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H_∞ Output Feedback Control for a Class of Nonlinear Systems

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***Abstract*— This paper proposes a convex approach to the H-infinity output feedback problem for a class of uncertain nonlinear systems in which the system matrices are allowed to be rational functions of the state and uncertain parameters. We derive sufficient linear matrix inequality (LMI) conditions for designing full-order output feedback controllers that assure the regional stability of the nonlinear system for a given energy bound on the disturbance input in the sense that the state stays inside a given region, and also minimize an upper-bound on the $\mathcal{L}_2$-gain of the input-output operator for the class of admissible disturbance signals. Numerical examples are used to illustrate the proposed methodology.**

I. Introduction

In the last ten years, the nonlinear $\mathcal{H}_\infty$ control has received great attention from the control community in a wide diversity of approaches, see for instance [1], [2], [3]. In particular, the nonlinear $\mathcal{H}_\infty$ control via measurement feedback can be solved by means of two (coupled) Hamilton-Jacobi equalities, HJEs, (or inequalities, HJIs) [4], [5]. However, HJEs are hard to solve and usually obtained by Taylor series approximations which can be impracticable for systems with more than few states [6].

On the other hand, taking into account the theory of gain-scheduled $\mathcal{H}_\infty$ control of linear parameter varying (LPV) systems [7], i.e., systems described by

$$\dot{x}(t) = A(\delta(t))x(t) + B_u(\delta(t))u(t),\ y(t) = C(\delta(t))x(t),$$

where the control computation incorporates the vector of time-varying parameters $\delta(t)$, several authors have generalized the above class of systems to deal with nonlinear ones leading to the quasi-LPV approach [8], [9]. In other words, the parameter $\delta(t)$ is allowed to be state dependent resulting in the following class of nonlinear systems

$$\dot{x}(t) = A(\delta(x))x(t) + B_u(\delta(x))u(t),\ y(t) = C(\delta(x))x(t),$$

where $\delta(x)$ belongs to a know polyhedral set. However, the *quasi-LPV* approach may lead to serious conservativeness since the nonlinearities of the system are considered as free time-varying parameters which are actually determined by the system trajectories [10]. In addition, the quasi-LPV approach via measurement feedback can be applied only for systems in which the nonlinearities are available on-line to the controller.

Alternatively to the quasi-LPV approach, we consider in this paper a more complex class of nonlinear systems in which the nonlinearities are concentrated in an auxiliary vector leading to the following differential-algebraic representation

$$\begin{cases} \dot{x} &=& A_1x + A_2\xi + B_u u, \\ y &=& C_1x + C_2\xi, \\ 0 &=& \Omega_1(x,\delta)x + \Omega_2(x,\delta)\xi, \end{cases}$$

which is linear with respect to the state $x(t)$ and the algebraic vector $\xi(x(t),\delta(t))$. It turns out that the above class of system requires only that $x(t)$ and $\delta(t)$ belong to bounded sets differing from the quasi-LPV approach that assumes that all nonlinearities are bounded which can be conservative and very demanding on numerical computations. Then, we propose sufficient conditions to the $\mathcal{H}_\infty$ output feedback problem in terms of linear matrix inequalities (LMIs) [11] that guarantee the regional closed-loop stability and minimize an upper-bound on $\mathcal{L}_2$-gain of the input-to-output operator. Numerical examples are presented to illustrate the above class of systems and also to demonstrate the approach.

The rest of the paper is organized as follows. Section II states the problem of interest, and Section III introduces some basic results. The main results are presented in Sections IV (closed-loop analysis) and V (control synthesis), and Section VI ends the paper.

