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Robust Order Reduction

A. Trofino and D. F. Coutinho

***Abstract*— This paper proposes an LMI approach to the robust order reduction problem. The technique can be applied for quadratically stable linear systems with polytopic uncertainties. Bounds on the magnitude of the model approximation error are characterized in terms of both the H_2 and H_∞ norms. Numerical examples are used to show the potential of the proposed approach.**

I. INTRODUCTION

The problem of model reduction has been largely studied in these last four decades. Different techniques and different measures for the approximation error were studied and many numerically reliable results are now available in the literature. See for instance [1] for a brief presentation of the main types of techniques and corresponding references. The key idea is to approximate a given high order (possibly uncertain) system by a reduced order nominal model such that an upper bound on the approximation error is guaranteed to be small enough.

More recently, the LMI framework has been considered to approach the model reduction problem for uncertain systems [2], [3]. Among the main motivations to use the LMI framework we could mention the flexibility it offers to cope with mixed problems and uncertain models [5] and also the existence of reliable tools that are available to numerically solve LMI problems. Unfortunately, the existing LMI based conditions for model reduction have an additional rank constraint [6] or are expressed in terms of bilinear matrix inequalities [4], [7] destroying the convexity [8], [9].

In this paper we propose a convex LMI characterization to the order reduction problem. The reduced order model is determined by solving an LMI optimization problem in which an upper bound on the H_2 or H_∞ norm of the approximation error is minimized. We first propose a solution to the full order case (approximate model) and then extend it to the reduced order case without rank constraints.

The rest of this paper is organized as follows. Section II states the problem of concern. Sections III and IV present some preliminary results regarding the parameterization of the model to be determined as well as the precise upper bounds on the H_2 and H_∞ norms to be minimized. The main result is then presented in Section V where is determined a reduced order model in a numerical tractable way. Numerical examples are presented in Section VI and Section VII ends the paper.

II. STATEMENT OF THE PROBLEM

Consider the system

$$\mathcal{S} \;:\; \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the state, $u \in \mathbb{R}^{n_u}$ the input vector, $y \in \mathbb{R}^{n_y}$ the output vector, A, B, C, D are real matrices of compatible dimensions. To represent some system dynamics and parameters that are not precisely known or are difficult to be exactly modelled, suppose the matrices of the system $\mathcal{S}$ can take any value in a given polytope Π as indicated below [5]:

$$\Pi = \mathbf{Co}\left\{ \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix},\ i = 1, \ldots, q \right\} \tag{2}$$

where $\mathbf{Co}\{\cdot\}$ refers to the convex hull of $\{\cdot\}$. For convenience, we may alternatively represent the uncertain system $\mathcal{S}$ by the notation $\mathcal{S} \in \mathbb{S}$ where the set $\mathbb{S}$ is as follows:

$$\mathbb{S} \;:= \left\{ \mathcal{S} \text{ in (1)} : \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Pi \right\} \tag{3}$$

Also, we assume that system $\mathcal{S}$ is quadratically stable, i.e.

$$\exists\ P = P' : P > 0,\ A_i' P + P A_i < 0,\ i = 1, \ldots, q$$

Then, the problem of concern in this paper is to design a fixed model of given order $n_f \le n$

$$\mathcal{M} \;:\; \begin{cases} \dot{x}_f = A_f x_f + B_f u \\ y_f = C_f x_f + D_f u \end{cases} \tag{4}$$

where $x_f \in \mathbb{R}^{n_f}$ denotes the state of the model and A_f, B_f, C_f, D_f are real matrices of compatible dimensions to be determined such that the model approximation error is as small as possible.

The mismatch between system $\mathcal{S}$ and model $\mathcal{M}$ can be represented as the output of the following augmented system hereafter referred to as error system.

$$\mathcal{T}(\mathcal{S}, \mathcal{M}) \;:\; \begin{cases} \dot{x}_a = A_a x_a + B_a u \\ e = C_a x_a + D_a u \end{cases} \tag{5}$$

where $e := y - y_f$ is error signal and

$$A_a = \begin{bmatrix} A & 0 \\ 0 & A_f \end{bmatrix}, B_a = \begin{bmatrix} B \\ B_f \end{bmatrix}, x_a := \begin{bmatrix} x \\ x_f \end{bmatrix},$$
$$C_a = \begin{bmatrix} C & -C_f \end{bmatrix},\ D_a = D - D_f.$$

The magnitude of the error signal may be measured in several ways. In this paper, we consider the following definitions.

