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Fixed Order $\mathscr{H}_\infty$-synthesis: Computing Optimal Values by Robust Performance Analysis

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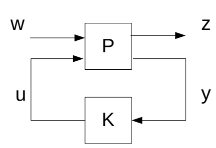
***Abstract*— The computation of optimal $\mathscr{H}_\infty$ controllers with a prescribed order is important for their real-time implementation. This problem is well-known to be non-convex, and only algorithms that compute upper bounds on the global optimal value are known. We present a method to compute lower bounds by re-formulating the problem as a robust analysis problem, where the controller variables are the "uncertain" parameters. This allows to apply the wide spectrum of robust analysis techniques to the fixed order controller design problem. The solution to the robust analysis problem is a global lower bound on the optimal closed loop $\mathscr{H}_\infty$ performance. We construct a family of robust analysis problems and relax them to convex optimization problems using the S-procedure. The optimal values of this family converge from below to the globally optimal fixed order $\mathscr{H}_\infty$-norm. This allows verification of global optimality of controllers. The number of complicating variables in our robust analysis problem is small if we optimize over a few controller parameters. The technique is therefore computationally feasible for optimization over few controller variables, e.g. PID-tuning of systems with large Mc-Millan degree. The method is applied to the tuning of two controller parameters of a 4-block $\mathscr{H}_\infty$ design of an active suspension system, with a Mc-Millan degree of the plant of 27.**

Keywords: $\mathscr{H}_\infty$ control, fixed-order, lower bounds, robust analysis.

I. INTRODUCTION

$\mathscr{H}_\infty$ controller synthesis is an attractive model-based control design tool. It allows incorporation of modelling uncertainties in control design. This paper considers the fixed order $\mathscr{H}_\infty$ synthesis problem. It is one of the most important open problems in control engineering, in the sense that until now there do not yet exist fast and reliable methods to compute optimal fixed order controllers. We consider the closed-loop interconnection as shown below, where the linear system P is the generalized plant and K is a linear controller.



The problem can be formulated as follows:

Problem 1: (Fixed-order $\mathscr{H}_\infty$ synthesis) Given a plant P, find a controller K of order n_c such that the closed loop interconnection is internally (asymptotically) stable and such that the $\mathscr{H}_\infty$-norm of the closed-loop transfer function from w to z is minimized.

The resulting optimization problem is non-convex and difficult to solve. Various approaches have been presented in the literature based on sequential solution of LMI's [1] [2], [3], [4], nonlinear SDP's [5], [6], [7] and branch and bound methods [8]. Except for the branch and bound method these algorithms can in general not guarantee convergence to the globally optimal solution. The method presented in this paper allows to add a stopping criterion to these algorithms with a guaranteed bound on the difference of the performance of the computed controller and the optimal fixed order $\mathscr{H}_\infty$ performance.

A trivial lower bound on the fixed order performance is of course the full order performance. Boyd and Vandenberghe [9] proposed lower bounds based on convex relaxations of the fixed order synthesis problem. These lower bounds are not asymptotically exact, i.e. they can not straightforwardly be improved to reduce the gap to the optimal fixed order performance.

In this paper we present a re-formulation of the fixed order synthesis problem into a robust analysis problem in which the controller variables are the "uncertain" parameters. This allows to apply the wide spectrum of robust analysis techniques to the fixed order controller design problem. The solution to the robust analysis problem is a global lower bound on the optimal closed loop $\mathscr{H}_\infty$ performance. We construct a family of robust analysis problems and relax them to convex optimization problems using the S-procedure. The optimal values of this family converge from below to the globally optimal fixed order $\mathscr{H}_\infty$-norm. This allows verification of global optimality of controllers. These convex optimization problems are LMI (Linear Matrix Inequality) problems. The size of the LMI problem is quadratic in the number of plant states, but grows exponentially with the number of controller variables. The method is therefore appropriate for the design of controllers with a few decision variables (PID with a few notches) for plants with large Mc-Millan degree.