The notation used in this paper is standard. $\mathbb{R}^n$ denotes the set of n-dimensional real vectors, $\mathbb{R}^{n\times m}$ is the set of $n\times m$ real matrices, I_n is the $n\times n$ identity matrix, $0_{n\times m}$ is the $n\times m$ matrix of zeros and 0_n is the $n\times n$ matrix of zeros. For a real matrix S, S' denotes its transpose and $S>0$ means that S is symmetric and positive-definite. The time derivative of a function $r(t)$ will be denoted by $\dot{r}(t)$ and the argument (t) is often omitted. For two sets $\Pi_a \subset \mathbb{R}^n$ and $\Pi_b \subset \mathbb{R}^m$, the notation $\Pi_a\times\Pi_b$ represents that $(\Pi_a\times\Pi_b) \subset \mathbb{R}^{(n+m)}$ is a meta-set obtained by the cartesian product, and $\mathcal{V}(\Pi_a\times\Pi_b)$ is the set of all vertices of $\Pi_a\times\Pi_b$. Matrix and vector dimensions are omitted whenever they can be inferred from the context.

II. Problem Statement

Consider the following nonlinear system:

$$\begin{cases} \dot{x} = f(x,\delta,w,u),\ y = g(x,\delta,w), \\ z = h(x,\delta,w,u),\ x(0) = 0 \end{cases} \tag{1}$$

where $x \in \mathcal{X} \subset \mathbb{R}^{n_x}$ denotes the state, $\delta \in \mathbb{R}^{n_\delta}$ the uncertain (time-varying) parameters, $w \in \mathbb{R}^{n_w}$ the disturbance signal, $u \in \mathbb{R}^{n_u}$ the control input, $y \in \mathbb{R}^{n_y}$ the measured signal, $z \in \mathbb{R}^{n_z}$ the performance output, and $\mathcal{X}$ is a polyhedral region of state containing the origin ($x=0$) with known vertices. To guarantee the existence and

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uniqueness of solution and assure that the control problem is well-posed, we assume for system (1) that:

A1 The uncertain parameters represented by δ and its time-derivative $\dot{\delta}$ lie in a given polytope Δ, i.e., $(\delta, \dot{\delta}) \in \Delta$. The notation $\delta \in \Delta$ means $(\delta, 0) \in \Delta$.

A2 The right-hand side of the differential equation is continuous and bounded for all x, δ, w and u of interest.

A3 The system origin $x = 0$ is an equilibrium point for all admissible uncertainty, i.e., $f(0, \delta, 0, 0) = 0, \ \forall \ \delta \in \Delta$.

For nonlinear systems which are not globally asymptotically stable (GAS), the disturbance signal can lead the system to instability even for vanishing perturbations. To characterize the system stability over disturbance signals with zero initial conditions, we define the system regional stability as follows

Definition 1: Consider the nonlinear system in (1), satisfying **A1-A3**, and a given set of disturbance signal $\mathcal{W}$. The system is called regionally stable (with respect to $\mathcal{W}$ and $\mathcal{X}$) if $x(t) \in \mathcal{X}$ for all $t \geq 0$ and all $w \in \mathcal{W}$. The corresponding set $\mathcal{W}$ is called a *set of admissible disturbance inputs*. □

Hereafter, we describe the class of admissible disturbance inputs as follows:

$$\mathcal{W} \triangleq \left\{ w(t) \ : \ \int_0^\infty w'(t) w(t) \, dt \ \leq \ \mu^2 \right\} \qquad (2)$$

where $\mu > 0$ represents the level of admissible disturbance signal[1].

This paper is concerned in regionally stabilizing system (1) via (full-order) measurement feedback with guaranteed input-to-output properties. To this end, consider the following definition of $\mathcal{L}_2$-gain.

Definition 2: Consider system (1) with **A1-A3**, and a given set of admissible disturbance signal $\mathcal{W}$. The $\mathcal{L}_2$-gain of the input-to-output operator, denoted by G_{wz}, of system (1) is given by

$$\|G_{wz}\|_\infty = \sup_{\substack{0 \neq w \in \mathcal{W} \\ \forall (\delta, \dot{\delta}) \in \Delta}} \frac{\|z\|_2}{\|w\|_2} \qquad (3)$$

□

The stability of system (1) will be achieved by designing the following linear map

$$u = C_c x_c, \ \dot{x}_c = A_c x_c + B_c y, x_c(0) = 0, \qquad (4)$$

where $x_c \in \mathbb{R}^{n_x}$ is the (full-order) control state, and A_c, B_c, C_c are constant matrices with appropriate dimensions to be determined.

