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Achieving diagonal dominance by frequency interpolation

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***Abstract*— This paper proposes a new method for systematic construction of dynamic pre-compensators to achieve better diagonal dominance. Curve fitting is used to construct a dynamic pre-compensator element-by-element, meaning unlike most minimization algorithms, there is no restriction on the structure of each element. To demonstrate its effectiveness, the method is applied to the Rolls-Royce Spey gas-turbine engine, which is a highly interacting and multivariable system.**

I. Introduction

For a matrix $A = [a_{ij}] \in \mathbf{C}^{n\times n}$, the radius of its column Gershgorin disks $C_j(A)$ also referred to as the *deleted absolute column sum* and row Gershgorin disks $R_i(A)$ also referred to as the *deleted absolute row sum* are defined respectively as

$$C_j(A) = \sum_{\substack{i=1\\ i\neq j}}^{n} |\, a_{ij} \,| \qquad i = 1,\ldots,n \tag{1}$$

$$R_i(A) = \sum_{\substack{j=1\\ i\neq j}}^{n} |\, a_{ij} \,| \qquad j = 1,\ldots,n. \tag{2}$$

Gershgorin's theorem [1] states that the eigenvalues of A lie inside the region defined by these disks centered on the diagonal entries of A, as given by

$$G_C(A) \equiv \bigcup_{j=1}^{n} \{s \in \mathbb{C} : |\, s - a_{jj} \,| \leq C_j(A)\}, \tag{3}$$

$$G_R(A) \equiv \bigcup_{i=1}^{n} \{s \in \mathbb{C} : |\, s - a_{ii} \,| \leq R_i(A)\}. \tag{4}$$

where $G_R(A)$ describes the row Gershgorin region and $G_C(A)$ is the column Gershgorin region. Notice both $G_R(A)$ and $G_C(A)$ must include the eigenvalues, hence their intersection $G_\mu(A) = G_R(A) \cap G_C(A)$ is the subset that the eigenvalues can truly exist in. $G_\mu(A)$ is referred to as a *minimal Gershgorin set* [2] and other minimal sets may be obtained by considering the intersection of all the Gershgorin sets corresponding to *similar* operators to A (e.g. $\tilde{A} = S^{-1}AS$). In this work, however, we are concerned with the standard Gershgorin sets. Rosenbrock [3] used Gershgorin's theorem to propose the first frequency-based linear multivariable controller design technique; namely, Diagonal Dominance. This is a design technique that converts a multivariable design problem into several single-loop design problems which can then be solved using any number of single-loop design techniques available. Given a linear multivariable system, $G(s) = [g_{ij}(s)] \in \mathbf{C}^{n\times n}$

$$G(s) = \begin{bmatrix} g_{11}(s) & \ldots & g_{1n}(s) \\ \vdots & \ddots & \vdots \\ g_{n1}(s) & \ldots & g_{nn}(s) \end{bmatrix}, \tag{5}$$

and a pre-compensator, $K(s) = [k_{ij}(s)] \in \mathbf{C}^{n\times n}$

$$K(s) = \begin{bmatrix} k_{11}(s) & \ldots & k_{1n}(s) \\ \vdots & \ddots & \vdots \\ k_{n1}(s) & \ldots & k_{nn}(s) \end{bmatrix}, \tag{6}$$

the open loop transfer function is

$$Q(s) = G(s)K(s) = \begin{bmatrix} \sum_{m=1}^{n} k_{m1}(s)g_{1m}(s) & \ldots & \sum_{m=1}^{n} k_{mn}(s)g_{1m}(s) \\ \vdots & \ddots & \vdots \\ \sum_{m=1}^{n} k_{m1}(s)g_{nm}(s) & \ldots & \sum_{m=1}^{n} k_{mn}(s)g_{nm}(s) \end{bmatrix}, \tag{7}$$

where the system would be column diagonal dominant as defined by Rosenbrock if

$$|\, q_{ii}(s) \,| \geq \sum_{\substack{j=1\\ j\neq i}}^{n} |\, q_{ji}(s) \,| \quad \forall s \text{ on } D_R \tag{8}$$