II. PROBLEM FORMULATION

Consider the $\mathscr{H}_\infty$-reduced order synthesis problem with parameterized closed-loop system described by $A(p)$, $B(p)$, $C(p)$ and $D(p)$, where p parametrizes the to-be-constructed controller and varies in the compact set $\mathscr{P}$. The compactness can for instance be realized by restricting the controller variables to certain intervals. In many cases these closed loop matrices depend affinely on the controller parameters. Consider for instance a plant of order n with state space

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description

$$\begin{pmatrix} \dot{x} \\ \hline z \\ y \end{pmatrix} = \left(\begin{array}{c|cc} A^{\rm ol} & B_1^{\rm ol} & B_2^{\rm ol} \\ \hline C_1^{\rm ol} & D_{11}^{\rm ol} & D_{12}^{\rm ol} \\ C_2^{\rm ol} & D_{21}^{\rm ol} & 0 \end{array}\right) \begin{pmatrix} x \\ \hline w \\ u \end{pmatrix}$$

where $(\cdot)^{\rm ol}$ stands for 'open loop' and $A^{\rm ol} \in \mathbb{R}^{n\times n}$, $B_1^{\rm ol} \in \mathbb{R}^{n\times m_1}$, $B_2^{\rm ol} \in \mathbb{R}^{n\times m_2}$, $C_1^{\rm ol} \in \mathbb{R}^{p_1\times n}$, $C_2^{\rm ol} \in \mathbb{R}^{p_2\times n}$, $D_{11}^{\rm ol} \in \mathbb{R}^{p_1\times m_1}$, $D_{12}^{\rm ol} \in \mathbb{R}^{p_1\times m_2}$ and $D_{21}^{\rm ol} \in \mathbb{R}^{p_2\times m_1}$. A controller of order n_c parameterized by its state-space matrices (A_k, B_k, C_k, D_k) yields a closed loop state-space representation with matrices

$$\left(\begin{array}{c|c} A(p) & B(p) \\ \hline C(p) & D(p) \end{array}\right) = \tag{1}$$

$$\left(\begin{array}{cc|c} A^{\rm ol}+B_2^{\rm ol}D_kC_2^{\rm ol} & B_2^{\rm ol}C_k^{\rm ol} & B_1^{\rm ol}+B_2^{\rm ol}D_kD_{21}^{\rm ol} \\ B_kC_2^{\rm ol} & A_k & B_k+B_2^{\rm ol}D_kD_{21}^{\rm ol} \\ \hline C_1^{\rm ol}+D_{12}^{\rm ol}D_kC_2^{\rm ol} & C_k & D_{11}^{\rm ol}+D_{12}^{\rm ol}D_kD_{21}^{\rm ol} \end{array}\right)$$

which depend affinely on the quadruple $p := (A_k, B_k, C_k, D_k)$. We intend to solve

$$\inf_{p\in\mathscr{P}, A(p)\,\text{stable}} \|D(p)+C(p)(sI-A(p))^{-1}B(p)\|_\infty$$

We assume that $(A(p),B(p))$ is controllable for all $p\in\mathscr{P}$. The bounded real lemma yields the following equivalent problem:

Problem 2:

$$\begin{array}{ll} \text{infimize} & \gamma \\ \text{subject to} & p\in\mathscr{P}, X\in\mathscr{S}^n, X\succ 0 \\ & B_\infty(X,p,\gamma)\prec 0 \end{array} \tag{2}$$

where $\mathscr{S}^n$ denotes the set of symmetric $n\times n$ matrices and

$$B_\infty(X,p,\gamma) := \begin{pmatrix} A(p)^TX+XA(p) & XB(p) & C(p)^T \\ B(p)^TX & -\gamma I & D(p)^T \\ C(p) & D(p) & -\gamma I \end{pmatrix}.$$

Let $t_{\rm opt}^p$ denote the primal optimal value of Problem 2. Problem 2 is a nonconvex problem due to the bilinear coupling of X and p. If n_p denotes the number of free controller variables, then the number of bilinearly coupled variables is $\frac{1}{2}(n+n_c)(n+n_c+1)+n_p$, which grows quadratically with the number of plant states n. It is therefore crucial to convert the fixed-order problem into a robust analysis problem with few complicating variables, i.e. only the controller variables. This is done in the next section by partial dualization.

III. Conversion to robust analysis problem

The key idea is to apply **partial** Lagrange dualization: we fix the controller variables and dualize with respect to the Lyapunov variable X. Optimizing the dual problem over all p leads to a robust analysis problem with p as the only complicating variables, i.e. the uncertain variables in the robustness test.