Using the above controller, we can describe the closed-loop system by means of the following augmented nonlinear map

$$z = h_a(x_a, \delta, w), \ \dot{x}_a = f_a(x_a, \delta, w), \ x_a(0) = 0 \qquad (5)$$

[1] I.e., μ controls the "size" of $\mathcal{W}$.

where the augmented vectors are given by

$$x_a = \begin{bmatrix} x \\ x_c \end{bmatrix}, f_a(\cdot) = \begin{bmatrix} f(x, \delta, w, C_c x_c) \\ A_c x_c + B_c g(x, \delta, w) \end{bmatrix},$$

and $h_a(\cdot) = h(x, \delta, w, C_c x_c)$.

We end this section introducing the following lemma based on the Lyapunov theory [12] that will provide the foundation of our control design.

Lemma 1: Consider system (5). Suppose there exists a function $V : \mathcal{X} \times \mathbb{R}^{n_x} \times \Delta \mapsto \mathbb{R}$ satisfying the following conditions for three positive scalars ϵ_1, ϵ_2 and γ:

$$\epsilon_1 x_a' x_a \leq V(x, x_c, \delta) \leq \epsilon_2 x_a' x_a \qquad (6)$$

$$\dot{V}(x, x_c, \delta) + \tfrac{1}{\gamma} z'(t) z(t) - \gamma w'(t) w(t) < 0, \ \forall \ t \geq 0 \qquad (7)$$

$$\mathcal{R} \triangleq \{(x, 0) : V(x, x_c, \delta) \leq 1\} \subset \mathcal{X} \qquad (8)$$

for all $x \in \mathcal{X}$, $(\delta, \dot{\delta}) \in \Delta$, and $x_c \in \mathbb{R}^{n_x}$. Then, $\mathcal{R}$ given above is an invariant set and the trajectory $x(t)$ driven by $w(t) \in \mathcal{W}$, where $\mu \geq 1/\gamma$, belongs to $\mathcal{X}$. Moreover, the $\mathcal{L}_2$-gain of system (5) satisfies

$$\|G_{wz}\|_\infty < \gamma, \ \forall \ (\delta, \dot{\delta}) \in \Delta, \ w \in \mathcal{W}. \qquad (9)$$

III. BASIC RESULTS

The idea considered in this paper for solving the nonlinear $\mathcal{H}_\infty$ output feedback control problem is to rewrite the stability conditions of Lemma 1 in terms of LMIs. To this end, we will represent system (1) by means of differential-algebraic equations. In order to decrease the well-known conservativeness of quadratic functions for nonlinear system analysis [10], we will employ the class of polynomial Lyapunov function originally presented in [13] in the stabilization conditions. We end this section discussing the problem of estimating the Domain of Attraction (DOA) for the class of admissible disturbance signals.

A. System Model

Certainly, the key idea in our approach is the modelling technique used to represent the nonlinear system. Roughly, we assume that system (1) can be described by a set of differential-algebraic equations that are linear with respect to x and also to auxiliary nonlinear vectors denoted by $\xi = \xi(x, \delta)$ and $\phi = \phi(x, \delta, w)$, i.e.