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A. Trofino is with the Department of Automation and Systems, Universidade Federal de Santa Catarina, PO BOX 476, 88040-900, Florianópolis, SC, Brazil. trofino@das.ufsc.br

D. F. Coutinho is with the Department of Electrical Engineering, Pontifícia Universidade Católica do Rio Grande do Sul, Av. Ipiranga 6681, 90619-900, Porto Alegre, RS, Brazil. daniel@ee.pucrs.br

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Definition 2.1: (H_2 norm) The H_2 norm of system $\mathcal{T}(\mathcal{S},\mathcal{M})$ is given by

$$\|\mathcal{T}(\mathcal{S},\mathcal{M})\|_2 := \sup_{\mathcal{S}\in\mathbb{S}} \|e\|_2 \tag{6}$$

where u is an unitary impulse, $x_a(0) = 0$ and $D_a = 0$. □

If the input signal u is a white noise with zero mean value and unitary power density spectra, we may interpret the H_2 norm as

$$\|\mathcal{T}(\mathcal{S},\mathcal{M})\|_2^2 := \sup_{\mathcal{S}\in\mathbb{S}} \mathcal{E}(e'e)$$

where $\mathcal{E}(e'e)$ denotes the mathematical expectation of the random variable $e'e$.

Alternatively, supposing $x_a(0) = 0$ the greatest energy gain that can be obtained from input signals $u \in L_2$ to the output e corresponds to the H_∞ norm of the uncertain system $\mathcal{T}(\mathcal{S},\mathcal{M})$ leading to the following definition.

Definition 2.2: (H_∞ norm) The H_∞ norm of system $\mathcal{T}(\mathcal{S},\mathcal{M})$ is given by

$$\|\mathcal{T}(\mathcal{S},\mathcal{M})\|_\infty := \sup_{\mathcal{S}\in\mathbb{S},\ u\in L_2} \frac{\|e\|_2}{\|u\|_2} \tag{7}$$

where L_2 denotes the space of square integrable vector functions on $[0,\infty)$. □

The following two sections introduce the foundations for the model reduction design. To be precise, we propose convex characterizations of the full order model approximation problem (i.e. $n_f = n$) in the H_2 (Section III) and H_∞ (Section IV) settings.

III. Guaranteed H_2 error bound

Suppose that model $\mathcal{M}$ has full order, i.e. $n_f = n$. Then, we can compute a bound on the H_2 norm of system $\mathcal{T}(\mathcal{S},\mathcal{M})$ by the following standard result from the LMI theory and observability grammian [5].

Lemma 3.1: Consider the error system $\mathcal{T}(\mathcal{S},\mathcal{M})$ with $D_a = D - D_f = 0$. Suppose there exist a model $\mathcal{M}$ with $n_f = n$ and symmetric matrices P and W of appropriate dimensions satisfying the following optimization problem for all $\mathcal{S}\in\mathbb{S}$.

$$\min_{\mathcal{M},P,W}\ trace(W) : \left\{ \begin{array}{l} P > 0,\ W - B_a' P B_a > 0, \\ A_a' P + P A_a + C_a' C_a < 0. \end{array} \right. \tag{8}$$

Then, $\mathcal{T}(\mathcal{S},\mathcal{M})$ is quadratically stable and the following holds:

$$\inf_{\mathcal{M}} \|\mathcal{T}(\mathcal{S},\mathcal{M})\|_2^2 \ <\ trace(W),\ \forall\ \mathcal{S}\in\mathbb{S}. \tag{9}$$

□

Remark 3.1: As A_a is block diagonal, observe that the quadratic stability of $\mathcal{T}(\mathcal{S},\mathcal{M})$ is equivalent to the quadratic stability of both $\mathcal{S}\in\mathbb{S}$ and $\mathcal{M}$. Moreover, if the class of systems $\mathbb{S}$ contains only one element, i.e. the nominal model, then $trace(W)$ can be done arbitrarily small with the choice $A_f = A$, $B_f = B$, $C_f = C$, $D_f = D$. □

To devise a convex formulation of Lemma 3.1, consider the notation below:

$$\begin{array}{c} P = \begin{bmatrix} S & T \\ T' & R \end{bmatrix}, N = \begin{bmatrix} I_n & 0_n \\ I_n & -TR^{-1} \end{bmatrix}, \\ Q = TR^{-1}T',\ S = S',\ R = R', \end{array} \tag{10}$$

where $S,T,R \in \mathbb{R}^{n\times n}$ are matrices to be determined. We assume in this paper that T is non-singular. As the matrix inequalities in this paper are strict, this assumption may be done without loss of generality by using small perturbation arguments.