Let the matrix Z be partitioned according to the partitioning of B_∞ in (2):

$$Z = \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{12}^T & Z_{22} & Z_{23} \\ Z_{13}^T & Z_{23}^T & Z_{33} \end{pmatrix}. \tag{3}$$

For fixed $p=p_0$, Problem 2 is an LMI problem in X and γ:

$$\begin{array}{ll} \text{infimize} & \gamma \\ \text{subject to} & X\in\mathscr{S}^n, \quad X\succ 0, \quad B_\infty(X,p_0,\gamma)\prec 0, \quad \gamma>0 \end{array} \tag{4}$$

where the redundant constraint $\gamma>0$ is added for technical reasons that will be clarified in Section IV. The Langrange dual of this problem reads as follows

Problem 3:

$$\begin{array}{ll} \text{maximize} & 2\mathrm{Tr}(Z_{13}C(p_0)+Z_{23}D(p_0)) \\ \text{subject to} & Z\succeq 0, \quad 1-\mathrm{Tr}(Z_{22})-\mathrm{Tr}(Z_{33})\geq 0 \\ & Z_{11}A(p_0)^T+A(p_0)Z_{11}+Z_{12}B(p_0)^T \\ & \qquad +B(p_0)Z_{12}^T\succeq 0 \end{array}$$

where $\mathrm{Tr}(A)$ denotes the trace of the matrix A. Let $t_{\rm opt}^d(p_0)$ denote the dual optimal value of Problem 3. In Appendix it is shown that Problem 3 is strictly feasible, in the sense that for all $p_0\in\mathscr{P}$ there exists a $W\succ 0$ such that

$$\begin{array}{rcl} W_{11}A(p_0)^T+A(p_0)W_{11}+ & & \\ +W_{12}B(p_0)^T+B(p_0)W_{12}^T & \succ & 0 \\ 1-\mathrm{Tr}(W_{22})-\mathrm{Tr}(W_{33}) & > & 0 \end{array}$$

are satisfied. This implies that strong duality holds, i.e. the dual and primal have the same optimal value $t_{\rm opt}^d(p_0) = t_{\rm opt}^p(p_0)$. Observe that for each $p_0\in\mathscr{P}$ there exists an optimal dual variable Z for Problem 3. Suppose that indeed we have found a dual variable $Z(p)$ for each $p\in\mathscr{P}$ such that

$$\begin{array}{rcll} Z_{11}(p)A(p)^T+A(p)Z_{11}(p)+ & & & \\ +Z_{12}(p)B^T(p)+B(p)Z_{12}(p)^T & \succ & 0 & (5) \\ 1-\mathrm{Tr}(Z_{22}(p))-\mathrm{Tr}(Z_{33}(p)) & > & 0 & (6) \\ 2\mathrm{Tr}(Z_{13}(p)C(p)+Z_{23}(p)D(p))>t, \quad Z(p) & \succ & 0 & (7) \\ \text{for all} \quad p\in\mathscr{P}. & & & \end{array}$$

Then clearly $t_{\rm opt}^d(p)\geq t$ for all $p\in\mathscr{P}$ and hence $t_{\rm opt}^p(p)\geq t$. This shows that t is a lower bound on the optimal control performance. Hence if we maximize t over $Z(\cdot)$ parameterized as, for instance, rational functions subject to (5), (6) and (7), we obtain a lower bound on the optimal controller performance. This maximization problem is a robust analysis problem with the uncertain variable p varying in the compact set $\mathscr{P}$. The distance of the lower bound to the optimal $\mathscr{H}_\infty$ performance is non-decreasing if we optimize over $Z(\cdot)$ in an increasing sequence of subspaces of matrix-valued functions. The following theorem shows that the lower bounds constructed in this way converge from below to the globally optimal $\mathscr{H}_\infty$ performance.

Theorem 4: Let γ_{opt} be the optimal solution to Problem 2. Let t_{opt} be the supremal t for which there exists a continuous function $Z(p)$ with (5), (6) and (7) for all $p \in \mathscr{P}$. Then $\gamma_{\text{opt}} = t_{\text{opt}}$.

Proof. We want to prove $\gamma_{\text{opt}} = t_{\text{opt}}$. On the previous page has been shown that $\gamma_{\text{opt}} \geq t_{\text{opt}}$. Now suppose $\gamma_{\text{opt}} \geq t_{\text{opt}} + \varepsilon$ for some $\varepsilon > 0$. Let us use for V, W, Y^0 and $Y(p)$ the same partitioning as for Z in (3) and fix $p_0 \in P$. Then Problem 2 has optimal value not smaller than $t_{\text{opt}} + \varepsilon$. Its dual is Problem 3. As shown in Appendix , the dual is strictly feasible and therefore there is no duality gap. Therefore there exists V with

$$\begin{aligned} V_{11}A(p_0)^T + A(p_0)V_{11} + & \\ +V_{12}B^T(p_0) + B(p_0)V_{12}^T &\succeq 0 \\ 1 - \text{Tr}(V_{22}) - \text{Tr}(V_{33}) &\geq 0 \\ V &\succeq 0 \\ 2\text{Tr}(V_{13}C(p) + V_{23}D(p)) - \varepsilon/2 &\geq t_{\text{opt}}. \end{aligned}$$

