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Characterization of the Optimal Disturbance Attenuation for Nonlinear Stochastic Partially Observable Uncertain Systems

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Abstract— This paper is concerned with stochastic optimal control systems, in which uncertainty is described by a relative entropy constraint between the nominal measure and the uncertain measure, while the pay-off is a functional of the uncertain measure. This is a minimax game, equivalent to the H^∞ optimal disturbance attenuation problem, in which the controller seeks to minimize the pay-off, while the disturbance described by a set of measures aims at maximizing the pay-off.

The objective of this paper is to apply the results of the abstract formulation to stochastic uncertain systems, in which the nominal and uncertain systems are described by conditional distributions. The results obtained include existence of the optimal control policy, explicit computation of the worst case conditional measure, and characterization of the optimal disturbance attenuation, for nonlinear partially observable systems. The linear case is presented to illustrate the concepts.

Key Words: Nonlinear Uncertain Stochastic Systems, Large Deviations, Relative Entropy, Minimax Games, Duality Properties.

I. Introduction

Subsequent to the publication of Zames [1] seminal paper on H^∞ control, several tools have been introduced to extend the H^∞ techniques to nonlinear deterministic and stochastic systems. Three pay-off functionals which received significant attention are deterministic minimax games [3], risk-sensitivity pay-offs [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [15], and stochastic minimax games [11], [12]. For nonlinear problems robustification is captured through dissipation inequalities [2].

Another class of problems aiming at robustification are described through relative entropy constraints, using duality relations between relative entropy and free energy. The problems are equivalent to risk-sensitive problems [18], [17], [15], and thus equivalent to the minimax game formulation [15] of the H^∞ disturbance attenuation problem. For linear fully observable or partially observable problems, the characterization of the optimal disturbance attenuation is expressed in terms of certain Ricatti equations. Unfortunately, in nonlinear problems such Ricatti equations are not available. Therefore, one considers the sub-optimal problem by either fixing the structure of the controller or by considering minimax dynamic games that result in strategies which ensure a certain dissipation inequality with respect to the supply rate (energy functional) [3]. Ideally, one would like to have an explicit expression which describes the optimal disturbance attenuaton level for general nonlinear fully observed and partially observed (output feedback) problems. Such an expression will be usefull in determining how far any sub-optimal solution is from the optimal one. However, there is another important issue associated with partially observable systems, namely, the fundamental question of existence of an optimal control law. To the best of the authors knowledge, no results have been presented addressing the existence of optimal control laws for nonlinear partially observable stochastic minimax games.

The abstract formulation is the following.

Let (Σ, d) denote a complete separable metric space (Banach Space), and $(\Sigma, \mathcal{B}(\Sigma))$ the corresponding measurable space in which $\mathcal{B}(\Sigma)$ are identified as the Borel sets generated by open sets in Σ. Let $\mathcal{M}(\Sigma)$ denote the set of probability measures on $(\Sigma, \mathcal{B}(\Sigma))$, $\mathcal{U}_{ad}$ the set of admissible controls, and let $\ell^u : \Sigma \to \Re$ be a real-valued measurable function, bounded from below, which depends on the control $u \in \mathcal{U}_{ad}$. Here, $\mathcal{M}(\Sigma)$ denotes the set of all possible measures induced by the stochastic systems, while ℓ^u denotes the energy function associated with a given choice of the control law $u \in \mathcal{U}_{ad}$.

Given a control law $u \in \mathcal{U}_{ad}$ and a nominal measure $\mu^u \in \mathcal{M}(\Sigma)$ induced by the nominal stochastic system, the pay-off is defined by

$$J(u, \nu^u) = E_{\nu^u}\left(\ell^u\right) = \int_\Sigma \ell^u d\nu^u \qquad \text{(I.1)}$$

Subject to fidelity $H(\nu^u|\mu^u) \leq R$ where $\nu^u << \mu^u$ and $R \in (0, \infty)$, $\mu^u \in \mathcal{M}(\Sigma)$ and $H(\nu^u|\mu^u)$ denotes the relative entropy between the uncertain measure ν^u

