{pmatrix} B_Q' & D_Q' \end{pmatrix}. \quad (5)$$

In order to achieve a finite RMS norm there must exist a matrix P^* such that $F - P^*D = 0$. Let $E^* = E - P^*C$. We can now define the following system:

$$\Sigma_Q : \begin{cases} \dot{\tilde{x}} = A\tilde{x} + B_Q\tilde{u}, \\ \tilde{y} = C\tilde{x} + D_Q\tilde{u}, \\ \tilde{z} = E^*\tilde{x}. \end{cases} \quad (6)$$

For Σ_Q, we consider filters of the form

$$\tilde{\Sigma}_f : \begin{cases} \dot{\xi} = L\xi + M\tilde{y} \\ \hat{z} = N\xi + (P - P^*)\tilde{y}. \end{cases} \quad (7)$$

We have the following results.

Theorem III.2 *Consider the systems Σ as in (1) with (C, A) detectable and $F = 0$ and Σ_Q as given by (6) with $P^* = 0$ (and thus $E^* = E$). Let the filter Σ_f be a strictly proper filter of the type given in (2) with $P = 0$. Then, the following two statements are equivalent:*

(i) *Σ_f is a strictly proper H_2 OID filter for the system Σ.*

(ii) *Σ_f is a strictly proper EID filter for the auxiliary system Σ_Q.*

Theorem III.3 *Consider a system Σ as in (1) with (C, A) detectable. Assume that $F - PD = 0$ has a solution for P and let P^* be any such solution, and then define the auxiliary* *system Σ_Q given by* (6). *Let Σ_f be a proper filter of the type given in* (2), *and $\tilde{\Sigma}_f$ be a proper filter of the type given in* (7). *Then, the following two statements are equivalent:*

(i) *Σ_f is a proper H_2 OID filter for the system Σ.*

(ii) *$\tilde{\Sigma}_f$ is a proper EID filter for the auxiliary system Σ_Q.*

Theorems III.2 and III.3 show that H_2 OID filtering problems for the given system Σ can be related to EID filtering problems for the auxiliary system Σ_Q. Along the same lines, one can show that H_2 SOID filtering problems for the given system Σ can be related to H_2 AID filtering problems for the auxiliary system Σ_Q. To do so, let us consider a family of parameterized filters of the form,

$$\Sigma_f^\varepsilon : \begin{cases} \dot{\xi} = L_\varepsilon \xi + M_\varepsilon y \\ \hat{z} = N_\varepsilon \xi + P_\varepsilon y \end{cases} \tag{8}$$

where L_ε, M_ε, N_ε, and P_ε are matrices parameterized in a positive parameter ε. We have the following theorem when the class of strictly proper filters are used.

Theorem III.4 *Consider a system Σ as in* (1) *with (C, A) detectable and $F = 0$. Let a family of filters Σ_f^ε be as given in* (8) *but with $P_\varepsilon = 0$. Also, let Σ_Q be the auxiliary system given in* (6) *with $P^* = 0$ and thus $E^* = E$. Then, the following two statements are equivalent:*

(i) *The family of filters Σ_f^ε is a family of strictly proper H_2 SOID filters for the system Σ.*

(ii) *The family of filters Σ_f^ε is a family of strictly proper H_2 AID filters for the auxiliary system Σ_Q.*

A result similar to the above theorem can be obtained when the class of proper filters are utilized. Before we state the result, consider the family of parameterized filters,

$$\tilde{\Sigma}_f^\varepsilon : \begin{cases} \dot{\xi} = L_\varepsilon \xi + M_\varepsilon \tilde{y} \\ \hat{z} = N_\varepsilon \xi + (P_\varepsilon - P^*) \tilde{y}. \end{cases} \tag{9}$$

We have the following theorem when the class of proper filters are used. We observe that H_2 SOID filtering problems can perhaps be solvable by proper filters (unlike in the case of strictly proper filters) even if $F \neq 0$.

Theorem III.5 *Consider a system Σ as in* (1) *with (C, A) detectable. Assume that $F - PD = 0$ has a solution for P and let P^* be any such solution, and then define the auxiliary system Σ_Q given by* (6). *Let a family of filters Σ_f^ε of the form* (8) *and the filters $\tilde{\Sigma}_f^\varepsilon$ of the form* (9). *Then, the following two statements are equivalent:*

(i) *The family of filters Σ_f^ε is a family of proper H_2 SOID filters for the system Σ.*

(ii) *The family of filters $\tilde{\Sigma}_f^\varepsilon$ is a family of proper H_2 AID filters for the auxiliary system Σ_Q.*

The above development transforms the H_2 OID (H_2 SOID) filtering problem for a given system to an EID (H_2 AID) filtering problem for another system. As we said earlier, such a transformation lays a road-map to study a number of issues related to H_2 OID (H_2 SOID) filtering problem. In the next three sections, one at a time, we concentrate on developing the existence and uniqueness conditions for H_2 OID (SOID) filters, computing γ_{sp}^* and γ_p^*, and determining the performance limitations of H_2 OID filtering owing to structural properties of a given system.

