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Optimization of Stochastic Uncertain Systems: Large Deviations and Robustness for Partially Observable Diffusions

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Abstract— This paper is concerned with stochastic control systems, in which the pay-off is described by the relative entropy between the nominal measure and the uncertain measure, while the uncertain measures satisfy certain energy inequality constraints. With respect to this formulation two problems are defined. The first, seeks to minimize the relative entropy over the set of unknown measures which satisfy inequality constraints. The second, seeks to maximize over the set of admissible control laws, the minimum value of relative entropy induced by the uncertain measures among those which satisfy inequality constraints. The second problem is equivalent to a minimax problem, while the first is an optimization problem with respect to a fix control law. Certain monotonicity properties of the optimal solution are discussed, while relations to the well-known Cramer's theorem of large deviations are introduced. In addition, connections to minimax games of partially observable stochastic systems and to risk-sensitive control problems are investigated.

Key Words: Uncertain Stochastic Systems, Large Deviations, Relative Entropy, Minimax Games, Cramer's Theorem.

I. Introduction

Since the publication of Zames [1] seminal paper, several approaches have been proposed to extend the techniques of robust controller design, with respect to unknown disturbances and unmodeled dynamics, to nonlinear deterministic and stochastic systems. Three pay-off functionals which received significant attention in achieving this goal are deterministic minimax games, risk-sensitivity pay-offs, and stochastic minimax games, because of their relations to attenuating disturbances to error signals, which is understood in the context of dissipation inequalities [2].

The deterministic formulation of minimax games is based on the assumption that the noises have finite energy, while the dissipation inequality is established through the value function of the dynamic games [3]. The stochastic analog of the deterministic minimax games is based on the assumption that the noises consist of color (or white noise) and finite energy disturbances. For fully observed dynamic games (both deterministic and stochastic), analysis and synthesis questions are discussed, in many places, for example, in [11], using the Isaacs equation. Unfortunately, this is not the case for stochastic partially observed nonlinear minimax games. In fact, very little work has been done in formulating and analyzing such classes of stochastic minimax problems from the control theoretic point of view. This is due to the difficulty in showing existence of optimal solutions, with respect to the class of control laws and disturbances which are described through output feedback control laws. If however, the concern about existence is discounted, connection between risk-sensitive pay-offs and minimax games are obtained by employing certain results from large deviations theory, or by recognizing similarity in the solutions, at least for problems whose solutions are known [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15].

This paper deviates from the above references, both in the problem formulation, and in the results sought, by introducing a class of uncertain stochastic control systems, in which the pay-off is described by the relative entropy between the nominal measure and the uncertain measure. The set of uncertain measures considered are described through energy inequality constraints expressed in terms of the uncertain measure. With respect to this formulation, a class of maximin problems is discussed. Specifically, the problems considered are the following.

Let (Σ, d) denote a complete separable metric space (Banach Space), and $(\Sigma, \mathcal{B}(\Sigma))$ the corresponding measurable space in which $\mathcal{B}(\Sigma)$ are identified as the Borel sets generated by open sets in Σ. Let $\mathcal{M}(\Sigma)$ denote the set of probability measures on $(\Sigma, \mathcal{B}(\Sigma)), \mathcal{U}_{ad}$ the set of admissible controls, and $B(\Sigma; \Re)$ the set of bounded real-valued measurable functions, $\ell^u : \Sigma \rightarrow \Re$ for a given $u \in \mathcal{U}_{ad}$. Here, $\mathcal{M}(\Sigma)$ denotes the set of all

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possible measures induced by the stochastic systems, while $\ell^u \in B(\Sigma;\Re)$ denotes the energy function or fidelity criterion associated with a given choice of the control law $u \in \mathcal{U}_{ad}$.

Given a nominal measure $\mu^u \in \mathcal{M}(\Sigma)$ induced by the nominal stochastic system, the problem is to find a control law $u^* \in \mathcal{U}_{ad}$ and a probability measure $\nu^{u,*} \in \mathcal{M}(\Sigma)$ which solve the following constrained optimization problem.

$$J(u^*, \nu^{u^*,*}) = \sup_{u \in \mathcal{U}_{ad}} \inf_{\nu^u \in \mathcal{A}(\mu^u)} H(\nu^u|\mu^u) \qquad \text{(I.1)}$$