Substitution for q in inequality (8) gives

$$\left| \sum_{m=1}^{n} k_{mi}(s) g_{im}(s) \right| \geq \alpha \sum_{\substack{j=1\\ j\neq i}}^{n} \left| \sum_{m=1}^{n} k_{mi}(s) g_{jm}(s) \right|. \tag{9}$$

where α is a scalar greater than one, which can be thought of as a weighting factor forcing the system to have higher dominance levels than would otherwise satisfy inequality (8). If such a $K(s)$ is found, $Q(s)$ may be approximated by $\tilde{Q}(s)$ where $\tilde{q}_{ij}(s) = q_{ij}(s)$ for $i = j$ and $\tilde{q}_{ij}(s) = 0$ otherwise. In the final stage, a diagonal controller $Kc(s) = [kc_{ii}(s)] \in \mathbf{C}^{n\times n}$ is found such

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that $\tilde{q}_{ii}(s)kc_{ii}(s) \simeq ref_i(s)/(1 - ref_i(s))$ where $ref_i(s)$ denotes the reference response function for loop i.

Although traditionally static pre-compensators tend to be used in dominance-based controller design, the subject of this work is dynamic pre-compensation, since numerous studies [4] have shown that in systems whose eigen-structure changes rapidly with frequency, constant pre-compensation can fail since its effect is only apparent in the vicinity of the frequency it was designed for. However, one can expect much better results with a dynamic pre-compensator since it can accommodate the rapid changes in the systems eigen-structure. Although there are established systematic methods for finding a constant pre-compensator, such as the Pseudo-diagonalisation method due to Hawkins [5] and the ALIGN algorithm due to Kouvartakis [6], there does not exist any established, or universally accepted, algorithm for design of dynamic pre-compensators.

One possible approach to finding $K(s)$ would be to first determine a fixed structure for the $k_{ij}(s)$, say a PI structure and then to feed inequality (9) into a minimization algorithm that will use k_p and k_i of each $k_{ij}(s)$ as minimization parameters to satisfy the inequality. Examples of such synthesis based techniques for finding the pre-compensator are the algorithms of Ford [7] and Edmunds [8]. This approach inevitably is not optimal by nature. When the structure of an element is decided upon without prior knowledge of how complex it actually is, one is forced to account for the maximum complexity it is predicted to take. Therefore thinking in terms of a whole column, this often translates into not enough dynamics for the complex elements, and unnecessary extra dynamics in other elements. In addition, as the size and order of $K(s)$ increases, a direct minimization of the parameters over some set of frequencies, becomes an increasingly impossible problem. In the next section, we present a simple design technique which overcomes this obstacle to a large extent (needless to say it will inevitably have its own weaknesses as well).

II. A SIMPLE DESIGN PROCEDURE

This technique is based on curve fitting for each element of the pre-compensator. The main steps involved in the design process are as follows.

Step 1

Evaluate $G(s)$ on the frequency vector ω_b, which denotes the frequency range over which it is desired to have diagonal dominance. A good guide for ω_b is the -3db frequency.

Step 2

For each $G(w_i)$, calculate its *(non-normalized) real inverse*. Given a matrix $A = [a_{ij}] \in \mathbf{C}^{n\times n}$ then $\tilde{A} = [\tilde{a}_{ij}] \in \mathbf{R}^{n\times n}$ is its non-normalized real inverse if it minimizes

$$\epsilon_n = \left|\left|\tilde{A}A - diag(\tilde{A}A)\right|\right|_2, \tag{10}$$

and $\tilde{A} = [\tilde{a}_{ij}] \in \mathbf{R}^{n\times n}$ would be the standard real inverse if it minimized

$$\epsilon_s = \left|\left|\tilde{A}A - I\right|\right|_2. \tag{11}$$

There are numerous algorithms to generate both the non-normalized real inverse and the standard real inverse. Pseudo-diagonalisation is an example of a non-normalized real inverse, whilst the ALIGN algorithm is an example of a standard real inverse algorithm. The reader should note that whilst there is freedom in the choice of the algorithm one uses, *the same* algorithm must be used for all w_i to avoid scaling problems arising. Upon completion of this step, one should be in possession of a set of (non-normalized) real inverses for $G(w_i)$ which will be denoted by $\tilde{G}(w_i),\ \forall w_i \in \omega_b$.