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and the nominal measure μ^u, and $<<$ denotes absolute continuity of $\nu^u \in \mathcal{M}(\Sigma)$ with respect to $\mu^u \in \mathcal{M}(\Sigma)$. For a given control law $u \in \mathcal{U}_{ad}$, and nominal measure $\mu^u \in \mathcal{M}(\Sigma)$, let $\nu^{u,*} \in \mathcal{M}(\Sigma)$ denote the measure which achieves the supremum in (I.1) defined by

$$J(u,\nu^{u,^*}) = \sup_{\nu^u \in \mathcal{A}(\mu^u)} \int_\Sigma \ell^u d\nu^u \qquad \text{(I.2)}$$

where

$$\mathcal{A}(\mu^u) = \{\nu^u \in \mathcal{M}(\Sigma); H(\nu^u|\mu^u) \leq R, \nu^u << \mu^u\}$$

and $R \in (0,\infty)$. The robust control problem is to find a control law $u^* \in \mathcal{U}_{ad}$ which impacts

$$\begin{aligned} J(u^*) &= \inf_{u \in \mathcal{U}_{ad}} J(u,\nu^{u,*}) \\ &= \inf_{u \in \mathcal{U}_{ad}} \Big(\sup_{\nu^u \in \mathcal{A}(\mu^u)} \int_\Sigma \ell^u d\nu^u \Big) \qquad \text{(I.3)} \end{aligned}$$

The important contribution of this paper is to transform nonlinear partially observable uncertain stochastic systems into separated minimax uncertain stochastic systems which are equivalent to (I.1)-(I.3). In the separated stochastic minimax systems the nominal measure is described by the conditional measure, while the uncertain measures are described by conditional measures which satisfy relative entropy constraints. All partially observable minimax problems can be transformed into separated problems provided the conditional distribution can be found.

In [20] some of the important mathematical tools and duality relations, which are employed in this paper, are discussed. The main properties associated with the dual functional and the equivalence of the unconstrained and constrained problems is also shown in [20] .

Section II, is concerned with application of the results of [20] to nonlinear stochastic optimal control systems with partial information. The problem is reformulated as a separated minimax game in which the nominal and uncertain measures are described by conditional distributions. Under this separated minimax game formulation it is shown that those results remain valid, while existence of an optimal control policy is shown. Finally, in Section II-D, closed form expressions are presented for the class of Gaussian nominal conditional measures, when the sample path pay-off has a quadratic form.

Due to the space limitation the derivations are not included (they are found in [21]).

II. Partially Observed Wide Sense Uncertain Control Systems

In this section the results derived in Lemma (3.2) and Theorem (3.3) in [20] are employed to address the partially observable problem. The difficulty of dealing with partially observable systems is overcome by introducing an equivalent separated minimax problem. In the separated minimax formulation, the nominal model is described by a conditional distribution while the uncertain system is described by a family of conditional distributions, which satisfy the relative entropy constraint. Thus, the problem is described through a posteriori information rather than a priori information. Under the equivalent separated minimax formulation, all previous results obtained in [20] remain valid. In addition, existence of an optimal control $u \in \mathcal{U}_{ad}$ is shown among the class of wide-sense control laws which are measures.

A. Problem Formulation

Let $\{x(t)\}_{t\geq 0}$ denote the state process which is subject to control, $\{y(t)\}_{t\geq 0}$ the observation process, and $\{u(t)\}_{t\geq 0}$ the control process, all defined for a fixed and finite interval of time $0 \leq t \leq T$.

For each $u \in \mathcal{U}_{ad}$ (in some admissible set which is defined shortly) the nominal state and observation process, giving rise to a nominal probability measure P^u are governed by the following Ito stochastic differential equations, in the probability space $(\Sigma, \mathcal{B}(\Sigma), P^u)$.

$$\begin{cases} dx(t) = f(x(t),u(t))dt + \sigma(x(t))dw(t), \quad x(0) \\ dy(t) = h(x(t))dt + Ndv(t), \quad y(0) = 0 \end{cases} \qquad \text{(II.4)}$$

Here $x(t) \in \Re^n, y(t) \in \Re^d, u(t) \in \mathcal{U} \subset \Re^k$, $\{w(t)\}_{t\geq 0}$ and $\{v(t)\}_{t\geq 0}$ are independent Brownian motions taking values in $\Re^m, \Re^d$, respectively, which are also independent of the Random Variable $x(0)$.