$$\begin{aligned} &\dot{x} = f(\cdot) = A_1 x + A_2 \xi + B_1 w + B_2 \phi + B_u u, \\ &y = g(\cdot) = C_1 x + C_2 \xi + D_1 w + D_2 \phi, \\ &z = h(\cdot) = E_1 x + E_2 \xi + F_1 w + F_2 \phi + F_u u, \\ &0 = \Omega_1(x, \delta) x + \Omega_2(x, \delta) \xi, \\ &0 = \Phi_1(x, \delta) w + \Phi_2(x, \delta) \phi, \end{aligned} \qquad (10)$$

where $\xi \in \mathbb{R}^{n_\xi}, \phi \in \mathbb{R}^{n_\phi}$ are nonlinear vector functions; $\Omega_1(x, \delta) \in \mathbb{R}^{m \times n_x}$, $\Omega_2(x, \delta) \in \mathbb{R}^{m \times n_\xi}$, $\Phi_1(x, \delta) \in \mathbb{R}^{q \times n_x}$ and $\Phi_2(x, \delta) \in \mathbb{R}^{q \times n_\phi}$ are affine matrix functions of x and δ; and $A_1, A_2, \ldots, F_2, F_u$ are constant matrices with appropriate dimensions. To simplify the notation, we may use the auxiliary matrices and vectors without explicitly mentioning their respective dependence on x, δ and w.

Basically, the above class of systems is essentially the same one proposed by El Ghaoui and co-authors in [14], [15] namely the linear fractional representation (LFR). The main difference between our technique and the LFR one is that we consider a differential-algebraic model and Ghaoui et al interpret the system as an interconnected system (i.e., a linear system with a feedback state-dependent connection between fictitious inputs and outputs). Similarly to the LFR technique, we can recover the original system by taking into account the following equalities

$$\xi = -(\Omega_2'\Omega_1)^{-1}\Omega_2'\Omega_1 x,\ \phi = -(\Phi_2'\Phi_1)^{-1}\Phi_2'\Phi_1 w. \quad (11)$$

In order to guarantee that representation (10) is well-posed, we further assume

A4 The matrices Ω_2 and Φ_2 are full column rank for all $x \in \mathcal{X}$ and $\delta \in \Delta$.

Considering Lemma 2.1 from [14] and (11), we can state the following proposition.

Proposition 1: For any rational matrix function $M : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$ with no singularities at origin there exist constant matrices M_1, M_2, and affine matrix functions $\Gamma_1(\sigma), \Gamma_2(\sigma)$ with appropriate dimensions such that $M(\sigma) = M_1 - M_2\left(\Gamma_2'(\sigma)\Gamma_1(\sigma)\right)^{-1}\Gamma_2'(\sigma)\Gamma_1(\sigma)$. □

In summary, from Proposition 1 we can infer that differential-algebraic representation (10) models the whole class of rational systems with no singularities at origin. To illustrate the proposed modelling technique, we give the following example.

Example 1: Consider the following scalar system

$$\dot{x} = \frac{x}{1+x^4} + (1+x^2)w + u,\ y = x + 0.1w,\ z = x \quad (12)$$

The above system can be rewritten as in (10) by $\Omega_1 = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \end{array}\right]$, and

$$\Omega_2 = \left[\begin{array}{cccc} -1 & 0 & 0 & 0 \\ x & -1 & 0 & 0 \\ 0 & x & -1 & 0 \\ 0 & 0 & x & -1 \end{array}\right], \xi = \left[\begin{array}{c} \frac{x}{1+x^4} \\ \frac{x^2}{1+x^4} \\ \frac{x^3}{1+x^4} \\ \frac{x^4}{1+x^4} \end{array}\right],$$
$$\phi = \left[\begin{array}{c} xw \\ x^2 w \end{array}\right], \Phi_1 = \left[\begin{array}{c} x \\ 0 \end{array}\right], \Phi_2 = \left[\begin{array}{cc} -1 & 0 \\ x & -1 \end{array}\right],$$

$A_1 = 0$, $A_2 = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \end{array}\right]$, $B_1 = 1$, $B_2 = \left[\begin{array}{cc} 0 & 1 \end{array}\right]$, $B_u = 1$, $C_1 = 1$, $D_1 = 0.1$, $E_1 = 1$, $C_2 = D_2 = E_2 = F_1 = F_2 = F_u = 0$.