Step 3

Calculate the two matrices Δ and Γ as follows

$$\Delta = \begin{pmatrix} \sum \frac{\Delta sgn\{\tilde{g}_{11}(w_i)\}}{\Delta w_i} & \dots & \sum \frac{\Delta sgn\{\tilde{g}_{1n}(w_i)\}}{\Delta w_i} \\ \vdots & \ddots & \vdots \\ \sum \frac{\Delta sgn\{\tilde{g}_{n1}(w_i)\}}{\Delta w_i} & \dots & \sum \frac{\Delta sgn\{\tilde{g}_{nn}(w_i)\}}{\Delta w_i} \end{pmatrix}, \tag{12}$$

where

$$sgn\{x\} \quad = \quad \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases} \tag{13}$$

and

$$\Gamma = \begin{pmatrix} sgn\{\tilde{g}_{11}(w_1)\} & \dots & sgn\{\tilde{g}_{1n}(w_i)\} \\ \vdots & \ddots & \vdots \\ sgn\{\tilde{g}_{n1}(w_i)\} & \dots & sgn\{\tilde{g}_{nn}(w_i)\} \end{pmatrix}. \tag{14}$$

If $\Delta = 0$, then proceed to step 5, if not proceed to the next step.

Step 4

The condition that Δ equals zero, is a requirement on the sign behavior of the elements of $\tilde{G}(w_i)$. Although each element of these matrices are real, they are not necessarily positive. If $\Delta = 0$, it means that a given element does not change its sign for the bandwidth of frequencies considered, and remains either positive or negative for this range. If the sign condition is violated for a given element difficulty arises, since a fit cannot be made simultaneously to the two sign value regions. In this case the designer has to make a decision relating to the size and nature of the violation. If the violation is small in magnitude and brief in frequency, then one may proceed by reversing the sign for that band. Otherwise, if it is judged to be too large, then unsatisfactory results will be obtained. In this case, one remedy is to calculate the (non-normalized) real inverses using a different algorithm because the tendency for this phenomenon to occur differs for different algorithms. In particular, standard real inverse algorithms are more prone to this than non-normalized real inverse algorithms. Indeed, this is why for this work they are favored to standard real inverse algorithms. Another difference is that standard real inverse algorithms tend to produce improper responses, as opposed to non-normalized real inverse algorithm which tend to produce proper or even strictly proper responses. The reason for this immediately follows from their basic definition. Standard real inverse algorithms try to minimize $\left\|\tilde{G}(w_i)G(w_i) - I\right\|_2$. For proper or strictly proper systems, as w_i increases, $G(w_i)$ approaches a singular matrix and this forces the element of its inverse to approach infinity (thus improper behavior) and this causes the real inverse to exhibit improper behaviour.

Suppose now for a given system, element $\tilde{g}_{kl}(w_i)$ is violating the sign condition, but it is judged that the violation is acceptable. In order to proceed, the corresponding element in the matrix Γ should be modified to

$$\gamma_{kl} = sgn\left\{\frac{1}{m}\sum_{i}^{m}\tilde{g}_{kl}(w_i)\right\}. \tag{15}$$

Step 5

Finally, construct a matrix $\tilde{K}(s) = [k_{ij}(s)] \in \mathbf{C}^{n\times n}$ where each element $k_{ij}(s)$ is designed by magnitude curve fitting to $\mid \tilde{g}(\omega_b)_{ij} \mid$. In the curve fitting stage, the following issues need to be kept in mind