Keeping in mind that the matrix N is nonsingular, the conditions in (8) are equivalent to

$$\begin{array}{c} NPN' > 0, N(A_a'P + PA_a + C_a'C_a)N' < 0, \\ W - B_a'PN'(NPN')^{-1}NPB_a > 0. \end{array}$$

Using the Schur complement and some straightforward algebraic manipulations the above inequalities can be rewritten as

$$\begin{array}{l} \Psi := \begin{bmatrix} \Psi_1 & \Psi_2 - A_m & C' \\ \star & \Psi_3 & C' + C_m' \\ \star & \star & -I_{n_y} \end{bmatrix} < 0, \\ \Psi_w := \begin{bmatrix} W & \star & \star \\ SB + B_m & S & \star \\ (S-Q)B & S-Q & S-Q \end{bmatrix} > 0, \end{array} \tag{11}$$

where $\star$ denotes the symmetric block outside the main diagonal and $\Psi_1 = SA + A'S$, $\Psi_2 = SA + A'(S-Q)$, $\Psi_3 = (S-Q)A + A'(S-Q)$, with A_m, B_m, C_m being related to the matrices of the model $\mathcal{M}$ through the following change of variables

$$A_m = TA_fR^{-1}T', B_m = TB_f, C_m = C_fR^{-1}T'. \tag{12}$$

Observe that the above relations are reversible, i.e. with $Q = TR^{-1}T'$ we get

$$\begin{array}{rcl} A_f &=& T^{-1}A_mQ^{-1}T, \\ B_f &=& T^{-1}B_m, \\ C_f &=& C_mQ^{-1}T, \end{array} \tag{13}$$

and the Lyapunov matrix P takes the form

$$P = \begin{bmatrix} S & T \\ T' & T'Q^{-1}T \end{bmatrix} \tag{14}$$

From above analysis, we obtain the following theorem.

Theorem 3.1: Consider the system $\mathcal{S}\in\mathbb{S}$, the model $\mathcal{M}$ and its parameterization as defined in (13) and (14), where T is any nonsingular matrix. Assume $D_f = D = 0$. Suppose there exist matrices W, S, Q, A_m, B_m, C_m solving the following convex optimization problem for all $\mathcal{S}\in\mathbb{S}$.

$$\min_{W,S,Q,A_m,B_m,C_m} trace(W) : \Psi < 0,\ \Psi_w > 0. \tag{15}$$

Then $\mathcal{M}$ is quadratically stable and the H_2 norm of the model approximation error satisfies (9). □

Proof. The first part of the proof follows directly from the equivalence between

$$\{\Psi < 0,\ \Psi_w > 0\}$$

and

$$\{A_a'P + PA_a + C_a'C_a < 0,\ W - B_a'PB_a > 0,\ P > 0\}$$

through the parameterization defined in (13) and (14). Then, the upper bound (9) follows from Lemma 3.1. ■

Corollary 3.1: Suppose that $\mathbb{S}$ has only one element and $D_f = D$. Then, $trace(W)$ in Theorem 3.1 is arbitrarily small with $A_m = \Psi_2 + C'(C + C_m)$. □

Proof. To show that $trace(W)$ becomes arbitrarily small whenever we can choose $A_m = \Psi_2 + C'(C + C_m)$, notice that with this choice $\Psi < 0$ becomes equivalent to $\Psi_1 < 0$ and $\Psi_3 < 0$. Next consider the choice $T = -Q$, $C_m = -C$, $B_m = -SB$. As $S - Q > 0$, choose in addition Q such that $S - Q$ becomes arbitrarily close to zero.