The following assumptions are introduced.

Assumptions 2.1: The nominal system satisfies the following assumptions.

1) The controls $\{u(t); t \in [0,T]\}$ take values in $\mathcal{U} \subset \Re^k$ which is compact and convex.
2) $f : \Re^n \times \mathcal{U} \to \Re^n, \sigma : \Re^n \to \mathcal{L}(\Re^n;\Re^m), f(x,u) = f_0(x) + f_1(x)u$, and f_0, f_1, σ are bounded and Lipschitz continuous.
3) $h : \Re^n \to \Re^d, h \in C_b^2(\Re^n)$.
4) $N \in \mathcal{L}(\Re^d;\Re^d)$ and $\exists \beta > 0$ such that $NN' \geq \beta I_d$.
5) The random variable $x(0)$ has distribution $\Pi_0(x)$.
6) $\lambda : \Re^n \times \mathcal{U} \to \Re, \kappa : \Re^n \to \Re$, ℓ, κ are continuous, bounded from below and from above, and $\lambda(\cdot,x)$ is convex in u for all $x \in \Re^n$.

Next, the problem is made precise by identifying the spaces on which the nominal system is defined and then introducing the precise definition of admissible controls. Consider the sample space

$$\Omega = \Omega^w \times \Omega^x \times \Omega^y \times \Omega^u$$

where

$$\Omega^w = C([0,T];\Re^m), \quad \Omega^x = C([0,T];\Re^n)$$
$$\Omega^y = C([0,T];\Re^d), \quad \Omega^u = L_2([0,T];\mathcal{U})$$

and $y(0)=0$ is assumed throughout. Here, $\Omega^w, \Omega^x, \Omega^y$ are endowed with the usual sup-norm topology, while Ω^u is endowed with the weak topology (which is metrizable and separable). A typical element of Ω is $\omega(t) = (w(t,\omega), x(t,\omega), y(t,\omega), u(t,\omega)), 0 \leq t \leq T$. Let $\Omega^{w,x} = \Omega^w \times \Omega^x, \Omega^{y,u} = \Omega^y \times \Omega^u$. Then Ω is provided with a filtration $\{\mathcal{F}_t; t \in [0,T]\}$ which is defined as follows.
Let $\mathcal{F}_t^w = \sigma\{w(s); 0 \leq s \leq t\}$, $\mathcal{F}_t^x = \sigma\{x(s); 0 \leq s \leq t\}$, $\mathcal{F}_t^y = \sigma\{y(s); 0 \leq s \leq t\}$, which may be regarded as the Borel $\sigma-$algebras on $C([0,T]; \Re^q), q = m, n, d$, respectively, and $\mathcal{F}_t^u = \sigma\{\int_0^s u(\tau)d\tau; 0 \leq s \leq t\}$, which is the Borel $\sigma-$algebra on Ω^u.
Then

$$\mathcal{F}_t = \mathcal{F}_t^{w,x} \times \mathcal{F}_t^{u,y},\ \mathcal{F}_t^{w,x} = \mathcal{F}_t^w \times \mathcal{F}_t^x,\ \mathcal{F}_t^{y,u} = \mathcal{F}_t^y \times \mathcal{F}_t^u$$

Fix a sample path for the observation and control process $y(\cdot,\omega), u(\cdot,\omega)$. Given the initial data $x(0) = x, y(0) = 0, w(0) = 0$, Assumptions 2.1 imply existence of a unique probability measure $\bar{P}_x^{y,u}$ on $(\Omega^{w,x}, \mathcal{F}_T^{w,x})$ which coincides with the law of $\{w(t), x(t); t \in [0,T]\}$, such that $\{w(t); t \in [0,T]\}$ and $\{v(t); t \in [0,T]\}$ are independent Wiener processes and in the probability space $(\Omega^{w,x}, \mathcal{F}_T^{w,x}, \bar{P}_x^{y,u})$ we have

$$\begin{cases} x(t) = x + \int_0^t f(x(s), u(s))ds + \int_0^t \sigma(x(s))dw(s) \\ \\ y(t) = Nv(t) \end{cases}$$

In addition, Assumptions 2.1, imply that $\bar{P}_x^{y,u} \in \mathcal{M}(\Omega^{w,x})$ depends continuously on $(u(\cdot,\omega), x)$.