Remark 1: The choice of matrices $A_1, A_2, \ldots, F_1, F_2$ in (10) is not unique, as a result the stability (and stabilization) results can be conservative. This problem is fully addressed in next section in which free multipliers are added to the problem decreasing the conservativeness. This problem is first discussed in [16], [9] for state-dependent algebraic Riccati equations and for the quasi-LPV approach. Trofino in [13] proposed a similar solution but from a different perspective leading to less conservative results. □

B. Lyapunov Function Candidate

Consider the following class of Lyapunov functions:

$$V(x,\delta,x_c) = \left[\begin{array}{c} x \\ x_c \end{array}\right]' \left[\begin{array}{cc} \mathcal{P}(x,\delta) & P_3 \\ P_3' & P_4 \end{array}\right] \left[\begin{array}{c} x \\ x_c \end{array}\right], \quad (13)$$

where

$$\mathcal{P}(x,\delta) = \left[\begin{array}{c} \Theta(x,\delta) \\ I_n \end{array}\right]' \left[\begin{array}{cc} P_2 & P_1' \\ P_1 & P_0 \end{array}\right] \left[\begin{array}{c} \Theta(x,\delta) \\ I_n \end{array}\right], \quad (14)$$

$P_0 = P_0' \in \mathbb{R}^{n_x\times n_x}, P_1 \in \mathbb{R}^{n_x\times n_\theta}, P_2 = P_2' \in \mathbb{R}^{n_\theta\times n_\theta}$, $P_3 \in \mathbb{R}^{n_x\times n_x}, P_4 = P_4' \in \mathbb{R}^{n_x\times n_x}$ are constant matrices to be determined; and $\Theta(x,\delta) \in \mathbb{R}^{n_\theta\times n_x}$ is a given affine matrix function of (x,δ).

For simplicity, define the following auxiliary notation:

$$\zeta = \left[\begin{array}{c} \zeta_1 \\ x \end{array}\right],\ \zeta_1 = \Theta(x,\delta)x,\ P = \left[\begin{array}{cc} P_2 & P_1' \\ P_1 & P_0 \end{array}\right]. \quad (15)$$

With above notation and taking into account the representation (10), the Lyapunov function $V(x,x_c,\delta)$ and its time-derivative $\dot{V}(x,x_c,\delta)$ can be written as follows:

$$V(\cdot) = \left[\begin{array}{c} \zeta_1 \\ x \\ x_c \end{array}\right]' \left[\begin{array}{ccc} P_2 & P_1' & 0 \\ P_1 & P_0 & P_3 \\ 0 & P_3' & P_4 \end{array}\right] \left[\begin{array}{c} \zeta_1 \\ x \\ x_c \end{array}\right] \quad (16)$$

$$\dot{V}(\cdot) = \left[\begin{array}{c} x \\ \xi \\ x_c \\ w \\ \phi \end{array}\right]' \left[\begin{array}{c} w_{ij} \end{array}\right]_{i,j=1,\ldots,5} \left[\begin{array}{c} x \\ \xi \\ x_c \\ w \\ \phi \end{array}\right] \quad (17)$$

where $w_{ij} = w_{ji}$, $w_{11} = A_1'\mathcal{P}(x,\delta) + \mathcal{P}(x,\delta)A_1 + \dot{\mathcal{P}}(x,\delta) + C_1'B_c'P_3' + P_3B_cC_1$, $w_{21} = A_2'\mathcal{P}(x,\delta) + C_2'B_c'P_3'$, $w_{31} = C_c'B_u'\mathcal{P}(x,\delta) + A_c'P_3' + P_3'A_1 + P_4B_cC_1$, $w_{32} = P_3'A_2 + P_4B_cC_2$, $w_{33} = A_c'P_4 + P_4A_c + C_c'B_u'P_3 + P_3'B_uC_c$, $w_{41} = B_1'\mathcal{P}(x,\delta) + D_1'B_c'P_3'$, $w_{43} = B_1'P_3 + D_1'B_c'P_4$, $w_{51} = B_2'\mathcal{P}(x,\delta) + D_2'B_c'P_3'$, $w_{53} = B_2'P_3 + D_2'B_c'P_4$ and the remaining ones are matrices of zeros with appropriate dimensions.