Subject to fidelity

$$E_{\nu^u}(\ell^u) = \int_\Sigma \ell^u d\nu^u \leq \gamma \qquad \text{(I.2)}$$

or

$$E_{\nu^u}(\ell^u) = \int_\Sigma \ell^u d\nu^u \geq \gamma \qquad \text{(I.3)}$$

where $\mathcal{A}(\mu^u) = \{\nu^u \in \mathcal{M}(\Sigma); \nu^u << \mu^u\}$ and $\gamma \in \Re$ and $H(\nu^u|\mu^u)$ denotes the relative entropy between the measure ν^u and the measure μ^u, and $<<$ denotes absolute continuity of $\nu^u \in \mathcal{M}(\Sigma)$ with respect to $\mu^u \in \mathcal{M}(\Sigma)$. The fidelity constraints $E_{\nu^u}(\ell^u) \leq \gamma$, $E_{\nu^u}(\ell^u) \geq \gamma$ represent average energy constraints with respect to the unknown measure $\nu^u \in \mathcal{M}(\Sigma)$, such as integral quadratic constraints, tracking errors, etc., while γ is a parameter which is in some relation with $m \triangleq E_{\mu^u}(\ell^u)$, that is, either $m > \gamma$ or $m < \gamma$. In particular, the case (I.1), (I.2), with $m > \gamma$ will correspond to the optimistic scenario (emphasizing the best cases) in which the strategies are risk-seeking, while the case (I.1), (I.3), with $m < \gamma$ will correspond to the pessimistic scenario (emphasizing the worst cases) in which the strategies are risk-averse.

In Section II, the problem of finding the control law which maximizes the dual problem corresponding to the minimum of the relative entropy is considered, for the class of nonlinear partially observable stochastic control problems. The problem is reformulated using separated policies, and existence of an optimal control policy is shown from the class of the so called wide-sense controls. Finally, in Section II-D, closed form expressions are presented for the class of Gaussian nominal conditional measures, when the inequality constraints have a quadratic form.

Due to space limitation, the main results will be stated without including the derivations.

II. Partially Observed Wide Sense Uncertain Control Systems

In this section the results derived in Lemma (3.1) and Theorem (3.2) in [21] are employed in addressing the partially observable problems. The difficulty of dealing with partially observable systems is overcome by introducing separated strategies, which are important in stochastic optimal control problems [17]. Employing separated strategies implies that for the current partially observable system, the nominal model should be described through conditional distributions, rather than a priori distributions.

Existence of the minimax strategies is shown and several properties of the optimal solution are presented, which are exactly equivalent to those under Lemma (3.1) and Theorem (3.2) in [21].

A. Problem Formulation

Let $\{x(t)\}_{t\geq 0}$ denote the state process which is subject to control, $\{y(t)\}_{t\geq 0}$ the observation process, and $\{u(t)\}_{t\geq 0}$ the control process, all defined for a fixed and finite time $0 \leq t \leq T$.

For each $u \in \mathcal{U}_{ad}$ (in some admissible set which is defined shortly) the nominal state and observation process, giving rise to a nominal probability measure P^u are governed by the following Ito stochastic differential equations, in the probability space $\Big(\Sigma, \mathcal{B}(\Sigma), P^u\Big)$

$$\begin{cases} dx(t) = f(x(t), u(t))dt + \sigma(x(t))dw(t), \quad x(0) \\ dy(t) = h(x(t))dt + N dv(t), \quad y(0) = 0 \end{cases} \qquad \text{(II.4)}$$

Here $x(t) \in \Re^n, y(t) \in \Re^d, u(t) \in \mathcal{U} \subset \Re^k$, $\{w(t)\}_{t\geq 0}$ and $\{v(t)\}_{t\geq 0}$ are independent Brownian motions taking values in $\Re^m, \Re^d$, respectively, which are also independent of the Random Variable $x(0)$. Given the nominal measure $P^u \in \mathcal{M}(\Sigma)$, find a $u^* \in \mathcal{U}_{ad}$ and a probability measure $Q^{u^*,*}$ which solve the following constrained optimization problem.

$$J(u^*, Q^{u^*,*}) = \sup_{u \in \mathcal{U}_{ad}} \inf_{Q^u \in \mathcal{A}(P^u)} H(Q^u|P^u) \qquad \text{(II.5)}$$

Subject to fidelity

$$E_{Q^u}\Big\{ \int_0^T \lambda(x(t), u(t))dt + \kappa(x(T)) \Big\} \leq \gamma, \qquad \text{(II.6)}$$

or subject to fidelity

$$E_{Q^u}\Big\{ \int_0^T \lambda(x(t), u(t))dt + \kappa(x(T)) \Big\} \geq \gamma, \qquad \text{(II.7)}$$

where $\mathcal{A}(P^u) = \{Q^u \in \mathcal{M}(\Sigma); \quad Q^u << P^u\}$ and $\gamma \in \Re$. The following assumptions are introduced.