Now recall that there exists some $W \succ 0$ with $W_{11}A(p_0)^T + A(p_0)W_{11} + W_{12}B^T(p_0) + B(p_0)W_{12}^T \succ 0$ and $1 - \text{Tr}(W_{22}) - \text{Tr}(W_{33}) = 0$. Therefore $Y^0 = \tau W + (1-\tau)V$ actually satisfies for some $\tau > 0$ close to zero

$$\begin{aligned} Y_{11}^0 A(p_0)^T + A(p_0)Y_{11}^0 + & \\ +Y_{12}B^T(p_0) + B(p)Y_{12}^{0\,T} &\succ 0, && (8) \\ 1 - \text{Tr}(Y_{22}) - \text{Tr}(Y_{33}) &> 0 \\ Y^0 &\succ 0 && (9) \\ 2\text{Tr}(Y_{13}^0 C(p_0) + Y_{23}^0 D(p_0)) - \varepsilon/4 &> t_{\text{opt}}. && (10) \end{aligned}$$

Since (8), (9) and (10) hold strictly, there exists a **continuous** function $Y(p)$ such that for all $p \in \mathscr{P}$

$$\begin{aligned} Y_{11}(p)A(p)^T + A(p)Y_{11}(p) + & \\ +Y_{12}(p)B^T(p) + B(p)Y_{12}(p)^T &\succ 0, \\ 1 - \text{Tr}(Y_{22}(p)) - \text{Tr}(Y_{33}(p)) &> 0 \\ Y(p) &\succ 0 \\ 2\text{Tr}(Y_{13}(p)C(p) + Y_{23}(p)D(p)) - \varepsilon/8 &> t_{\text{opt}} \end{aligned}$$

This finishes the proof.

■

Remark on strict feasibility. Existence of a strict feasible W is guaranteed by our controllability assumption on the pair $A(p), B(p)$ for all $p \in \mathscr{P}$. We used this to show that $Y(p)$ may be chosen to be continuous. Continuity of $Y(p)$ is important to show exactness for the rational approximation of $Y(p)$ that is presented in the next section.

The theorem shows that maximizing t over $Z(p)$ satisfying (5), (6) and (7) for all $p \in P$ gives lower bounds on the closed loop $\mathscr{H}_\infty$ performance, and these lower bounds can be improved to arbitrary accuracy. We have reduced the $\mathscr{H}_\infty$ synthesis problem to a robust analysis problem with complicating variables p and nice variables $Z(p)$. The constraints (5), (6) and (7) can be tested for all $p \in \mathscr{P}$ via a robustness analysis test based on the S-procedure.

IV. SOLUTION OF THE ROBUST ANALYSIS PROBLEM

In this paper we consider a set of functions $Z(p)$ that are rational in p, and we apply the S-procedure to solve the robust analysis problem. Our approach is, however, not restricted to these choices. We can use in fact any parameterization that is representable as a Linear Fractional Representation (LFR) and apply any other robust analysis test. In future work we will investigate the effect of alternative parameterizations and analysis techniques on the size of the LMI problems.

Let $Z(p,z)$ be parameterized as a rational matrix function in $p \in \mathbb{R}^{n_p}$ without poles in $\mathscr{P}$ and an affine function in $z \in \mathbb{R}^N$, i.e. $Z(\cdot,\cdot) \in \mathscr{Z}_N$, where $\mathscr{Z}_N$ is the space

$$\mathscr{Z}_N := \left\{ \sum_{j=1}^{N} z_j Z_j(p) | z \in \mathbb{R}^N \right\},$$

$Z_j : \mathbb{R}^{n_p} \mapsto \mathscr{S}^{n+m_1}, j \in \{1,2,\ldots,N\}$ are matrix-valued rational functions in p without poles in $\mathscr{P}$, and the free coefficients $z_j \in \mathbb{R}, j \in \{1,2,\ldots,N\}$ are collected in a single vector z.

Remark on (6)**.** Since $\gamma > 0$ is redundant in (4), Lagrange dualization implies that (6) can equivalently be replaced by the equation

$$1 - \text{Tr}(Z_{22}) - \text{Tr}(Z_{33}) = 0. \tag{11}$$

We used the formulation with the inequality to simplify the proof of Theorem 4. For implementation it is however useful to use the version with the equation constraint, since then we can eliminate it by a simple modification of $Z(\cdot,\cdot)$: express the $(1,1)$ element of $Z_{22}(\cdot,\cdot)$ explicitly in terms of the remaining diagonal elements of $Z_{22}(\cdot,\cdot)$ and the diagonal elements of $Z_{33}(\cdot,\cdot)$ using (11). When we discuss the LFR construction later in this section it is tacitly assumed that this elimination has been realized.