Note from above that we have to compute the term $x'\dot{\mathcal{P}}(x,\delta)x$. To this end, consider the following structure for $\Theta(x,\delta)$:

$$\Theta(x,\delta) = \sum_{i=1}^{n_x} R_i x_i + \sum_{i=1}^{n_\delta} S_i\delta_i + T \quad (18)$$

where R_i (for $i = 1,\ldots,n_x$), S_i (for $i = 1,\ldots,n_\delta$) and T are constant matrices with the same dimensions of $\Theta(x,\delta)$, and x_i, δ_i are respectively the entries of the vectors x and δ. Considering (18), we get:

$$x'\dot{\mathcal{P}}(x,\delta)x = 2\zeta'P\left[\begin{array}{c} \tilde{\Theta}(x)\dot{x} + \hat{\Theta}(\dot{\delta})x \\ 0 \end{array}\right] \quad (19)$$

where the matrices $\tilde{\Theta}(x)$ and $\hat{\Theta}(\dot{\delta})$ are given by

$$\tilde{\Theta}(x) = \sum_{i=1}^{n_x} R_i x r_i \quad \text{and} \quad \hat{\Theta}(\dot{\delta}) = \sum_{i=1}^{n_\delta} S_i\dot{\delta}_i \quad (20)$$

with r_i denoting the i-th row of the identity matrix I_n.

Considering the above notation, we get for $\dot{V}(x, x_c, \delta)$ the following:

$$\dot{V}(\cdot) = \begin{bmatrix} \zeta \\ \xi \\ x_c \\ w \\ \phi \end{bmatrix}' \begin{bmatrix} v_{ij} \end{bmatrix}_{i,j=1,\ldots,5} \begin{bmatrix} \zeta \\ \xi \\ x_c \\ w \\ \phi \end{bmatrix} \tag{21}$$

where $v_{ij} = v_{ji}$, $v_{11} = A_{a1}'P + PA_{a1} + U'(C_1'B_c'P_3' + P_3B_cC_1)U$, $v_{21} = A_{a2}'P + C_2'B_c'P_3'U$, $v_{31} = C_c'B_a'P + (A_c'P_3' + P_3'A_1 + P_4B_cC_1)U$, $v_{32} = w_{32}$, $v_{33} = w_{33}$, $v_{41} = B_{a1}'P + (D_1'B_c'P_3')U$, $v_{43} = w_{43}$, $v_{51} = B_{a2}'P + (D_2'B_c'P_3')U$, $v_{53} = w_{53}$, $U = \begin{bmatrix} 0_{n\times n_\theta} & I_n \end{bmatrix}$,

$$A_{a1} = \begin{bmatrix} 0_{n_\theta} & \Theta_x A_1 \\ 0 & A_1 \end{bmatrix}, A_{a2} = \begin{bmatrix} \Theta_x A_2 \\ A_2 \end{bmatrix},$$

$$B_a = \begin{bmatrix} \Theta_x B_u \\ B_u \end{bmatrix}, B_{a1} = \begin{bmatrix} \Theta_x B_1 \\ B_1 \end{bmatrix}, B_{a2} = \begin{bmatrix} \Theta_x B_2 \\ B_2 \end{bmatrix},$$

$\Theta_x = \Theta(x, \delta) + \tilde{\Theta}(x)$, and the remaining ones are matrices of zeros.

Remark 2: Matrix $\Theta(x, \delta)$ plays an important rule on the conservativeness of our approach, since it defines the complexity of the Lyapunov function. Generally, as large is its dimension as accurate will be the method at the cost of extra computations. □

C. Estimating the Domain of Stability

From the theory of nonlinear systems a level set of the Lyapunov function is normally used as an estimate of DOA, see e.g. [14]. The idea is as follows.