Thus within an arbitrary small error tolerance we get $S = Q$, $A_f = Q^{-1}A_m = A$, $B_f = -Q^{-1}B_m = B$, $C_f = -C_m = C$. Finally observe that Ψ_w becomes arbitrarily close to a block diagonal matrix which allows in turn $trace(W)$ to be arbitrarily small. ■

Lemma 3.1 is defined in terms of the inequalities associated with the observability grammian. This lemma could be alternatively represented by means of the controllability grammian leading to the following optimization problem:

$$\min_{\substack{\mathcal{M}, P, W,\\ \forall\, \mathcal{S} \in \mathbb{S}}} \ trace(W) : \begin{cases} P > 0,\ W - C_aPC_a' > 0, \\ A_aP + PA_a' + B_aB_a' < 0, \end{cases} \tag{16}$$

where P and W are symmetric matrices.

The same quadratic stability properties of $\mathcal{T}(\mathcal{S},\mathcal{M})$ and the upper bound (9) hold for Lemma 3.1 with (16) instead of (8). As a result, Theorem 3.1 can be recast in accordance with (16) by using the dual system, i.e. the expressions in (11) are redefined with $A = A'$, $B = C'$ and $C = B'$ and to get a coherent notation also replace A_m, B_m, C_m by A_m', C_m', B_m', respectively. These change of variables in (11) correspond to the parameterization defined in (12), (13) and (14) to represent Lemma 3.1 with (16). The idea of dual system is interesting when the matrix P in (16) or (8) cannot come arbitrarily close to its respective grammian. In this case Lemma 3.1 with (8) or (16) has different properties and then Theorem 3.1 and its dual version lead to different upper bounds on $\|\mathcal{T}(\mathcal{S},\mathcal{M})\|_2$. As a consequence, we can use them as alternative approaches to the model approximation problem.

IV. Guaranteed H_∞ error bound

Suppose that model $\mathcal{M}$ has full order, i.e. $n_f = n$, and consider the following version of the bounded real lemma [10] for an H_∞ characterization of the error system.

Lemma 4.1: Consider the error system $\mathcal{T}(\mathcal{S},\mathcal{M})$. Suppose there exist a model $\mathcal{M}$ with $n_f = n$, a matrix $P = P'$ of appropriate dimension and a positive scalar γ satisfying the following optimization problem for all $\mathcal{S} \in \mathbb{S}$.

$$\min_{\mathcal{M},P,\gamma} \ \gamma \ : \ P > 0,\ \Phi(\mathcal{M}, P, \gamma) < 0, \tag{17}$$

where

$$\Phi(\mathcal{M}, P, \gamma) = \begin{bmatrix} A_a'P + PA_a & PB_a & C_a' \\ B_a'P & -\gamma I_{n_u} & D_a' \\ C_a & D_a & -\gamma I_{n_y} \end{bmatrix}.$$

Then, the error system $\mathcal{T}(\mathcal{S},\mathcal{M})$ is quadratically stable and

$$\inf_{\mathcal{M}} \|\mathcal{T}(\mathcal{S},\mathcal{M})\|_\infty \ < \ \gamma,\ \forall\ \mathcal{S} \in \mathbb{S}. \tag{18}$$

□

Remark 4.1: From the same arguments of Remark 3.1, Lemma 4.1 implies the quadratic stability of both $\mathcal{S} \in \mathbb{S}$ and $\mathcal{M}$. Moreover, when the class of systems $\mathbb{S}$ contains only one element the scalar γ can be done arbitrarily small with the choice $A_f = A$, $B_f = B$, $C_f = C$ and $D_f = D$. □

Using the same procedure of Section III, we can obtain an LMI version to Lemma 4.1. To this end, consider the notation defined in (10). Then, the conditions in (17) are equivalent to $NPN' > 0$ and $\Psi_\gamma < 0$, where

$$\Psi_\gamma := diag\{N, I_{n_u}, I_{n_y}\}\Phi(\mathcal{M}, P, \gamma)diag\{N', I_{n_u}, I_{n_y}\}.$$