IV. Existence and uniqueness of H_2 OID and SOID filters

In what follows, we develop the conditions for the existence of H_2 OID (H_2 SOID) filters for a given system. Also, we develop here the conditions under which H_2 OID filters are unique whenever they exist. We observe that the notion of uniqueness of an H_2 OID filter can be viewed either in the sense of its transfer function, or in the sense of its state space realization with a fixed architecture. Here we view the uniqueness of an H_2 OID filter in the sense of its transfer function and not in the sense of its state space realization.

We have the following result regarding H_2 OID filters.

Theorem IV.1 *Consider a system Σ as in* (1). *Let the matrix pair (C, A) be detectable, and B_Q and D_Q be as in* (5). *Then, the following hold:*

(i) *There exists a linear unbiased strictly proper filter of the form* (2) *with $P = 0$ which solves the H_2 OID filtering problem for Σ if and only if $F = 0$ and $\mathcal{S}^-(A, B_Q, C, D_Q) \subseteq \ker E$.*

(ii) *There exists a linear unbiased proper filter of the form* (2) *that solves the H_2 OID filtering problem for Σ if and only if $\ker D \subseteq \ker F$ and*

$$\mathcal{S}^-(A, B_Q, C, D_Q) \cap C^{-1}\{\operatorname{im} D_Q\} \subseteq \ker(E - P^*C).$$

where P^* satisfies $F - P^*D = 0$.

Moreover, whenever they exist, the above filters are unique if and only if the subsystem characterized by (A, B, C, D) is right invertible.

Remark IV.2 Let the pair (C, A) be detectable. Then, the regular H_2 OID filtering problem is solvable via a linear strictly proper filter if $F = 0$ and via a proper filter if $\ker D \subseteq \ker F$.

The following result pertains to the H_2 SOID filters.

Theorem IV.3 *Consider a system as in (1). Let the matrix pair (C, A) be detectable. Then, the following results hold:*

(i) *The H_2 SOID filtering problem via a family of linear unbiased strictly proper filters is solvable if and only if $F = 0$.*

(ii) *The H_2 SOID filtering problem via a family of linear unbiased proper filters is solvable if and only if $\ker D \subseteq \ker F$.*

We note that, by definition, a family of H_2 SOID filters is non-unique.

V. Computation of γ_{sp}^* and γ_p^*

In this section, we compute γ_{sp}^* and γ_p^*. We have the following result.

Theorem V.1 *Consider a system as in* (1). *Let the matrix pair* (C, A) *be detectable. Let* Q *be the unique semi-stabilizing solution of the CLMI* (4). *We have the following:*

(*i*) *The* H_2 *OID filtering performance via linear unbiased strictly proper filters (denoted by* γ_{sp}^* *as formulated in Definition II.5) is finite if and only if* $F = 0$, *and is given by*

$$\gamma_{sp}^* = \left(\mathrm{trace}(EQE')\right)^{1/2}. \qquad (10)$$

(*ii*) *The* H_2 *OID performance via linear unbiased proper filters (namely* γ_p^* *as defined in Definition II.5) is finite if and only if* $\ker D \subseteq \ker F$, *and it is given by*

$$\gamma_p^* = \left(\mathrm{trace}((E - P^*C)Q(E - P^*C)')\right)^{1/2}$$

where P^* *satisfies* $F - P^*D = 0$.

Remark V.2 The above expression for γ_p^* appears to be dependent on the choice of P^*. However, it can be shown that a different choice of P^* satisfying $F - P^*D = 0$ does not affect the expression for γ_p^*.

VI. Relationship between γ_{sp}^* and γ_p^* and the structural properties of Σ

The purpose of this section is to relate γ_{sp}^* and γ_p^* to the structural constraints imposed by the subsystem characterized by (A, B, C, D). Such a relationship shows the structural limitations imposed by the given system. Our aim is to study the solution Q of the CLMI (4) since Q basically defines γ_{sp}^* and γ_p^* as seen from Theorem V.1. To do so, consider a system $\bar{\Sigma}$ characterized by $(\bar{A}, \bar{B}, \bar{C}, \bar{D}, \bar{E})$, where

$$\bar{A} = A', \quad \bar{B} = C', \quad \bar{C} = B', \quad \bar{D} = D' \text{ and } \bar{E} = E'. \qquad (11)$$