Assumptions 2.1: The nominal system satisfies the following assumptions.

1) The controls $\{u(t); t \in [0,T]\}$ take values in $\mathcal{U} \subset \Re^k$ which is compact and convex.
2) $f : \Re^n \times \mathcal{U} \rightarrow \Re^n, \sigma : \Re^n \rightarrow \mathcal{L}(\Re^n;\Re^m), f(x,u) = f_0(x) + f_1(x)u$, and f_0, f_1, σ are bounded and Lipschitz continuous.
3) $h : \Re^n \rightarrow \Re^d, h \in C_b^2(\Re^n)$.
4) $N \in \mathcal{L}(\Re^d;\Re^d)$ and $\exists \beta > 0$ such that $NN' \geq \beta I_d$.
5) The random variable $x(0)$ has distribution $\Pi_0(x)$.

6) $\lambda : \Re^n \times \mathcal{U} \rightarrow \Re, \kappa : \Re^n \rightarrow \Re$, ℓ, κ are continuous, bounded from below and from above, and $\lambda(\cdot, x)$ is convex in u for all $x \in \Re^n$.

Next, the problem is made precise by identifying the spaces on which the nominal system is defined and then introducing the precise definition of admissible controls. Consider the sample space

$$\Omega = \Omega^w \times \Omega^x \times \Omega^y \times \Omega^u$$

where

$$\Omega^w = C([0,T]; \Re^m), \quad \Omega^x = C([0,T]; \Re^n)$$
$$\Omega^y = C([0,T]; \Re^d), \quad \Omega^u = L_2([0,T]; \mathcal{U})$$

and $y(0) = 0$ is assumed throughout. Here, $\Omega^w, \Omega^x, \Omega^y$ are endowed with the usual sup-norm topology, while Ω^u is endowed with the weak topology (which is metrizable and separable). A typical element of Ω is $\omega(t) = \Big(w(t,\omega), x(t,\omega), y(t,\omega), u(t,\omega)\Big), 0 \leq t \leq T$. Let $\Omega^{w,x} = \Omega^w \times \Omega^x, \Omega^{y,u} = \Omega^y \times \Omega^u$. Then Ω is provided with a filtration $\{\mathcal{F}_t; t \in [0,T]\}$ which is defined as follows.

Let $\mathcal{F}_t^w = \sigma\{w(s); 0 \leq s \leq t\}$, $\mathcal{F}_t^x = \sigma\{x(s); 0 \leq s \leq t\}$, $\mathcal{F}_t^y = \sigma\{y(s); 0 \leq s \leq t\}$, which may be regarded as the Borel $\sigma-$algebras on $C([0,T]; \Re^q), q = m, n, d$, respectively, and $\mathcal{F}_t^u = \sigma\{\int_0^s u(\tau)d\tau; 0 \leq s \leq t\}$, which is the Borel $\sigma-$algebra on Ω^u.

Then

$$\mathcal{F}_t = \mathcal{F}_t^{w,x} \times \mathcal{F}_t^{u,y}, \;\; \mathcal{F}_t^{w,x} = \mathcal{F}_t^w \times \mathcal{F}_t^x, \; \mathcal{F}_t^{y,u} = \mathcal{F}_t^y \times \mathcal{F}_t^u$$

Fix a sample path for the observation and control process $y(\cdot, \omega), u(\cdot, \omega)$. Given the initial data $x(0) = x, y(0) = 0, w(0) = 0$, Assumptions 2.1 imply existence of a unique probability measure $\bar{P}_x^{y,u}$ on $\Big(\Omega^{w,x}, \mathcal{F}_T^{w,x}\Big)$ which coincides with the law of $\{w(t), x(t); t \in [0,T]\}$, such that $\{w(t); t \in [0,T]\}$ and $\{v(t); t \in [0,T]\}$ are independent Wiener processes and

$$\Big(\Omega^{w,x}, \mathcal{F}_T^{w,x}, \bar{P}_x^{y,u}\Big) :$$
$$\begin{cases} x(t) = x + \int_0^t f(x(s), u(s))ds + \int_0^t \sigma(x(s))dw(s) \\ y(t) = Nv(t) \end{cases}$$

In addition, Assumptions 2.1, imply that $\bar{P}_x^{y,u} \in \mathcal{M}(\Omega^{w,x})$ depends continuously on $\Big(u(\cdot,\omega), x\Big)$.