Using this parameterization we observe that the inequalities (5), (6) and (7) are rational inequalities in p depending affinely on the coefficients z in the parameterization of Z. We construct an LFR

$$\begin{aligned} L(z,p) &= \Delta(p) \star \underbrace{\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C}(z) & \mathbf{D}(z) \end{pmatrix}}_{H(z)} \\ &= \mathbf{D}(z) + \mathbf{C}(z)\Delta(p)(I - \mathbf{A}\Delta(p))^{-1}\mathbf{B}, \end{aligned}$$

where $\star$ denotes the Redheffer star-product, $\Delta(p)$ is linear in p and $I - \mathbf{A}\Delta(p)$ is nonsingular for all $p \in \mathscr{P}$. L can be constructed such that (5), (6) and (7) are equivalent to

$$\begin{pmatrix} I \\ \Delta(p) \star H(z) \end{pmatrix}^T M_t \begin{pmatrix} I \\ \Delta(p) \star H(z) \end{pmatrix} \prec 0 \tag{12}$$

where

$$M_t := \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & -I_n & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_{n+p_1+m_1} & 0 \\ 0 & 0 & t & 0 & 0 & -1 \\ \hline -I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_{n+p_1+m_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{array}\right)$$

Since $\mathscr{P}$ is compact we can apply the S-procedure [10], [11], to infer that our problem is equivalent to minimizing γ over $z \in \mathbb{R}^N$ and multipliers M with

$$\begin{pmatrix} \Delta(p) \\ I \end{pmatrix}^T M \begin{pmatrix} \Delta(p) \\ I \end{pmatrix} \succ 0 \text{ for all } p \in \mathscr{P}, \quad (13)$$

$$\begin{pmatrix} I & 0 \\ \mathbf{A} & \mathbf{B} \end{pmatrix}^T M \begin{pmatrix} I & 0 \\ \mathbf{A} & \mathbf{B} \end{pmatrix} + \\ + \begin{pmatrix} 0 & I \\ \mathbf{C}(z) & \mathbf{D}(z) \end{pmatrix}^T M_t \begin{pmatrix} 0 & I \\ \mathbf{C}(z) & \mathbf{D}(z) \end{pmatrix} \prec 0$$

Equation (13) is a semi-infinite constraint on the multiplier M. To render the lower bound computation tractable we make a (standard) inner approximation of the set of multipliers. If $\mathscr{P}$ equals $co\{p_1, p_2, \ldots, p_m\}$, the set $\mathbf{M}$ of multipliers M such that

$$\begin{pmatrix} I \\ 0 \end{pmatrix}^T M \begin{pmatrix} I \\ 0 \end{pmatrix} \prec 0,$$

$$\begin{pmatrix} \Delta(p_j) \\ I \end{pmatrix}^T M \begin{pmatrix} \Delta(p_j) \\ I \end{pmatrix} \succ 0, \quad j = 1, \ldots, m$$

is an inner approximation for $\{M | M \text{ satisfies (13)}\}$, as can be shown by an elementary convexity argument. If we optimize over many controller variables, the number of generators m of p may be large. To avoid explosion of the size of the relaxed problem, we can exploit the block diagonal structure in $\Delta(p)$. The exactness of the test can then be guaranteed using higher order relaxations of the robustness problem. The reader is referred to [12] for details on this approach.

Remark on elimination of parameters. If $A(p)$ is stable for all $p \in P$, we can eliminate several parameters in the problem. Inequality (5) then reduces to

$$Z_{11}(p)A(p)^T + A(p)Z_{11}(p) + \\ + Z_{12}(p)B(p)^T + B(p)Z_{21}^T(p) = 0,$$

and $Z_{11}(p)$ can be explicitly described as a rational function in terms of $Z_{12}(p)$. This drastically reduces the number of free variables in the polynomial Z. The function $Z_{11}(p)$ constructed in this fashion has a rational dependence on the controller variables and affine dependence on $Z_{12}(p)$.