To view the structural details of the dual system $\bar{\Sigma}$, we can transform its subsystem characterized by $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ to a Special Coordinate Basis (SCB) as given in [4]. Let $(\bar{\Gamma}_s, \bar{\Gamma}_i, \bar{\Gamma}_o)$ be the related state, input, and output transformation matrices. Let

$$\bar{\Gamma}_s^{-1}\bar{E} = \left((\bar{E}_a^-)' \quad (\bar{E}_a^0)' \quad (\bar{E}_a^+)' \quad (\bar{E}_b)' \quad (\bar{E}_c)' \quad (\bar{E}_d)'\right)',$$

$$\bar{A}_s := \begin{pmatrix} \bar{A}_{aa}^+ & \bar{L}_{ab}^+\bar{C}_b \\ 0 & \bar{A}_{bb} \end{pmatrix}, \quad \bar{B}_s := \begin{pmatrix} \bar{B}_{a0}^+ & \bar{L}_{ad}^+ \\ \bar{B}_{b0} & \bar{L}_{bd} \end{pmatrix},$$

$$\bar{C}_s := \bar{\Gamma}_o \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \bar{C}_b \end{pmatrix}, \quad \bar{D}_s := \bar{\Gamma}_o \begin{pmatrix} I_{\bar{m}_0} & 0 \\ 0 & \bar{C}_d\bar{C}_d' \\ 0 & 0 \end{pmatrix},$$

and $\bar{E}_s' = ((\bar{E}_a^+)' \quad (\bar{E}_b)')$. We remark that various sub-matrices in the above definitions come from the Special Coordinate Basis (see [4]) as applied to the subsystem characterized by $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$. The above definitions of matrices lead us to define a standard linear quadratic optimization problem given by

$$\bar{\Sigma}_{sub}: \begin{cases} \dot{\bar{x}}_s = \bar{A}_s\bar{x}_s + \bar{B}_s\bar{u}_s \\ \bar{z}_s = \bar{C}_s\bar{x}_s + \bar{D}_s\bar{u}_s, \end{cases} \quad J_{sub} = \int_0^\infty \bar{z}_s'\bar{z}_s \, dt \qquad (12)$$

where $\bar{x}_s' = ((\bar{x}_a^+)' \quad \bar{x}_b')$. When $\bar{x}_s(0) = \bar{E}_{si}$ where $\bar{E}_{si}$ is the i-th column of $\bar{E}_s$, the infimum of J_{sub} over all possible state feedback controllers is given by

$$J_{sub}^*(x_s(0)) = J_{sub}^*(\bar{E}_{si}) = \bar{E}_{si}'Q_s\bar{E}_{si} \qquad (13)$$

where Q_s is the stabilizing solution of H_2 algebraic Riccati equation

$$0 = Q_s\bar{A}_s + \bar{A}_s'Q_s + \bar{C}_s'\bar{C}_s - (Q_s\bar{B}_s + \bar{C}_s'\bar{D}_s)(\bar{D}_s'\bar{D}_s)^{-1}(\bar{B}_s'Q_s + \bar{D}_s'\bar{C}_s). \qquad (14)$$

In view of equations (10), (13), and (14), we have the following lemma that relates γ_{sp}^* to the structural properties of the given system.

Lemma VI.1 *Consider a system as in* (1) *with* $F = 0$. *Let the matrix pair* (C, A) *be detectable. Also, let* $J_{sub}^*(\bar{E}_{si})$ *be as in* (13). *Then, the* H_2 *OID filtering performance via linear unbiased strictly proper filters, namely* γ_{sp}^*, *is given by*

$$(\gamma_{sp}^*)^2 = \sum_{i=1}^{q} J_{sub}^*(\bar{E}_{si}) = \sum_{i=1}^{q} \bar{E}_{si}'Q_s\bar{E}_{si} \qquad (15)$$

where q *is the dimension of the output* z *in* (1).

Let us develop a result similar to the above, however, for γ_p^*. This can be done by slightly modifying the development given above. The required modification is to be made in equation (11) by replacing the matrix E by $E^* = E - P^*C$ with P^* being any solution of the equation $F - PD = 0$ for P. We have the following result.

Lemma VI.2 *Consider a system as in* (1). *Let the matrix pair* (C, A) *be detectable. Assume that* $\ker D \subseteq \ker F$. *Moreover, in equation* (11), *let* E *be replaced by* $E^* = E - P^*C$ *with* P^* *being any solution of the equation* $F - PD = 0$ *for* P. *Also, following the development subsequent to the equation* (11) *and culminating in equation* (13), *obtain* $J_{sub}^*(\bar{E}_{si}^*)$ *as in* (13) *(Note that in this case* $\bar{E}_{si}^*$ *replaces* $\bar{E}_{si}$*). Then, the* H_2 *OID filtering performance via linear unbiased proper filters, namely* γ_p^*, *is given by*

$$(\gamma_p^*)^2 = \sum_{i=1}^{q} J_{sub}^*(\bar{E}_{si}^*) = \sum_{i=1}^{q} (\bar{E}_{si}^*)'Q_s\bar{E}_{si}^* \qquad (16)$$

where q *is the dimension of the output* z *in* (1).