Definition 2.2: The set of admissible controls denoted by $\mathcal{U}_{ad}$ consists of measures π on $\Big(\Omega^{y,u}, \mathcal{F}_T^{y,u}\Big)$, that is, $\pi \in \mathcal{M}(\Omega^{y,u})$, such that $\{y(t); t \in [0,T]\}$ is $\mathcal{F}_T^{y,u} - \pi$-a.s. Brownian motion.

The projection $\Big(y(\cdot,\omega), u(\cdot,\omega)\Big) \mapsto y(\cdot,\omega)$ maps $\pi \in \mathcal{M}(\Omega^{y,u})$ onto a Wiener measure, and for all $t \in [0,T]$, $u(t)$ and $\sigma\{y(r) - y(t); 0 \leq t \leq r \leq T\}$ are independent under π.

Given the measure $\Pi_0 \in \mathcal{M}(\Re^n)$ of $x(0)$, by Baye's rule

$$\bar{P}^{y,u}(A) = \int_{\Re^n} \bar{P}_x^{y,u}(A) d\Pi_0(x), \qquad A \in \mathcal{F}_T^{w,x} \quad \text{(II.8)}$$

which is the unique joint distribution measure of $\{x(t), w(t); t \in [0,T]\}$ given $\Big(y(\cdot,\omega), u(\cdot,\omega\Big)$.

For each $\pi \in \mathcal{U}_{ad}$ define the joint distribution measure $\tilde{P}^\pi$ on $\Big(\Omega, \mathcal{F}_T\Big)$ by

$$\tilde{P}^\pi(dw, dx, du, dy) \triangleq \bar{P}^{y,u}(dw, dx) \times \pi(dy, du)$$

Notice that the projection $\Big(w(\cdot,\omega), x(\cdot,\omega), y(\cdot,\omega), u(\cdot,\omega)\Big) \mapsto \Big(y(\cdot,\omega), u(\cdot,\omega)\Big)$ under $\tilde{P}^\pi \in \mathcal{M}(\Omega)$ is $\pi \in \mathcal{M}(\Omega^{y,u})$.

Finally, define the nominal measure P^π as follows. Introduce the $\Big(\{\mathcal{F}_t; t \in [0,T]\}, \tilde{P}^\pi\Big)$-adapted exponential martingale process

$$\Lambda^u(t) = \exp\Big\{ \int_0^t h'(x(s))\Big(NN'\Big)^{-1} dy(s) - \frac{1}{2}\int_0^t h'(x(s))\Big(NN'\Big)^{-1} h(x(s))ds \Big\} \qquad \text{(II.9)}$$

Define the nominal measure through the Radon-Nikodym derivative

$$\frac{dP^\pi(w,x,y,u)}{d\tilde{P}^\pi(w,x,y,u)}|_{\mathcal{F}_T} \triangleq \Lambda^u(T) \qquad \text{(II.10)}$$

Then, Assumptions 2.1 imply that $P^\pi(\Omega) = E_{\tilde{P}^\pi}\Big\{\Lambda^u(t)\Big\} = 1, \forall t \in [0,T]$, and thus $P^\pi \in \mathcal{M}(\Omega)$. Moreover, by Girsanov's theorem, $v^\pi(t) \triangleq \int_0^t N^{-1}dy(s) - \int_0^t N^{-1}h(x(s))ds$ is a standard Wiener process under $P^\pi \in \mathcal{M}(\Omega)$, and the distribution of $\{w(t), x(t); t \in [0,T]\}$ is invariant under the measure change of (II.10). Thus, under the measure $P^\pi \in \mathcal{M}(\Omega)$, the processes $\{v^\pi(t); t \in [0,T]\}$ and $\{w(t); t \in [0,T]\}$ are independent Wiener processes.

Thus, for each $\pi \in \mathcal{U}_{ad}$, there exist a unique nominal measure $P^\pi \in \mathcal{M}(\Omega)$ on which the state $\{x(t); t \in [0,T]\}$ and observation process $\{y(t); t \in [0,T]\}$ satisfy (II.4).

The following is shown in [19].

Lemma 2.3: The set of admissible controls $\mathcal{U}_{ad}$ is compact under weak sequential convergence.

The precise problem statement should thus, be as follows.