Using the Schur complement and some straightforward algebraic manipulations we may rewrite these conditions as follows

$$Q > 0, \quad S - Q > 0, \text{ and}$$

$$\Psi_\gamma := \begin{bmatrix} \Psi_1 & \Psi_2 - A_m & \Psi_4 \\ \star & \Psi_3 & \Psi_5 \\ \star & \star & -\Psi_6 \end{bmatrix} < 0, \tag{19}$$

where Ψ_1, Ψ_2, Ψ_3 are as defined in (11) and

$$\begin{aligned} \Psi_4 &:= \begin{bmatrix} SB + B_m & C' \end{bmatrix}, \\ \Psi_5 &:= \begin{bmatrix} (S-Q)B & C' + C_m' \end{bmatrix}, \\ \Psi_6 &:= \begin{bmatrix} \gamma I_{n_u} & -D_a' \\ -D_a & \gamma I_{n_y} \end{bmatrix}, \end{aligned}$$

with A_m, B_m, C_m being related to the matrices of $\mathcal{M}$ regarding the change of variables defined in (12), (13) and (14).

Now, we are ready to state the following result.

Theorem 4.1: Consider the uncertain system $\mathcal{S} \in \mathbb{S}$, the model $\mathcal{M}$ and its parameterization as defined in (13) and (14) with T being any nonsingular matrix. Suppose there exist matrices S, Q, A_m, B_m, C_m, D_f and a positive scalar γ solving the following optimization problem for all $\mathcal{S} \in \mathbb{S}$.

$$\min_{S,Q,A_m,B_m,C_m,D_f,\gamma} \gamma : Q > 0, S - Q > 0, \Psi_\gamma < 0. \tag{20}$$

Then, $\mathcal{M}$ is quadratically stable and the H_∞ norm of $\mathcal{T}(\mathcal{S},\mathcal{M})$ satisfies the bound (18). □

Proof. The parameterization defined in (13) and (14) implies the equivalence between the conditions

$$\{\Psi_\gamma < 0,\ Q > 0,\ S - Q > 0\}$$

$$\text{and} \quad \{\Phi_\gamma < 0,\ P > 0\}.$$

Then, from Lemma 4.1 it follows that $\mathcal{T}(\mathcal{S}, \mathcal{M})$ is quadratically stable and the upper bound (18) is satisfied. ■

Corollary 4.1: Suppose that $\mathbb{S}$ has only one element. Then, γ in Theorem 4.1 is arbitrarily small with $A_m = \Psi_2 + \Psi_4\Psi_6^{-1}\Psi_5'$. □

Proof. Notice with

$$A_m = \Psi_2 + \Psi_4\Psi_6^{-1}\Psi_5'$$

that Ψ_γ becomes equivalent to

$$\Psi_1 + \gamma^{-1}C'C < 0 \text{ and } \Psi_3 < 0.$$

Next consider the choice

$$T = -Q,\ C_m = -C,\ B_m = -SB,\ D_f = D.$$

As $S - Q > 0$, also choose $Q > 0$ such that $S - Q$ becomes arbitrarily close to zero and $\Psi_3 < 0$ (this is always possible provided A is quadratically stable).

Thus, within an arbitrary small error tolerance that is proportional to $\|S - Q\|$, we get

$$S = Q,\ A_f = Q^{-1}A_m = A,$$
$$B_f = -Q^{-1}B_m = B,\ C_f = -C_m = C.$$

The desired result follows by observing that in this case $\Psi_\gamma < 0$ becomes equivalent to $\gamma\Psi_1 + C'C < 0$ that can always be satisfied by simply re-scaling the matrix S. ■

V. Reduced order models

The convex techniques developed in Sections III and IV cannot be used in a straightforward way to the model reduction problem, i.e. when the order of the model to be determined is strictly smaller than system order. The reason is the existence of a rank constraint associated to the parameterization defined in (10) and (12). To overcome this non-convex rank condition, consider in this section that the error system $\mathcal{M}$ defined in (5) has a particular structure $\mathcal{M}_{rg}$ indicated below.

$$\mathcal{M}_{rg} : \begin{cases} \dot{x}_f &= \begin{bmatrix} A_r & 0 \\ A_{gr} & A_g \end{bmatrix} x_f + \begin{bmatrix} B_r \\ B_g \end{bmatrix} u, \\ y_f &= \begin{bmatrix} C_r & 0 \end{bmatrix} x_f + D_f u \end{cases} \quad (21)$$

where the dimensions of the matrices are in accordance with the partition

$$x_f = \begin{bmatrix} x_r \\ x_g \end{bmatrix} \begin{cases} x_r \in \mathbb{R}^{n_r}, \\ x_g \in \mathbb{R}^{n_g}, \\ x_f \in \mathbb{R}^{n}, \end{cases}$$

with $n = n_r + n_g$.