The above lemmas clearly show the roles played by $\bar{\Sigma}_{sub}$ and $\bar{E}_s$ or $\bar{E}_s^*$ in dictating the values of γ_{sp}^* and γ_p^*. The subsystem $\bar{\Sigma}_{sub}$ has two types of dynamics. The first type of dynamics is represented by the state $\bar{x}_a^+$ which is often called the *unstable zero dynamics*. It is present only when the subsystem characterized by (A, B, C, D) (and hence the dual subsystem characterized by $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$) has invariant zeros in the open right half plane. Such invariant zeros are given by the eigenvalues of $\bar{A}_{aa}^{+}$. The second type of dynamics is represented by the state $\bar{x}_b$, and it is present only when the subsystem characterized by (A, B, C, D) is non-left invertible, and hence the dual subsystem characterized by $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is non-right invertible.

Let us first assume that the subsystem characterized by (A, B, C, D) is left invertible. Consequently, $\bar{x}_b$ does not exist. In this case, the subsystem $\bar{\Sigma}_{sub}$ of (12) simplifies to

$$\bar{\Sigma}_{sub} : \begin{cases} \dot{\bar{x}}_a^{+} = \bar{A}_{aa}^{+}\bar{x}_a^{+} + \bar{B}_s\bar{u}_s \\ \bar{z}_s \ = \ \bar{\Gamma}_o\bar{u}_s. \end{cases} \tag{17}$$

Also, in this case, since $\bar{z}_s = \bar{\Gamma}_o\bar{u}_s$, the performance measure J_{sub} has the interpretation of being the energy of control input.

The above simplification, and the results of Lemmas VI.1 and VI.2 enable us to interpret γ_{sp}^{*} and γ_{p}^{*} as explained in the following remark.

Remark VI.3 (Energy interpretation) Whenever the subsystem characterized by (A, B, C, D) is left invertible, $(\gamma_{sp}^{*})^2$ (or $(\gamma_{p}^{*})^2$) equals the sum of minimum energies required to stabilize the unstable zero dynamics of the subsystem characterized by $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ for all initial conditions $\bar{x}_a^{+}(0) = \bar{E}_{si}$, $i = 1$ to q (or $\bar{x}_a^{+}(0) = \bar{E}_{si}^{*}$, $i = 1$ to q).

The above remark considers the case when the subsystem characterized by (A, B, C, D) is left invertible. In addition, let us consider next the case of the same subsystem being at most weakly non-minimum phase (i.e. it has no invariant zeros in the open right-half plane $\mathbb{C}^{+}$), then the state $\bar{x}_a^{+}$ is non-existent as well. As such both γ_{sp}^{*} and γ_{p}^{*} will then equal zero.

The above study can be viewed from a different angle. This time by looking at the roles played by the matrices E and F which dictate $\bar{E}_{si}$ and $\bar{E}_{si}^{*}$ in equations (15) and (16). We note that the matrices E and F are defined by the output z that is to be estimated. When viewed from the view point of $\bar{E}_{si}$ and $\bar{E}_{si}^{*}$ that appear in (15) and (16), the fundamental limitations to the H_2 OID filtering performance arise from the inclusion of two types of dynamics in the output z that is to be estimated, one is the unstable zero dynamics of the subsystem characterized by (A, B, C, D) (as represented here by the dynamics of the state $\bar{x}_a^{+}$), and the other is the non-left invertible dynamics of the subsystem characterized by (A, B, C, D) (as represented here by the dynamics of the state $\bar{x}_b$). In the absence of both of these dynamics in the output z, the H_2 OID filtering performance γ_{sp}^{*} and γ_{p}^{*} simply equal zero.

REFERENCES

[1] M. Hautus, "Strong detectability and observers," *Lin. Alg. Appl.*, vol. 50, pp. 353–368, 1983.

[2] A. Saberi, A. Stoorvogel, and P. Sannuti, "Exact, almost, and optimal input decoupled (delayed) observers," *Int. J. Contr.*, vol. 73, no. 7, pp. 552–582, 2000.

[3] H. Sorenson, Ed., *Kalman filtering : theory and application*, ser. IEEE Press selected reprint series. New York: IEEE Press, 1985.

[4] A. Saberi, P. Sannuti, and B. M. Chen, *H_2 Optimal Control*. Englewood Cliffs: Prentice Hall, 1995.