Remark. There are two sources of conservatism in our approach. The first is the approximation of the matrix valued function $Z(p)$ by a finite-order rational matrix function. To proof its asymptotic exactness we need Weierstrass' theorem on the approximation of continuous functions by polynomials [13]. There exists an increasing sequence of finite dimensional spaces $\mathscr{Z}_N, N \in \mathbb{N}$ of matrix-valued rational functions in p, such that for every continuous matrix polynomial $Y(\cdot)$

$$\lim_{N\to\infty} \inf_{Z\in\mathscr{Z}_N} \sup_{p\in\mathscr{P}} \sigma_{\max}(Z(p) - Y(p)) = 0,$$

where $\sigma_{\max}$ denotes the maximum singular value. Since this holds in particular for the continuous matrix polynomial in Theorem 4, the sequence of optimal values of

$$\begin{array}{rl} \text{supremum} & t \\ \text{over} & Z \in \mathscr{Z}_N \\ \text{subject to} & (5), (6) \text{ and } (7) \text{ for all } p \in \mathscr{P} \end{array}$$

converges to γ_{opt} for $N \to \infty$. The size of the LMI problems of this family is increasing due to the increase of the dimension of $\mathscr{Z}_N$ for larger N.

The second source of conservatism is the error made by the inner approximation of the multipliers. These errors can be reduced by using an asymptotically exact family of multiplier parameterizations [12], again by introducing more variables in the LMI problem.

The methods presented in [12] allow to check the exactness of the test for the robust analysis problem, i.e. they allow to verify if the second source did not introduce conservatism. If this is the case, then it is possible to construct a worst-case perturbation, i.e. in our case a "worst-case" controller. This controller is optimal if and only if neither the first source introduced conservatism. This can easily be verified, since the closed loop $\mathscr{H}_\infty$ norm of this controller equals the lower bound in case of optimality. If this is not the case, the lower bound improves for increasing N.

Remark on LFR construction. For $H(z)$ and $\Delta(p)$ with

$$\Delta(p) \star H(z) = G(p,z) := \\ \begin{bmatrix} G_{11}(p,z) & 0 & 0 \\ 0 & Z(p,z) & 0 \\ 0 & 0 & G_{33}(p,z) \end{bmatrix}$$

where

$$G_{11}(p,z) : = Z_{11}(p,z)A(p)^T + A(p)Z_{11}(p,z) + \\ + Z_{12}(p,z)B(p)^T + B(p)Z_{12}^T(p,z)^T$$

and

$$G_{33}(p,z) : = Z_{13}(p)C(p) + Z_{23}(p)D(p) + \\ C(p)^T Z_{13}(p)^T + D(p)^T Z_{23}(p)^T - V$$

where $V = V^T$ is an auxiliary variable, existence of a V and z with (12) and $\text{Tr}(V) > t$ is equivalent to (5), (6) and (7). We construct H by diagonal augmentation of the LFRs for each row of G. To keep the LMIs small it is important to construct the LFR with a small sized uncertainty block $\Delta(p)$. Decompose $G(p)$ as $G(p) = \sum_{i=1}^{3} L_i Z(p) R_i(p)$, where

$$L_1 = \begin{pmatrix} I_n & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, L_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, L_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & I_{m_1} \end{pmatrix}$$

$$R_1(p) = \begin{pmatrix} A(p)^T & 0 & 0 \\ B(p)^T & 0 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & I & 0 \end{pmatrix},$$

$$R_3(p) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & C(p) & D(p) \end{pmatrix}.$$

The LFR's for R_1 and R_2 can simply be derived of the LFR of the closed-loop state-space matrices in (1), which depends on the specific dependence of these matrices on p and will typically have an uncertainty block of size equal to the number of variables in p. If we assume that its uncertainty block is square and of size $r_{\mathrm{cl}} \times r_{\mathrm{cl}}$, then the sizes of the uncertainty blocks of the LFR's of R_1 and R_3 are at most $r_{\mathrm{cl}} \times r_{\mathrm{cl}}$ each.

Next we construct the LFR for $Z(p)$. The size $r_Z \times r_Z$ of its (again assumed square) uncertainty block depends on the degree of numerator and denominator of $Z(p)$. We compose each term $L_i Z(p) R_i(p)$ by multiplication of the LFR $Z(p)$ with the LFR $R_i(p)$. Multiplication of two LFR's yields a new LFR with the uncertainty blocks of the two factors diagonally concatenated as the new uncertainty block. The size of the uncertainty block of the LFR of G, i.e. the size of $\Delta(p)$ in (12) is therefore in our construction at most of size $(2r_{\mathrm{cl}} + 3r_Z) \times (2r_{\mathrm{cl}} + 3r_Z)$.