Definition 2.4: Given the nominal measure $P^\pi \in \mathcal{M}(\Omega)$, find a $\pi^* \in \mathcal{U}_{ad}$ and a probability measure $Q^{\pi^*,*} \in \mathcal{M}(\Omega)$ which solve the following constrained optimization problem.

$$J(\pi^*, Q^{\pi^*,*}) = \sup_{\pi \in \mathcal{U}_{ad}} \inf_{Q^\pi \in \mathcal{A}(P^\pi)} H(Q^\pi|P^\pi) \quad \text{(II.11)}$$

Subject to fidelity

$$E_{Q^\pi}\Big\{ \int_0^T \lambda(x(t), u(t))dt + \kappa(x(T)) \Big\} \leq \gamma, \quad \text{(II.12)}$$

or subject to fidelity

$$E_{Q^\pi}\Big\{ \int_0^T \lambda(x(t), u(t))dt + \kappa(x(T)) \Big\} \geq \gamma, \quad \text{(II.13)}$$

where $\gamma \in \Re$ and $\mathcal{A}(P^\pi) = \{Q^\pi \in \mathcal{M}(\Sigma); \quad Q^\pi << P^\pi\}$.

B. Duality of Wide Sense Separated Uncertain Systems

In this section, separated strategies are introduced by describing the nominal systems using conditional distributions rather than a priori distributions. In the theory of partially observable systems such strategies are called separated strategies. They are important in answering questions about existence of optimal policies, and in reformulating partially observable problems as fully observable problems in which the role of a state variable is played by conditional distributions.

For each $\Big(u(\cdot,\omega), y(\cdot,\omega)\Big)$ introduce the multiplicative functional

$$\chi^u(t) \stackrel{\triangle}{=} \exp\Big(s\int_0^T \lambda(x(t), u(t))dt\Big) \quad \text{(II.14)}$$

Define the measure-valued process $M_t^{y,u}(\phi), \phi \in C_b(\Re^n)$ by

$$M_t^{y,u}(\phi) \stackrel{\triangle}{=} E_{\bar{P}^{y,u}}\Big\{\phi(x(t))\chi^u(t)\Lambda^u(T)\Big\} \quad \text{(II.15)}$$

Denote the kernel associated with $M_t^{y,u}(\phi)$ by $(y,u) \mapsto M^{y,u}(dx, t|y,u) \stackrel{\nabla}{=} M^{y,u}(dx,t) \in \mathcal{M}(\Re^n)$.

Then

$$M_t^{y,u}(\phi) = <\phi, M_t^{y,u}> = \int_{\Re^n} \phi(z)dM^{y,u}(z,t), \qquad \phi \in C_b(\Re^n) \quad \text{(II.16)}$$

It can be shown that the following separated maximin games are equivalent to the original problems.

Theorem 2.5: The two problems of Definition 2.4 are equivalent to the following separated maximin problems.

Define

$$\mu^\pi(T,x,y,u) = M^{y,u}(t,x)\pi(y,u) \in \mathcal{M}(\Omega^{y,u} \times \Re^n)$$

$$\nu^\pi(T,x,y,u) = N^{y,u}(t,x)\pi(y,u) \in \mathcal{M}(\Omega^{y,u} \times \Re^n)$$

Given a nominal conditional measure valued-process $M_t^{y,u} \in \mathcal{M}(\Re^n)$ find a $\pi^* \in \mathcal{U}_{ad}$ and a conditional measure-valued process $N_t^{y,u,*} \in \mathcal{M}(\Re^n)$ which solve the following constrained optimization problems.

1)

$$J(\pi^*, \nu^{\pi,*}) = \sup_{\pi \in \mathcal{U}_{ad}} \inf_{\nu^\pi \in \mathcal{A}} H(\nu^\pi|\mu^\pi) \quad \text{(II.17)}$$

where $\mu^\pi \in \mathcal{M}(\Omega^{y,u} \times \Re^n)$ and

$$\mathcal{A} = \Big\{\nu^\pi \in \mathcal{M}(\Omega^{y,u} \times \Re^n); \int_{\Omega^{y,u}\times\Re^n} \kappa(z)d\nu^\pi(T,x,y,u) \leq \gamma, \quad \nu^\pi << \mu^\pi\Big\}$$

Equivalently,

$$J(\pi^*, \nu^{\pi,*}) = \sup_{\pi \in \mathcal{U}_{ad}} \inf_{N^{y,u} \in \mathcal{B}} H\Big(N^{y,u} \times \pi | M^{y,u} \times \pi\Big) \quad \text{(II.18)}$$

where

$$\mathcal{B} = \Big\{N^{y,u} \in \mathcal{M}(\Re^n); \int_{\Omega^{y,u}} <\kappa, N_T^{y,u}> d\pi(y,u) \leq \gamma, \quad N_t^{y,u} << M_t^{y,u}\Big\}$$

for the following two cases.