The interest of the above structure is that the model $\mathcal{M}_{rg}$ can be viewed as a cascade connection of two auxiliary sub-models, namely $\mathcal{M}_r$ and $\mathcal{M}_g$, and hereafter refereed to as reduced-order model and pseudo-model, respectively, which are given by:

$$\begin{aligned} \mathcal{M}_r &: \begin{cases} \dot{x}_r = A_r x_r + B_r u \\ y_r = C_r x_r + D_f u \end{cases}, \\ \mathcal{M}_g &: \{\ \dot{x}_g = A_g x_g + B_g u + A_{gr} x_r, \end{aligned} \quad (22)$$

where $x_r \in \mathbb{R}^{n_r}$ is the state of the reduced-order model, $x_g \in \mathbb{R}^{n_g}$ the state of the pseudo-model with $n_g = n - n_f$ and A_r, B_r, C_r, D_f and A_g, A_{gr}, B_g being real matrices of compatible dimensions to be determined such that the model approximation error is minimized as in Sections III and IV.

Observe from (22) and (5) that the error signal $e := y - y_f$ associated with the model estimation problem is now given in terms of the state of the reduced-order model since $y_f = y_r$. Moreover, it can be seen that the dynamics of the pseudo-model, which are driven by the system input vector and the states of the reduced-order model, neither affect the dynamics of the reduced-order model $\mathcal{M}_r$ nor the error signal. In fact, the dynamics of the pseudo-model $\mathcal{M}_g$ are non-observable at the output error signal. In other words, under zero initial conditions the full order error system $\mathcal{T}(\mathcal{S}, \mathcal{M}_{rg})$ and the reduced-order error system $\mathcal{T}(\mathcal{S}, \mathcal{M}_r)$, obtained from $(\mathcal{S}, \mathcal{M}_{rg}, \mathcal{M}_r)$ according to (5), have the same response for the same input. Also, the quadratic stability of $\mathcal{T}(\mathcal{S}, \mathcal{M}_{rg})$ implies the stability of both the reduced-order model and the pseudo-model as well, i.e the non-observable dynamics of $\mathcal{M}_g$ are stable.

The main reason for considering the reduced-order model approximation problem based on the full-order error system $\mathcal{T}(\mathcal{S}, \mathcal{M}_{rg})$ is that we can handle it by means of Theorems 3.1 and 4.1 provided that we take the particular structure of the model $\mathcal{M}_{rg}$ in (21) into account.

To obtain from the parameterization (13) the matrices A_f, B_f, C_f, with the structure indicated in (21), the free matrix T is defined as follows:

$$T = QT_0T_{rg},\ T_{rg} = \begin{bmatrix} T_r & 0_{n_r \times n_g} \\ 0_{n_g \times n_r} & T_g \end{bmatrix} \quad (23)$$

where T_r, T_g are any nonsingular matrices and T_0 is a given nonsingular matrix.

Now, for a given order n_r of $\mathcal{M}_r$, consider the following convex constraints on A_m, C_m and Q:

$$\Upsilon(A_m, C_m, Q) = \begin{bmatrix} \Omega_1\, A_m\, T_0\, \Omega_2' \\ \Omega_1\, Q\, T_0\, \Omega_2' \\ C_m\, T_0\, \Omega_2' \end{bmatrix} = 0, \quad (24)$$

where Ω_1 and Ω_2 are as follows

$$\Omega_1 = \begin{bmatrix} I_{n_r} & 0_{n_r \times n_g} \end{bmatrix},\ \Omega_2 = \begin{bmatrix} 0_{n_g \times n_r} & I_{n_g} \end{bmatrix}.$$

It can be easily checked that under the above constraints we get the following

$$\begin{aligned} A_m T_0 &= \begin{bmatrix} A_{m_1} & 0_{n_r \times n_g} \\ A_{m_2} & A_{m_3} \end{bmatrix}, \\ Q T_0 &= \begin{bmatrix} Q_1 & 0_{n_r \times n_g} \\ Q_2 & Q_3 \end{bmatrix}, \\ C_m T_0 &= \begin{bmatrix} C_{m_1} & 0_{n_y \times n_g} \end{bmatrix} \end{aligned} \quad (25)$$

for some matrices A_{m_i}, Q_i $(i = 1, 2, 3)$ and C_{m_1}.