Definition 2.2: The set of admissible controls denoted by $\mathcal{U}_{ad}$ consists of measures π on $(\Omega^{y,u}, \mathcal{F}_T^{y,u})$, that is, $\pi \in \mathcal{M}(\Omega^{y,u})$, such that $\{y(t); t \in [0,T]\}$ is $\mathcal{F}_T^{y,u} - \pi$-a.s. Brownian motion.

The projection $(y(\cdot,\omega), u(\cdot,\omega)) \mapsto y(\cdot,\omega)$ maps $\pi \in \mathcal{M}(\Omega^{y,u})$ onto a Wiener measure, and for all $t \in [0,T]$, $u(t)$ and $\sigma\{y(r) - y(t); 0 \leq t \leq r \leq T\}$ are independent under π.
Given the measure $\Pi_0 \in \mathcal{M}(\Re^n)$ of $x(0)$, by Baye's rule

$$\bar{P}^{y,u}(A) = \int_{\Re^n} \bar{P}_x^{y,u}(A) d\Pi_0(x), \qquad A \in \mathcal{F}_T^{w,x} \quad \text{(II.5)}$$

which is the unique joint distribution measure of $\{x(t), w(t); t \in [0,T]\}$ given $(y(\cdot,\omega), u(\cdot,\omega))$.
For each $\pi \in \mathcal{U}_{ad}$ define the joint distribution measure $\tilde{P}^\pi$ on $(\Omega, \mathcal{F}_T)$ by

$$\tilde{P}^\pi(dw, dx, du, dy) \triangleq \bar{P}^{y,u}(dw, dx) \times \pi(dy, du) \in \mathcal{M}(\Omega) \quad \text{(II.6)}$$

Notice that the projection $\Big(w(\cdot,\omega), x(\cdot,\omega), y(\cdot,\omega), u(\cdot,\omega)\Big) \mapsto \Big(y(\cdot,\omega), u(\cdot,\omega)\Big)$ under $\tilde{P}^\pi \in \mathcal{M}(\Omega)$ is $\pi \in \mathcal{M}(\Omega^{y,u})$.
Finally, define the nominal measure P^π as follows.

Introduce the $(\{\mathcal{F}_t; t \in [0,T]\}, \tilde{P}^\pi)$-adapted exponential martingale process

$$\Lambda^u(t) = \exp\Big\{ \int_0^t h'(x(s))(NN')^{-1} dy(s) - \frac{1}{2}\int_0^t h'(x(s))(NN')^{-1} h(x(s))ds \Big\} \quad \text{(II.7)}$$

Define the nominal measure through the Radon-Nikodym derivative

$$\frac{dP^\pi(w,x,y,u)}{d\tilde{P}^\pi(w,x,y,u)}|_{\mathcal{F}_T} \triangleq \Lambda^u(T) \quad \text{(II.8)}$$

Then, Assumptions 2.1 imply that $P^\pi(\Omega) = E_{\tilde{P}^\pi}\Big\{\Lambda^u(t)\Big\} = 1, \forall t \in [0,T]$, and thus $P^\pi \in \mathcal{M}(\Omega)$. Moreover, by Girsanov's theorem, $v^\pi(t) \triangleq \int_0^t N^{-1}dy(s) - \int_0^t N^{-1}h(x(s))ds$ is a standard Wiener process under $P^\pi \in \mathcal{M}(\Omega)$, and the distribution of $\{w(t), x(t); t \in [0,T]\}$ is invariant under the measure change of (II.8). Thus, under the measure $P^\pi \in \mathcal{M}(\Omega)$, the processes $\{v^\pi(t); t \in [0,T]\}$ and $\{w(t); t \in [0,T]\}$ are independent Wiener processes.
Thus, for each $\pi \in \mathcal{U}_{ad}$, there exists a unique nominal measure $P^\pi \in \mathcal{M}(\Omega)$ on which the state $\{x(t); t \in [0,T]\}$ and observation process $\{y(t); t \in [0,T]\}$ satisfy (II.4).
The following is shown in [19].