Case 1. $m \stackrel{\triangle}{=} \int_{\Omega^{y,u}} <\kappa, N_T^{y,u}> d\pi(y,u) > \gamma$;

Case 2. $m \stackrel{\triangle}{=} \int_{\Omega^{y,u}} <\kappa, N_T^{y,u}> d\pi(y,u) < \gamma$;

2)

$$J(\pi^*, \nu^{\pi,*}) = \sup_{\pi \in \mathcal{U}_{ad}} \inf_{\nu^\pi \in \mathcal{C}} H(\nu^\pi|\mu^\pi) \quad \text{(II.19)}$$

where

$$\mathcal{C} = \Big\{\nu^\pi \in \mathcal{M}(\Omega^{y,u} \times \Re^n); \int_{\Omega^{y,u}\times\Re^n} \kappa(z)d\nu^\pi(T,x,y,u) \geq \gamma, \quad \nu^\pi << \mu^\pi\Big\}$$

and $\mu^\pi \in \mathcal{M}(\Omega^{y,u} \times \Re^n)$. Equivalently,

$$J(\pi^*, \nu^{\pi,*}) = \sup_{\pi \in \mathcal{U}_{ad}} \inf_{N^{y,u} \in \mathcal{D}} H\Big(N^{y,u} \times \pi | M^{y,u} \times \pi\Big) \quad \text{(II.20)}$$

where

$$\mathcal{D} = \Big\{N^{y,u} \in \mathcal{M}(\Re^n); \int_{\Omega^{y,u}} <\kappa, N_T^{y,u}> d\pi(y,u) \geq \gamma \quad N_t^{y,u} << M_t^{y,u}\Big\}$$

for the following two cases.

Case 1. $m \stackrel{\triangle}{=} \int_{\Omega^{y,u}} <\kappa, N_T^{y,u}> d\pi(y,u) < \gamma$;

Case 2. $m \stackrel{\triangle}{=} \int_{\Omega^{y,u}} <\kappa, N_T^{y,u}> d\pi(y,u) > \gamma$;

Similarly to the previous section, the above problems can be reformulated using the dual functional as follows. For every $s \in \Re$ define the Lagrangian

$$J^{s,\gamma}(\pi, \nu^\pi) \stackrel{\triangle}{=} H(\nu^\pi|\mu^\pi) - s\Big(\int_{\Omega^{y,u}} <\kappa, N_T^{y,u}> d\pi(y,u) - \gamma\Big) \quad \text{(II.21)}$$

and its associated dual functional

$$J^{s,\gamma}(\pi, \nu^{\pi,*}) = \inf_{\nu^\pi \in \mathcal{A}} J^{s,\gamma}(\pi, \nu^\pi) \quad \text{(II.22)}$$

where $\mathcal{A} = \{\nu^\pi \in \mathcal{M}(\Omega^{y,u} \times \Re^n); \nu^\pi << \mu^\pi\}$. In addition define the quantity

$$\varphi^{s^*}(\pi, \gamma) \stackrel{\triangle}{=} \sup_{s \in \Re} J^{s,\gamma}(\pi, \nu^{\pi,*}) \quad \text{(II.23)}$$

which may or may not exist.
Then, the statements of Lemma (3.1) and Theorem (3.2) in [21] hold, provided the appropriate changes are made, which are made precise in the next Corollary.

Corollary 2.6: Let $\pi \in \mathcal{U}_{ad}$ be fixed.
The statements of Lemma (3.1) and Theorem (3.2) in [21] hold with the following changes.

$$\begin{aligned} &\Sigma \mapsto \Omega^{y,u} \times \Re^n; \quad u \mapsto \pi \in \mathcal{M}(\Omega^{y,u}); \\ &\mu^u \mapsto \mu^\pi \in \mathcal{M}(\Omega^{y,u} \times \Re^n); \\ &\nu^u \mapsto \nu^\pi \in \mathcal{M}(\Omega^{y,u} \times \Re^n); \quad \ell^u \mapsto \kappa \end{aligned}$$