- **Improper responses**. Some elements of the real inverses might exhibit improper behavior with respect to frequency. In these cases, once a good fit is obtained in the chosen bandwidth, a pole at w_m may be added to the transfer function of those elements to roll off the responses and make them proper. The same objective may be met by incorporating integral action directly in the pre-compensator. This has the added advantage of allowing the controller to take non-PI(D) structures, such as pure leads or lags, without having to worry about steady state errors. This does not effect the dominance levels of the lower frequencies.
- **Fitting accuracy**. Judgement needs to be exercised on the merits of obtaining the closet fit possible, irrespective of the order of the fitted polynomials. In otherwords, since this is a design technique as opposed to a synthesis technique, the trade-off compromise between the complexity of each element versus the improvements obtained is a matter for the designer to decide. Design studies of applying this technique show that in most cases second or even first order transfer functions are sufficient for a very good response and one rarely needs to consider more complex dynamics.

Once the design of $\tilde{K}(S)$ is complete, the final pre-compensator is $K(S) = \Gamma \otimes \tilde{K}(S)$ where $\otimes$ denotes element by element multiplication. We present a brief example next, before further discussing the design technique.

III. An Example

The Rolls-Royce spey gas-turbine engine is used here as an example of a highly interacting and non-linear system. The model has been linearized at several operating points: the model linearized at 74% sea-level thrust is the subject of this example. The state space composite model (engine + actuators) contains 21 states. It has three inputs; Fuel Flow (FF), Inlet Guide Vanes (IGVs) and Nozzle Area (NA); and three outputs; Low pressure spool speed (%NL), High pressure spool speed (%NH) and Surge margin (SM). The control exercise is to control the three outputs with the three inputs in the order given as this is the optimal I/O pairing found through dynamic-RGA analysis. There are also reference performance functions available for each loop. These are

$$Sys_r(s) = \begin{pmatrix} \frac{1}{0.35s+1} & 0 & 0 \\ 0 & \frac{1}{1.2s+1} & 0 \\ 0 & 0 & \frac{1}{0.18s+1} \end{pmatrix}. \tag{16}$$

The direct Nyquist array plot of the composite system is shown in Figure 1. The interactions are clearly visible and can be confirmed in the open loop step response of the system, shown in Figure 5.a. Surge margin (SM) is heavily interacting and similarly is the high pressure spool speed (NH%). Following the steps outlined previously one can now design a dynamic pre-compensator to achieve a high amount of diagonal dominance.

Figure 2 shows the magnitude bode plots of the elements of $\mid \tilde{G}(w_i) \mid$ for $w \le \omega_b$. Initially, all three columns were calculated using the ALIGN algorithm. However, the second column was violating the sign condition and it was decided to switch to the Pseudo-diagonalization algorithm to generate the second column. The dashed lines show the response of the function fitted to the data. The matrices $\tilde{K}(s)$ and Γ where in this case found to be

$$\tilde{K}(s) = \begin{pmatrix} \frac{0.00072(s+9)}{s} & \frac{0.88(s+0.35)}{s(s+3)} & \frac{0.0046(s+70)}{s} \\ \frac{0.25(s+5)}{s} & \frac{6.37(s+19)}{s} & \frac{1.05(s+28)}{s} \\ \frac{0.00052(s+40)}{s} & \frac{0.56(s+0.5)}{s(s+8)} & \frac{0.02(s+36)}{s} \end{pmatrix}, \tag{17}$$

$$\Gamma = \begin{pmatrix} +1 & +1 & +1 \\ -1 & +1 & -1 \\ -1 & -1 & +1 \end{pmatrix}. \tag{18}$$

Resulting in the following pre-compensator, which also contains integral action

$$K(s) = \tag{19}$$
$$\begin{pmatrix} \frac{0.00072(s+9)}{s} & \frac{0.88(s+0.35)}{s(s+3)} & \frac{0.0046(s+70)}{s} \\ -\frac{0.25(s+5)}{s} & \frac{6.37(s+19)}{s} & -\frac{1.05(s+28)}{s} \\ -\frac{0.00052(s+40)}{s} & -\frac{0.56(s+0.5)}{s(s+8)} & \frac{0.02(s+36)}{s} \end{pmatrix}.$$