Lemma 2.3: The set of admissible controls $\mathcal{U}_{ad}$ is compact under weak sequential convergence.
The precise problem statement should thus, be as follows.

Definition 2.4: Given the nominal measure $P^\pi \in \mathcal{M}(\Omega)$, find a $\pi^* \in \mathcal{U}_{ad}$ and a probability measure $Q^{\pi^*,*} \in \mathcal{M}(\Omega)$ which solve the following constrained optimization problem.

$$J(\pi^*, Q^{\pi^*,*}) = \inf_{\pi \in \mathcal{U}_{ad}} \sup_{Q^\pi \in \mathcal{A}(P^\pi)} \Big(E_{Q^\pi}\Big\{ \int_0^T \lambda(x(t), u(t))dt + \kappa(x(T))\Big\}\Big)$$

subject to fidelity

$$H(Q^\pi | P^\pi) \leq R, \quad R \in (0, \infty) \quad \text{(II.9)}$$

where $\mathcal{A}(P^\pi) = \{Q^\pi \in \mathcal{M}(\Omega); Q^\pi << P^\pi\}$

B. Duality of Wide Sense Separated Uncertain Systems

Similar to [20], the problem is reformulated using the dual functional. However, in partially observable problems it is crucial to reformulate them as completely observable problems, using separated laws, in which the minimizing and maximizing players are functionals of conditional distributions, rather than a priori distributions. For the specific problem under investigation, the maximizing measure $Q^{\pi,*}$ should be restricted on $(\Omega^{y,u}, \mathcal{F}_T^{y,u})$, because this is the only information

available to both controller and minimizing measure. This approach is considered next.
For each $\Big(u(\cdot,\omega),y(\cdot,\omega)\Big)$ introduce the multiplicative functional

$$\chi^u(t) \triangleq \exp\Big(\frac{1}{s}\int_0^T \lambda(x(t),u(t))dt\Big) \quad \text{(II.10)}$$

Define the measure-valued process $M_t^{y,u}(\phi), \phi \in C_b(\Re^n)$ by

$$M_t^{y,u}(\phi) \triangleq E_{\bar{P}^{y,u}}\Big\{\phi(x(t))\chi^u(t)\Lambda^u(T)\Big\} \quad \text{(II.11)}$$

Then

$$M_t^{y,u}(\phi) = <\phi, M_t^{y,u}> = \int_{\Re^n}\phi(z)dM^{y,u}(z,t), \qquad \phi \in C_b(\Re^n) \quad \text{(II.12)}$$

It can be shown that the following separated minimax game is equivalent to the original problem.

Theorem 2.5: The problem of Definition 2.4 is equivalent to the following separated minimax problem.
Define

$$\mu^\pi(T,x,y,u) \triangleq M^{y,u}(t,x)\pi(y,u) \in \mathcal{M}(\Omega^{y,u}\times\Re^n)$$

$$\nu^\pi(T,x,y,u) \triangleq N^{y,u}(t,x)\pi(y,u) \in \mathcal{M}(\Omega^{y,u}\times\Re^n)$$

Given a nominal conditional measure valued-process $M_t^{y,u} \in \mathcal{M}(\Re^n)$ find a $\pi^* \in \mathcal{U}_{ad}$ and a conditional measure-valued process $N_t^{y,u,*} \in \mathcal{M}(\Re^n)$ which solve the following constrained optimization problem.

$$J(\pi^*,\nu^{\pi^*,*}) = \inf_{\pi\in\mathcal{U}_{ad}}\sup_{\nu^\pi\in\mathcal{A}(\mu^\pi)}\int_{\Omega^{y,u}\times\Re^n}\kappa(z)d\nu^\pi(T,x,y,u) \quad \text{(II.13)}$$

where

$$\mathcal{A}(\mu^\pi) = \{\nu^\pi \in \mathcal{M}(\Omega^{y,u}\times\Re^n); H(\nu^\pi|\mu^\pi) \le R, \nu^\pi << \mu^\pi\}$$