V. Application

For a first order plant given by

$$\left[\begin{array}{c|c|c} a & b_1 & b_2 \\ \hline c_1 & d_{11} & d_{12} \\ \hline c_2 & d_{21} & d_{22} \end{array}\right] = \left[\begin{array}{c|c|c} -7 & 9 & 2 \\ \hline -10 & 0 & 3 \\ \hline .8 & 3 & 0 \end{array}\right],$$

we computed lower bounds for a static controller. The LMI's are solved with SeDuMi of Jos Sturm [14]. Lower bounds are shown in Figure 1 for two different choices of Z:

- a first order $Z = Z^{(0)} + Z^{(1)} p$ and
- a second order matrix polynomial $Z(p) = Z^{(0)} + Z^{(1)} p + Z^{(2)} p^2$.

The elements of the matrices $Z^{(i)} \in \mathscr{S}^{n+m_1}, i \in \{0,1,2\}$ form the decision variables z. In these cases $r_Z = 1, r_{\mathrm{cl}} = 1$ and $r_Z = 3, r_{\mathrm{cl}} = 1$, resulting in $\Delta(p)$ in (12) of sizes 5×5 and 11×11 for the first order and second order polynomial respectively. For the i^{th} lower bound the controller parameter was restricted to lie in the compact set $2k + [-1,1], k \in \{-5,-4,\ldots,4,5\}$. The bounds for the second order parameterization are clearly better than the bounds for the first order ones. We also observe that the second order dotted curve is quite close to the upper bound (dashed) curve.

As a second example we consider the control of an active suspension system, which has been a benchmark system of a special issue of the European Journal of Control on fixed-order controller synthesis [15], see Figure 2. The goal is to compute a low-order discrete-time controller which minimizes the sensitivity to two resonance frequencies. The system has 17 states and the weights of our 4-block $\mathscr{H}_\infty$ design contributed with 10 states, which add up to

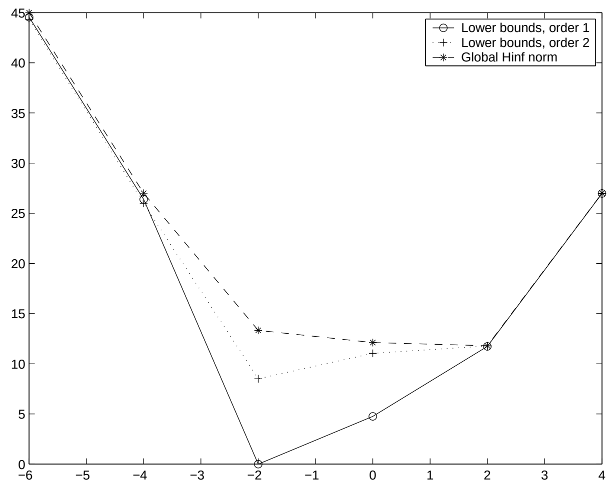

Fig. 1. Lower bounds and closed loop $\mathscr{H}_\infty$-norm for first order system: first order lower bounds (o solid), second order lower bounds (+ dotted) and $\mathscr{H}_\infty$ performance ($*$ dashed)

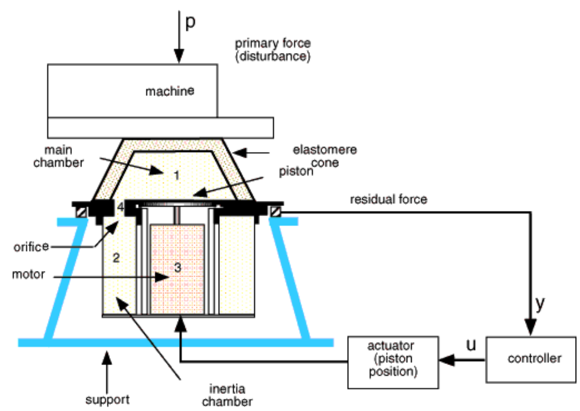

Fig. 2. Active suspension system

27 states of the generalized plant. The full order design leads to a closed loop $\mathscr{H}_\infty$-norm of 2.48. We computed a 5^{th} order controller by closed-loop balanced residualization with performance 3.41. For more details on the fixed order $\mathscr{H}_\infty$-design the reader is referred to [7]. We computed lower bounds for changes in two diagonal elements of the state-space matrices of the controller

$$\begin{pmatrix} A_k(p) & B_k \\ C_k & D_k \end{pmatrix}$$

where

$$A_k(p) = \begin{bmatrix} -78.2 & 1129.2 & 173.24 & -97.751 & -130.36 \\ -1240.9 & -78.2 + p_1 & 111.45 & 125.12 & 76.16 \\ 0 & 0 & -6.03 + p_2 & 164.81 & 159 \\ 0 & 0 & 0 & -204.56 & 49.031 \\ 0 & 0 & 0 & -458.3 & -204.56 \end{bmatrix},$$

$$B_k = \begin{pmatrix} 6.6086 & 21.4447 & -11.1262 & -12.4050 & -9.4469 \end{pmatrix}^T,$$