Then, from (13) and (23) the matrices A_f, B_f, C_f take the form indicated bellow:

$$\begin{aligned} A_f &= T_{rg}^{-1} (QT_0)^{-1} (A_m T_0) T_{rg}, \\ B_f &= T_{rg}^{-1} (QT_0)^{-1} B_m, \; C_f = C_m T_0 T_{rg}. \end{aligned} \quad (26)$$

Note that the above definition leads to the desired structure defined in (21).

Applying the matrix inversion lemma to $(QT_0)^{-1}$ and (25), we obtain the following matrices for $\mathcal{M}_r$:

$$\begin{aligned} A_r &= T_r^{-1} Q_1^{-1} A_{m_1} T_r, \\ B_r &= T_r^{-1} Q_1^{-1} \Omega_1 B_m, \; C_r = C_{m_1} T_r, \end{aligned} \quad (27)$$

and to the pseudo-model $\mathcal{M}_g$: $A_g = T_g^{-1} Q_3^{-1} A_{m_3} T_g$ and

$$B_g = T_g^{-1} (-Q_3^{-1} Q_2 Q_1^{-1} \Omega_1 B_m + Q_3^{-1} \Omega_2 B_m),$$

$$A_{gr} = T_g^{-1} (-Q_3^{-1} Q_2 Q_1^{-1} A_{m_1} + Q_3^{-1} A_{m_2}) T_r.$$

From above, the main result of this paper is stated as follows.

Theorem 5.1: Let T_0 be any given non-singular matrix and n_r a given integer such that $n_r < n$. There exists a model $\mathcal{M}_r$ of order n_r with an H_2 (or H_∞) bound on the error signal of system $\mathcal{T}(\mathcal{S}, \mathcal{M}_r)$, if the conditions of Theorem 3.1 (or Theorem 4.1) are satisfied for all $\mathcal{S} \in \mathbb{S}$ with the additional convex constraint $\Upsilon(A_m, C_m, Q) = 0$ given by (24). In the affirmative case, $\mathcal{M}_r$ indicated in (22), is quadratically stable and its matrices are given by (27). □

Remark 5.1: In the full-order model estimation problem the matrices A_m, B_m, C_m, Q, S in Theorem 3.1 (or Theorem 4.1) as well as the free matrix T have to be constant because $\mathcal{M}$ and P are the same for all element $\mathcal{S} \in \mathbb{S}$. However, for reduced-order models the matrices associated with the pseudo-model $\mathcal{M}_g$ are not required to be constant because they do not affect the model estimation error $e := y - y_r$. As a result, we may let them to depend on the uncertain parameters to reduce the conservativeness of the LMI conditions. □

Remark 5.2: If the system model depends on a set of parameters, one may wish to estimate a reduced (or full-order) model that is itself dependent on the same set (or sub-set) of these parameters. In this case we just let the matrices A_m, C_m, B_m, D_f be affine functions of the sub-set of desired parameters. As a result the matrices of the model we get will also be an affine function of this same sub-set of parameters through the relation (13). □

VI. Numerical examples

To illustrate the proposed results[1], we present three numerical examples. The first one is based on the TGEN system described in [11], the second one is the example proposed in [3] and the third is the second example of [4].

Example 6.1: Consider the TGEN system described in [11]. It has the form (1) with $A \in \mathbb{R}^{6\times6}, B \in \mathbb{R}^{6\times2}, C \in \mathbb{R}^{2\times6}$. The nominal system is Hurwitz but has 3 poles close to the imaginary axis, namely -0.231, $-0.351 \pm j6.34$. To illustrate the robustness feature let $a_{(i,j)}$ to represent the (i,j) element of A and suppose $a_{(6,6)} = c_0 + c_0\,\delta$ where c_0 is the nominal value and δ is a uncertain parameter satisfying $|\delta| \le \alpha$, for a given α. The nominal system described in [11] corresponds to $\delta = 0$.