Figure 3 shows the open loop direct Nyquist array plot of the compensated engine with the dynamic pre-compensator of Equation 17. Note that the integrators are omitted for the open loop tests. Figure (5.b) shows the closed loop step response of the plant with this pre-compensator. The dashed lines show the reference outputs. As indicated from the direct Nyquist array, interactions are heavily suppressed and the system is highly diagonal dominant. However, at this stage, the responses are not necessarily the same as the required responses. Finally, to match the loop responses with those of the reference functions, three single-loop lead controllers were designed. This diagonal controller was found to be

$$C(s) = \begin{pmatrix} 27\frac{s+5}{s+45} & 0 & 0 \\ 0 & 19\frac{s+1}{s+23} & 0 \\ 0 & 0 & 10\frac{s+20}{s+40} \end{pmatrix}, \tag{20}$$

where the final multivariable controller is $K(s)C(s)$. Figure 4 shows the DNA of the closed loop system, and Figure 5.c shows the step responses together with the reference responses. Note that, using only second order pre-compensation and first order controllers, the interactions are reduced to less than 3% and all of the system outputs match the reference outputs extremely closely. These results show enormous improvements over dominance-based multivariable controllers designed for the Spey engine using the traditional constant pre-compensators [9]. This technique has been applied to other benchmark dominance multivariable problems including a 4 by 4 reheat furnace [3] and a 2 by 2 automotive gas-turbine [10], with extremely good results.

IV. Concluding remarks

In this paper, a technique has been proposed for the design of dynamic pre-compensators which had two major advantages when compared to previous proposals. First, the pre-compensator does not ‘self-impose’ any structural constraints as it does in other proposed techniques. By this we mean, if for example element (1,1) needs a third order fit, it does not imply that the remaining elements of the first column of the compensator would have a minimum degree of third order. In other words, the order of the elements of the pre-compensator are not interlinked (this, as mentioned previously is a common problem with minimization techniques). Secondly, since this is a design technique as opposed to a synthesis technique, the designer makes the choice on the point of compromise between complexity and performance. As an example, consider Figure 2 which shows the magnitude fit for the Spey example. It is obvious

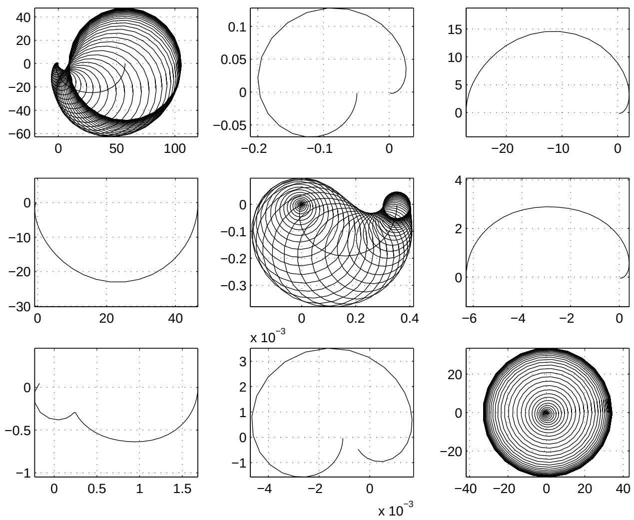

Fig. 1. Open loop DNA of Spey engine

that the fits are ‘crude’ by any standards. However, the gain in performance did not justify higher order fits. Indeed, even with the basic fits, the results are satisfactory.

This design technique has been applied very successfully to several other real-life examples including a two-input two-output automotive gas-turbine and a four-input four-output re-heat furnace, and also to some fictitious systems. In all the cases, first or second order pre-compensators were obtained which were able to achieve considerably high levels of diagonal-dominance. Further work on this topic would be to try to derive a formal proof for the technique, which the authors do acknowledge at this point as being only an ‘experimental’ technique.