In particular,

$$\begin{aligned} J^{s,\gamma}(\pi, \nu^{\pi,*}) &= s\gamma - \log E_{\nu^\pi}\Big\{e^{s\kappa}\Big\} \\ &= s\gamma - \log \int_{\Omega^{y,u} \times \Re^n} e^{s\kappa(z)} dN_T^{y,u}(T,z) \times d\pi(y,u) \\ &\stackrel{\nabla}{=} s\gamma - \Psi_{\nu^\pi}(s) \end{aligned} \quad \text{(II.24)}$$

where the infimum in (II.22) is attained by $\nu^{\pi,*} \in \mathcal{M}(\Omega^{y,u} \times \Re^n)$ given by

$$d\nu^{\pi,*}(T,z,y,u) = \frac{e^{s\kappa(z)} d\mu^\pi(T,z,y,u)}{\int_{\Omega^{y,u} \times \Re^n} e^{s\kappa(z)} d\mu^\pi(T,z,y,u)} \text{(II.25)}$$

or equivalently,

$$\begin{aligned} &dN^{y,u,*}(T,z) \\ &= \frac{e^{s\kappa(z)} dM^{y,u}(T,z)}{\int_{\Omega^{y,u} \times \Re^n} e^{s\kappa(z)} dM^{y,u}(T,z,y,u) \times d\pi(y,u)} \end{aligned} \quad \text{(II.26)}$$

Proof. This follows from the above arguments a complete proof is found in [20]).

The existence of the optimal control policy $\pi^* \in \mathcal{U}_{ad}$ can be shown, using the property that the measure valued process $M_t^{y,u}$ is a continuous function of $\Big(y(\cdot,\omega), u(\cdot,\omega)\Big), \forall t \in [0,T]$ (see in [19] for details).

Theorem 2.7: Consider any s in the admissible intervals, $(0, s^*]$ or $[s^*, 0)$, and the resulting pay-off corresponding to the minimizing measure, namely,

$$\begin{aligned} &J^{s,\gamma}(\pi, \nu^{\pi,*}) = J^{s,\gamma}(\pi, M^{y,u}) \\ &= s\gamma - \log \int_{\Omega^{y,u}} \Big\{ \int_{\Re^n} e^{s\kappa(z)} dM_T^{y,u}(T,z) \Big\} \times d\pi(y,u) \end{aligned}$$

Then
1) $J^{s,\gamma}(\pi, \nu^{\pi,*})$ is upper-semi continuous on $\mathcal{U}_{ad}$.
2) There exists a $\pi^* \in \mathcal{U}_{ad}$ such that

$$J^{s,\gamma}(\pi^*, \nu^{\pi^*,*}) \geq J^{s,\gamma}(\pi, \nu^{\pi,*}), \qquad \forall \pi \in \mathcal{U}_{ad}$$

Proof. This is similar to in [19].

C. Evolution of the Density of the Minimum Measure

Introducing some additional regularities on σ, Π_0 would imply that the measure valued processes $N_t^{y,u}(\phi), M_t^{y,u}(\phi)$ have densities. The following are sufficient to show that such densities exists

7) $n = m, a(x) \stackrel{\triangle}{=} \sigma(x)\sigma'(x) \geq I_n\alpha, \alpha > 0, \forall x \in \Re^n$, $\frac{\partial}{\partial x_j} a_{i,j} \in L^\infty(\Re^n), \forall i, j$.

8) Π_0 has a density $p_0(x)$ and $p_0 \in L_2(\Re^n)$.

Under Assumptions 2.1 and 7), 8), it can be shown that $M_t^{y,u}$ has an unnormalized density and thus

$$\begin{aligned} &M_t^{y,u}(\phi) \\ &= \int_{\Re^n} \phi(z) e^{y'(t)h(z)} q^{y,u}(t,z) dz, \ \phi \in C_b(\Re^n) \end{aligned} \quad \text{(II.27)}$$

Moreover, $q^{y,u}(\cdot, z)$ is the solution of the following partial differential equation

$$\begin{aligned} &\frac{\partial}{\partial t} q^{y,u}(t,x) \\ &= A^*(y(t)) q^{y,u}(t,x) + e(x, y(t), u(t)) q^{y,u}(t,x) \end{aligned}$$

where $(t,x) \in (0,T] \times \Re^n$ and the initial condition is

$$q^{y,u}(0,x) = p_0(x), \qquad x \in \Re^n$$

and $A^*(y)$ is the adjoint operator of $A(y)$ (e.g., with respect to the duality product

$$< A(y)\phi, \psi > = < \phi, A^*(y)\psi >, \qquad \phi, \psi \in C_b^2(\Re^n)$$