The table shows the guaranteed cost obtained with the Theorems 3.1 and 4.1. Each column of the table corresponds to a given model order. The rows correspond to the design requirements: (i) nominal guaranteed cost on the model reduction error; and (ii) robust guaranteed cost for the above uncertainty set. The results were obtained by using $T_0 = I_n$ and $\alpha = 0.2$. The last row displays the model reduction error obtained from the Hankel-norm technique applied to the nominal system.

	n_r	6	5	4	3	2
$trace(W)$	(i)	0	1.09	2.48	2.94	4.33
$trace(W)$	(ii)	0.07	1.52	2.91	3.37	4.75
γ	(i)	0	2.11	2.11	2.11	2.13
γ	(ii)	0.78	2.62	2.62	2.62	2.69
hankel	(i)	0	0.18	1.26	3.25	5.31

Example 6.2: Consider the following uncertain system as proposed in [3]:

$$\dot{x} = Ax + Bu, \; y = Cx,$$

where the system matrices are given by

$$\begin{aligned} A &= \begin{bmatrix} -2 & 3 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & a_{33} & 12 \\ 0 & 0 & 0 & -4 \end{bmatrix} \\ B &= \begin{bmatrix} -2.5 & b_{12} & -1.2 \\ 1.3 & -1 & 1 \\ 1.6 & 2 & 0 \\ -3.4 & 0.1 & 2 \end{bmatrix} \\ C &= \begin{bmatrix} -2.5 & 1.3 & 1.6 & -3.4 \\ 0 & -1 & 2 & 0.1 \\ -1.2 & 1 & 0 & 2 \end{bmatrix}. \end{aligned}$$

and $a_{33} \in [-3.5, -2.5]$, $b_{12} \in [-0.5, 0.5]$.

The problem of interest in this example is to obtain a reduced-order model with $n_f = 2$ in an H_∞ sense. To this end, consider the following definition for the matrix T_0:

[1]The numerical results were obtained with the free software Scilab available at the site www-rocq.inria.fr/scilab.

$$T_0 = \left[\begin{array}{cc} I_2 & I_2 \\ 0_2 & I_2 \end{array} \right].$$

Then, applying Theorem 5.1 to the above system yields an upper-bound $\gamma = 5.54$ on H_∞ norm of the error signal. This result is conservative with respect to the H_∞ bound obtained in [3] ($\gamma = 3.79$). In this reference the model is itself dependent of the uncertain parameters and an alternating projection method is used to handle a non-convex rank constraint. In contrast, our approach considers a fixed model (parameter independent) with a much lower computational effort than the result obtained in [3].

Example 6.3: Consider the following time-invariant system [4]:

$$\mathcal{G} : \left\{ \begin{array}{l} \dot{x} = Ax + Bu, \\ y = Cx \end{array} \right.$$

where the system matrices are as follows

$$A = \left[\begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -0.007 & -0.114 & -0.85 & -2.8 & -4.45 & -3.4 \end{array} \right]$$

$$B = \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 1 \end{array}{}'$$

$$C = \begin{array}{cccccc} 0.007 & 0.014 & 0 & 0 & 0 & 0 \end{array}$$

The objective in this example is to determine a first order model $\mathcal{G}_r$ for system $\mathcal{G}$. Applying Theorem 5.1, we get $trace(W) = 0.0205$ with $T_0 = I_6$ which is considerably better than the result proposed in [4] where was obtained $0.0557 \leq \|\mathcal{T}(\mathcal{G}, \mathcal{G}_r)\|_2^2 \leq 0.0616$.

VII. Conclusions

In this paper, we have proposed sufficient convex LMI conditions to the order-reduction problem. In contrast with the techniques found in the literature, as for instance the ones in the references [2], [3], [6] and [4], our method is convex (does not depend on rank constraint conditions) and the model obtained is fixed (parameter independent) even if the system is uncertain (parameter dependent). The technique can be applied to quadratically stable linear systems with polytopic uncertainties, and the model is determined by minimizing upper bounds on the H_2 or H_∞ norms of the error signal between the model and its approximation. The stability of the model to be determined is shown through a suitable Lyapunov function, and the results are based on the parameterization of the model matrices and the Lyapunov function as defined in (13) and (14), respectively. The numerical examples have shown that the proposed methodology yields a good compromise between accuracy and computational effort. The best choice of matrix T_0 used in the order-reduction case (Section V) is an interesting problem that remains open.

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