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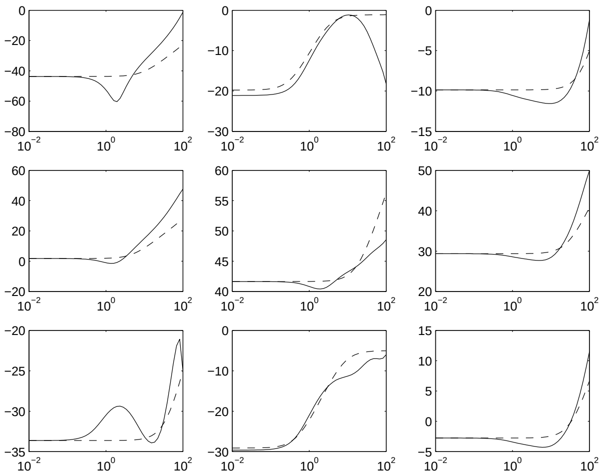

Fig. 2. Curve fitting to find $K(s)$ - log-log scale

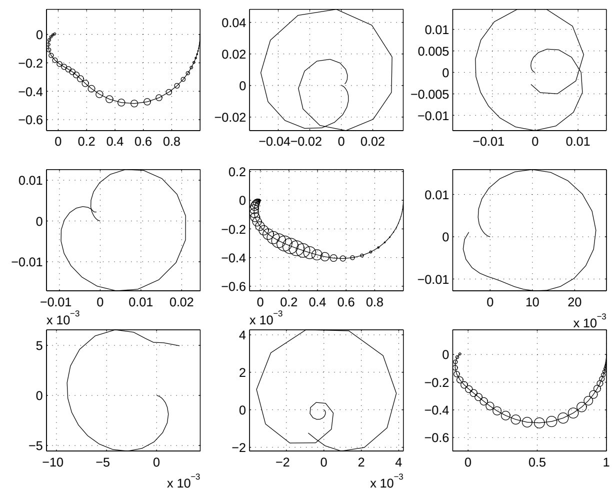

Fig. 4. DNA of closed loop compensated Spey + controllers

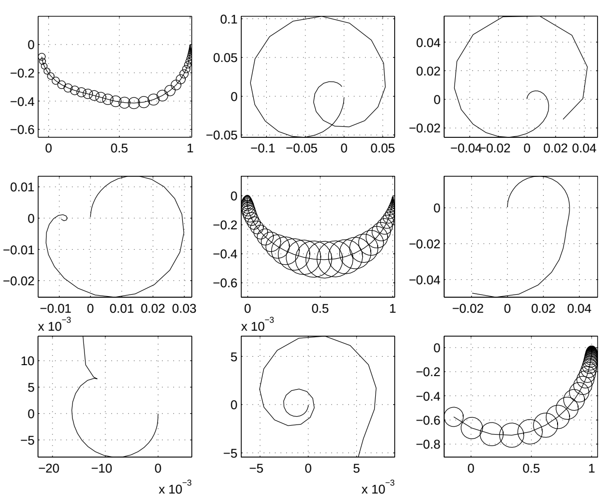

Fig. 3. DNA of open loop compensated Spey

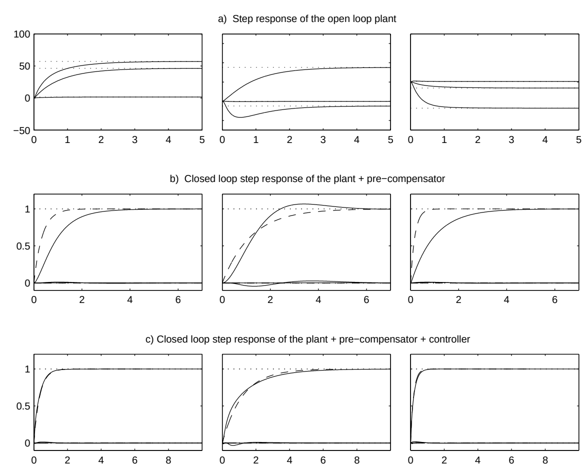

Fig. 5. Responses of the Spey