$$\begin{aligned} A(y) = \quad & \frac{1}{2} \sum_{i,j=1}^n a_{i,j}(x,y) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x,y,u) \frac{\partial}{\partial x_i} \\ & - \sum_{i=1}^n (ay \cdot \nabla h)_i \frac{\partial}{\partial x_i} \end{aligned}$$

$$e(x,y,u) = \frac{1}{2}(ay \cdot \nabla h, y \cdot \nabla h) - y \cdot A(y)h - ||h||^2_{\Re^d}$$

in which $\cdot, (,)$ are the dot products in $\Re^d$, $\Re^n$, respectively.
Using the density of the measure valued process the maximization with respect to $\pi \in \mathcal{U}_{ad}$ can be expressed as

$$\begin{aligned} &J^{s,\gamma}(\pi, \nu^{\pi,*}) = s\gamma \\ &- \log \int_{\Omega^{y,u}} \int_{\Re^n} e^{s\kappa(z)} e^{y'(t)h(z)} q^{y,u}(t,z) dz d\pi(y,u) \\ &\stackrel{\nabla}{=} J^{s,\gamma}(\pi, q^{y,u,*}) \end{aligned} \quad \text{(II.28)}$$

Therefore, existence of the optimal control policy $\pi^* \in \mathcal{U}_{ad}$ can be shown, following the partial differential equations arguments (see [19] for details).

Theorem 2.8: Consider any s in the admissible intervals, $(0, s^*]$ or $[s^*, 0)$ specified by the solution of the minimizing measure.
Then
1) The functional $J^{s,\gamma}(\pi, q^{y,u,*})$ is upper semicontinuous on the set $\mathcal{U}_{ad}$
2) There exists an optimal control policy $\pi^* \in \mathcal{U}_{ad}$.
Proof. This is similar to the results found in [19].

D. Solvable Partially Observable Problems

For the purpose of illustrating the concepts presented earlier, the following linear dynamics and observations are considered, with quadratic constraints.

Assumptions 2.9: The coefficients of (II.4), the density of $x(0)$, and the constraint are given by

$$f(x,u) = FxBu, \quad \sigma(t,x) = G, \quad h(x) = Hx,$$
$$2\lambda(x,u) = x'Qx + u'Ru, \quad 2\kappa(x) = x'Mx$$
$$p_0(x) = \frac{\exp(-\frac{1}{2}|P_0^{-\frac{1}{2}}(x-\xi)|^2)}{(2\pi)^{\frac{n}{2}}|P_0|^{\frac{1}{2}}}, \quad P_0 = P_0' \geq 0,$$

each element having appropriate dimensions.
Under Assumptions 2.9, it can be shown that $M_t^{y,u}$ has a density $m^{y,u}(x,t)$ given by

$$\begin{aligned} dM^{y,u}(t,x) &= m^{y,u}(x,t)dx \\ &= \nu_{0,t}^u \times \frac{\exp\left(-\frac{1}{2}|P(t)^{-\frac{1}{2}}(x-r(t))|^2\right)}{(2\pi)^{\frac{n}{2}}|P(t)|^{\frac{1}{2}}} \\ &\times \exp\frac{s}{2}\left(\mathcal{C}_{0,t}^u\right) \times \mathcal{I}_{0,t}dx, \end{aligned} \tag{II.29}$$

where

$$\nu_{0,t}^u = \exp\Big(\int_0^t (Hr)'(NN')^{-1}dy - \frac{1}{2}\int_0^t ||N^{-1}Hr||_{\Re^d}^2 ds\Big)$$
$$\mathcal{C}_{0,t}^u \doteq \int_0^t (r'Qr + u'Ru)ds$$
$$\mathcal{I}_{0,t} \doteq \exp\frac{s}{2}\Big(\int_0^t Tr(PQ)ds\Big)$$

where $P : [0,T] \to \mathcal{L}(\Re^n;\Re^n), P = P' \geq 0$ and $r : [0,T]\times\Omega \to \Re^n$ are given by

$$\dot{P} = FP + PF' + sPQP + GG' - PH'\Big(NN'\Big)^{-1}HP,$$
$$P(0) = P_0,$$

and

$$dr = (F + sPQ)\,rdt + Budt + PH'\Big(NN'\Big)^{-1}\Big(dy - (Hr+h)dt\Big),$$
$$r(0) = \xi,$$

where $y(\cdot)$ is an $\{\mathcal{F}_t^{y,u}; t \in T\}$ −adapted Wiener process with correlation NN'. Suppose the class of strict-sense control laws is considered.Then the minimizing density associated with the measure valued process $N_t^{y,u}$ can be easily found.

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