TABLE I

LOWER BOUNDS FOR VARIOUS CONTROLLER INTERVALS

Interval p_1	Interval p_2	Lower bound
$[-0.1, 0.1]$	$[-0.1, 0.1]$	3.41
$[-2.5, 2.5]$	$[-2.5, 2.5]$	3.19
$[-2.8, 2.8]$	$[-2.8, 2.8]$	-25565
$[-0.1, 0.1]$	$[-4, 4]$	$-9.62e5$
$[-4, 4]$	$[-0.1, .1]$	3.18

$$C_k = \begin{pmatrix} -0.067565 & 0.198 & -1.00 & -0.0697 & 0.193 \end{pmatrix}$$

and $D_k = 0.00629$ and p_1 and p_2 are free scalar controller variables. Lower bounds for various intervals of the controller variables p_1 and p_2 are shown in Table I. These lower bounds have been computed with Z being independent of p. The table should be read as follows: the lower bound 3.19 in the second row shows that there does not exist $p_1 \in [-2.5, 2.5]$ and $p_2 \in [-2.5, 2.5]$ such that the closed-loop $\mathscr{H}_\infty$-norm is smaller than 3.19. The lower bounds are good up to intervals within $[-2.5, 2.5]$, in the sense that the computed lower bounds are much larger than the full order performance 2.48. But for larger parameter sets, such as $p_1, p_2 \in [-2.8, 2.8]$, the lower bounds are useless. This must probably be attributed to the zero-order polynomial parameterization of Z, which is not flexible enough for good lower bounds on larger compact sets $\mathscr{P}$. As shown in the first order example, we could establish better lower bounds on larger controller parameter sets using higher order polynomials in Z.

The size of $\Delta(p)$ is 4×4 in this example, since $r_Z = 0$ and $r_{\mathrm{cl}} = 2$. The number of LMI variables in this example is 631. If one constructs the LFR without care, one may easily get a $\Delta(p)$ of size 50×50 and hence a computationally intractable LMI problem.

VI. CONCLUDING REMARKS

Asymptotically exact global lower bounds have been presented in this paper. The bounds are based on reformulating the fixed order control problem as a robust analysis test. An example for a first order system illustrated that the lower bounds converge to the exact value. A 27^{th} order example has been presented to show the applicability to tuning few controller parameters in high-order systems.

APPENDIX

Problem 3 is strictly feasible, in the sense that for all $p_0 \in \mathscr{P}$ there exists a W such that

$$\begin{aligned} W_{11}A(p_0)^T + A(p_0)W_{11} + & \\ +W_{12}B(p_0)^T + B(p_0)W_{12}^T &\succ 0 \\ 1 - \mathrm{Tr}(W_{22}) - \mathrm{Tr}(W_{33}) &> 0 \qquad (14) \\ W &\succ 0 \end{aligned}$$

are satisfied. This is shown as follows. Consider an arbitrary $p_0 \in \mathscr{P}$. Since by assumption $A(p_0), B(p_0)$ is controllable, there exists an anti-stabilising state-feedback, i.e. a K with $A(p_0) + B(p_0)K$ is strictly anti-stable. Then there exists $P = P^T \succ 0$ such that

$$(A(p_0) + B(p_0)K)P + P(A(p_0) + B(p_0)K)^T \succ 0. \qquad (15)$$

Since (15) is homogeneous in P we conclude that if P satisfies (15), rP, $r > 0$ also satisfies (15). Now set $W_{11}(r) = rP$, $W_{12}(r)^T = rKP$ and $W_{22}(r) = W_{12}(r)^T W_{11}(r)^{-1} W_{12}(r) - \varepsilon r$ with $\varepsilon < 0$. Now choose $r > 0$ such that $\mathrm{Tr}(W_{22}(r)) < 1$, choose $W_{33} > 0$ such that (14) is satisfied and choose $W_{31} = 0$ and $W_{32} = 0$. This construction can always be done and leads to a strictly feasible W for problem 3. Since p_0 is arbitrary this implies that Problem 3 is strictly feasible for all $p \in \mathscr{P}$. This finishes the proof.

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