and $\mu^\pi \in (\Omega^{y,u}\times\Re^n)$. Equivalently,

$$J(\pi^*,\nu^{\pi^*,*}) = \inf_{\pi\in\mathcal{U}_{ad}}\sup_{N^{y,u}\in\mathcal{A}(M^{y,u})}\int_{\Omega^{y,u}} <\kappa, N_T^{y,u}> d\pi(y,u) \quad \text{(II.14)}$$

where

$$\mathcal{A}(M^{y,u}) = \{N^{y,u} \in \mathcal{M}(\Re^n); H(N^{y,u}\times\pi|M^{y,u}\times\pi) \le R, N_t^{y,u} << M_t^{y,u}\}$$

and $M_t^{y,u} \in \mathcal{M}(\Re^n)$.
Similar to [20], the above problems can be reformulated using the dual functional as follows.
For every $s \in \Re$ define the Lagrangian

$$J^{s,R}(\pi,\nu^\pi) \triangleq \int_{\Omega^{y,u}} <\kappa, N_T^{y,u}> d\pi(y,u) - s\Big(H(\nu^\pi|\mu^\pi) - R\Big) \quad \text{(II.15)}$$

and its associated dual functional

$$J^{s,R}(\pi,\nu^{\pi,*}) = \sup_{\nu^\pi\in\mathcal{A}(\mu^\pi)} J^{s,R}(\pi,\nu^\pi) \quad \text{(II.16)}$$

where $\mathcal{A}(\mu^\pi) = \{\nu^\pi \in \mathcal{M}(\Omega^{y,u}\times\Re^n); \nu^\pi << \mu^\pi\}$ and $\mu^\pi \in \mathcal{M}(\Omega^{y,u}\times\Re^n)$. In addition define the quantity

$$\varphi^{s^*}(\pi,R) \triangleq \inf_{s>0} J^{s,R}(\pi,\nu^{\pi,*}) \quad \text{(II.17)}$$

which may or may not exist.
Then, the statements of Lemma (3.2) and Theorem (3.3) in [20] hold, provided the appropriate changes are made, which are made precise in the next Corollary.

Corollary 2.6: Let $\pi \in \mathcal{U}_{ad}$ be fixed.
The statements of Lemma (3.2) and Theorem (3.3) in [20] hold with the following changes.

$$\Sigma \mapsto \Omega^{y,u}\times\Re^n;\quad u \mapsto \pi \in \mathcal{M}(\Omega^{y,u});$$
$$\mu^u \mapsto \mu^\pi \in \mathcal{M}(\Omega^{y,u}\times\Re^n);$$
$$\nu^u \mapsto \nu^\pi \in \mathcal{M}(\Omega^{y,u}\times\Re^n);\quad \ell^u \mapsto \kappa$$

In particular,

$$J^{s,R}(\pi,\nu^{\pi,*}) = sR + s\log E_{\nu^\pi}\Big\{e^{\frac{1}{s}\kappa}\Big\} = sR + s\log\int_{\Omega^{y,u}\times\Re^n} e^{\frac{1}{s}\kappa(z)}dN_T^{y,u}(T,z)\times d\pi(y,u) \overset{\nabla}{=} sR + s\Psi_{\mu^\pi}(\frac{1}{s}) \quad \text{(II.18)}$$

where the supremum in (II.16) is attained by $\nu^{\pi,*} \in \mathcal{M}(\Omega^{y,u}\times\Re^n)$ given by

$$d\nu^{\pi,*}(T,z,y,u) = \frac{e^{\frac{1}{s}\kappa(z)}d\mu^\pi(T,z,y,u)}{\int_{\Omega^{y,u}\times\Re^n} e^{\frac{1}{s}\kappa(z)}d\mu^\pi(T,z,y,u)} \quad \text{(II.19)}$$

or equivalently,

$$dN^{y,u,*}(T,z,y,u) = \frac{e^{\frac{1}{s}\kappa(z)}dM^{y,u}(T,z,y,u)}{\int_{\Omega^{y,u}\times\Re^n} e^{\frac{1}{s}\kappa(z)}dM^{y,u}(T,z,y,u)\times d\pi(y,u)} \quad \text{(II.20)}$$

Proof. The derivation is found in [21].
Using the results of Corollary 2.6 the existence of the optimal control policy $\pi^* \in \mathcal{U}_{ad}$ can be shown, using the property that the measure valued process $M_t^{y,u}$ is a continuous function of $\Big(y(\cdot,\omega),u(\cdot,\omega)\Big), \forall t \in [0,T]$.

Theorem 2.7: Consider any s in the admissible intervals, $(0,s^*]$, and the resulting pay-off corresponding to the maximizing measure, namely,

$$J^{s,R}(\pi,\nu^{\pi,*}) = J^{s,R}(\pi,M^{y,u}) = sR + s\log\int_{\Omega^{y,u}}\Big\{\int_{\Re^n} e^{\frac{1}{s}\kappa(z)}dM_T^{y,u}(T,z)\Big\}\times d\pi(y,u) \quad \text{(II.21)}$$

Then
1) $J^{s,R}(\pi,\nu^{\pi,*})$ is lower-semi continuous on $\mathcal{U}_{ad}$.
2) There exists a $\pi^* \in \mathcal{U}_{ad}$ such that $J^{s,R}(\pi^*,\nu^{\pi^*,*}) \le J^{s,R}(\pi,\nu^{\pi,*}), \forall\pi \in \mathcal{U}_{ad}$
Proof. The methodology is similar to that found in [19] (the complete derivation is found in [21]).

C. Evolution of the Density of the Minimum Measure

Introducing some additional regularities on σ, Π_0 would imply that the measure valued processes $N_t^{y,u}(\phi), M_t^{y,u}(\phi)$ have densities. The following are sufficient to show that such densities exists

7) $n = m, a(x) \stackrel{\triangle}{=} \sigma(x)\sigma'(x) \geq I_n\alpha, \alpha > 0, \forall x \in \Re^n$, $\frac{\partial}{\partial x_j} a_{i,j} \in L^\infty(\Re^n), \forall i, j.$

8) Π_0 has a density $p_0(x)$ and $p_0 \in L_2(\Re^n)$.

Under Assumptions 2.1 and 7), 8), it can be shown that $M_t^{y,u}$ has an unnormalized density and thus

$$M_t^{y,u}(\phi) = \int_{\Re^n} \phi(z) e^{y'(t)h(z)} q^{y,u}(t,z)dz, \ \phi \in C_b(\Re^n) \tag{II.22}$$

Moreover, $q^{y,u}(\cdot, z)$ is the solution of the following partial differential equation

$$\frac{\partial}{\partial t} q^{y,u}(t,x) = A^*(y(t))q^{y,u}(t,x) + e(x, y(t), u(t))q^{y,u}(t,x)$$

where $(t,x) \in (0,T] \times \Re^n$ and $q^{y,u}(0,x) = p_0(x)$.

where $A^*(y)$ is the adjoint operator of $A(y)$ (e.g., with respect to the duality product $< A(y)\phi, \psi >=< \phi, A^*(y)\psi >, \phi, \psi \in C_b^2(\Re^n)$)

$$A(y) = \frac{1}{2}\sum_{i,j=1}^n a_{i,j}(x,y)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x,y,u)\frac{\partial}{\partial x_i} - \sum_{i=1}^n (ay \cdot \nabla h)_i \frac{\partial}{\partial x_i}$$

$$e(x,y,u) = \frac{1}{2}(ay \cdot \nabla h, y \cdot \nabla h) - y \cdot A(y)h - ||h||^2_{\Re^d}$$

in which $\cdot, (,)$ are the dot products in $\Re^d$, $\Re^n$, respectively.

Using the density of the measure valued process the maximization with respect to $\pi \in \mathcal{U}_{ad}$ can be expressed as

$$J^{s,R}(\pi, \nu^{\pi,*}) = sR + s\log\int_{\Omega^{y,u}} \Big\{ \int_{\Re^n} e^{\frac{1}{s}\kappa(z)} e^{y'(t)h(z)} q^{y,u}(t,z)dz \Big\} \times d\pi(y,u) \stackrel{\nabla}{=} J^{s,R}(\pi, q^{y,u,*}) \tag{II.23}$$

Therefore, existence of the optimal control policy $\pi^* \in \mathcal{U}_{ad}$ can be shown, through the partial differential equations arguments.

Theorem 2.8: Consider any s in the admissible intervals, $(0, s^*]$ specified by the solution of the minimizing measure.

Then

1) The functional $J^{s,R}(\pi, q^{y,u,*})$ is upper semi continuous on the set $\mathcal{U}_{ad}$

2) There exists an optimal control policy $\pi^* \in \mathcal{U}_{ad}$.

Proof. This is similar to the proof found in [19] (a complete proof is given in [21]).

D. Solvable Partially Observable Problems

For the purpose of illustrating the concepts presented earlier, the following linear dynamics and observations are considered, with quadratic constraints.

Assumptions 2.9: The coefficients of (II.4), the density of $x(0)$, and the constraint are given by

$$f(x,u) = Fx + Bu, \quad \sigma(t,x) = G, \quad h(x)Hx,$$
$$2\lambda(x,u) = x'Qx + u'Ru, \quad 2\kappa(x) = x'Mx$$

$$p_0(x) = \frac{\exp(-\frac{1}{2}|P_0^{-\frac{1}{2}}(x-\xi)|^2)}{(2\pi)^{\frac{n}{2}}|P_0|^{\frac{1}{2}}}, \quad P_0 = P_0' \geq 0,$$

each element having appropriate dimensions.

Under Assumptions 2.9, it can be shown that $M_t^{y,u}$ has a density $m^{y,u}(x,t)$ given by

$$dM^{y,u}(t,x) = m^{y,u}(x,t)dx = \nu_{0,t}^u \times \frac{\exp\left(-\frac{1}{2}|P(t)^{-\frac{1}{2}}(x - r(t))|^2\right)}{(2\pi)^{\frac{n}{2}}|P(t)|^{\frac{1}{2}}} \times \exp\frac{1}{2s}\left(\mathcal{C}_{0,t}^u\right) \times \mathcal{I}_{0,t}dx, \tag{II.24}$$

where

$$\nu_{0,t}^u = \exp\Big(\int_0^t (Hr)'\Big(NN'\Big)^{-1}dy - \frac{1}{2}\int_0^t ||N^{-1}Hr||^2_{\Re^d}ds\Big)$$

and

$$\mathcal{C}_{0,t}^u \doteq \int_0^t (r'Qr + u'Ru)ds$$
$$\mathcal{I}_{0,t} \doteq \exp\frac{1}{2s}\Big\{\int_0^t Tr(PQ)ds\Big\}$$

and $P : [0,T] \to \mathcal{L}(\Re^n; \Re^n), P = P' \geq 0, r : [0,T] \times \Omega \to \Re^n$, are given by

$$\dot{P} = FP + PF' + \frac{1}{s}PQP + GG' - PH'\Big(NN'\Big)^{-1}HP$$

$$P(0) = P_0,$$
$$dr = (F + \frac{1}{s}PQ)rdt + Budt + PH'\Big(NN'\Big)^{-1}\Big(dy - (Hr + h)dt\Big),$$
$$r(0) = \xi, \tag{II.25}$$

where $y(\cdot)$ is an $\{\mathcal{F}_t^{y,u}; t \in T\}$-adapted Wiener process with correlation NN'. Suppose the class of strict-sense control laws is considered.

Then the minimizing density associated with the measure valued process $N_t^{y,u}$ is given by

$$dM^{y,u,*}(t,x) = \frac{\exp\left(-\frac{1}{2}|P(t)^{-\frac{1}{2}}(x-r(t))|^2\right)}{(2\pi)^{\frac{n}{2}}|P(t)|^{\frac{1}{2}}}$$
$$\times \frac{|I-\frac{1}{s}P(T)M(T)|^{\frac{1}{2}}}{\exp\frac{1}{2s}\left(r'(T)\left(I-\frac{1}{s}P(T)M(T)\right)^{-1}M(T)r(T)\right)}dx$$

where $dI^u dy - Hrdt - hdt$ is the innovations process with covariance NN' defined on the space $(P^u; \mathcal{F}_t^{r,u})$, $\mathcal{F}_t^{r,u} \doteq \sigma\{r(t), u(t); t \in [0,T]\}$.
Moreover, the optimal s^* corresponds to the value of s for which the relative entropy between $M_t^{y,u,*}$ and $N_t^{y,u,*}$ equals R.

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