Without loss of generality, we assume that $\mathcal{X}$ is represented in terms of the following constraints:

$$\mathcal{X} = \{x : a_k'x \leq 1,\ i = 1, \ldots, n_e\} \tag{22}$$

where $a_k \in \mathbb{R}^{n_x}$ (for $i = 1, \ldots, n_e$) are given constant vectors associated with the n_e edges of $\mathcal{X}$. It turns out that $\mathcal{X}$ can be equivalently represented by its vertices.

Using the $\mathcal{S}$-procedure (see Sections 2.6 and 5.2 of [11]), the condition $\mathcal{R} \subset \mathcal{X}$ is satisfied if the following inequality is satisfied for all k:

$$2 - 2a_k'x + (V(x, x_c, \delta) - 1) \geq 0,\ k = 1, \ldots, n_e \tag{23}$$

Taking into account (13) and (15), the above is equivalent to the following:

$$\begin{bmatrix} 1 \\ \zeta_1 \\ x \\ x_c \end{bmatrix}' \begin{bmatrix} 1 & \begin{bmatrix} 0 & a_k' & 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ a_k \\ 0 \end{bmatrix} & \begin{bmatrix} P_2 & P_1' & 0 \\ P_1 & P_0 & P_3 \\ 0 & P_3' & P_4 \end{bmatrix} \end{bmatrix} \begin{bmatrix} 1 \\ \zeta_1 \\ x \\ x_c \end{bmatrix} \geq 0 \tag{24}$$

for $k = 1, \ldots, n_e$, where $\begin{bmatrix} \zeta_1' & x' \end{bmatrix}'$ satisfies

$$\begin{bmatrix} I_{n_\theta} & -\Theta(x, \delta) \end{bmatrix} \begin{bmatrix} \zeta_1 \\ x \end{bmatrix} = 0. \tag{25}$$

If the Lyapunov function candidate satisfies the conditions (6) and (7) of Lemma 1 jointly with (24) then $\mathcal{R}$ as defined in (8) is an invariant set for all $\delta \in \Delta$ and $w \in \mathcal{W}$.

IV. Output Feedback Analysis

This section presents an LMI technique for analyzing the stability of closed-loop system (5) providing the foundation for the output feedback control design which is given in next section.

From the expressions of $V(\cdot)$, $\dot{V}(\cdot)$ in (16) and (21), respectively, and also (24), we have associated with them the equality constraints (25) and the following

$$\begin{bmatrix} \Omega_1 & \Omega_2 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} = 0, \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix} \begin{bmatrix} w \\ \phi \end{bmatrix} = 0.$$

The basic idea for incorporating the above constraints in the conditions of Lemma 1 is to consider the Finsler's lemma [11] and thus free multipliers are added to the state-dependent LMIs. For instance, consider condition (24) and its constraint in (25). Applying the Finsler's lemma, we get the following:

$$\begin{bmatrix} 1 & \begin{bmatrix} 0 & a_k' & 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ a_k \\ 0 \end{bmatrix} & \begin{bmatrix} \Pi_{ij} \end{bmatrix} \end{bmatrix} \geq 0,\ \forall\ k \tag{26}$$

where $\Pi_{ij} = \Pi_{ji}$, $\Pi_{11} = P_2 + N_{1k} + N_{1k}'$, $\Pi_{21} = P_1 - \Theta' N_{1k}' + N_{2k}$, $\Pi_{22} = P_0 - N_{2k}\Theta - \Theta' N_{2k}'$, $\Pi_{31} = N_{3k}$, $\Pi_{32} = P_3' - N_{3k}\Theta$, $\Pi_{33} = P_4$, $\Theta = \Theta(x, \delta)$ and N_{1k}, N_{2k}, N_{3k} $(k = 1, \ldots, n_e)$ are free multipliers to be determined.

Also, the test of parameterized inequalities of the form

$$\sigma'\Gamma(\sigma)\sigma > 0,\ \forall\ \sigma \in \Sigma, \tag{27}$$

where $\sigma \in \mathbb{R}^{n_\sigma}$ is a general parameter belonging to a polytope Σ with known vertices and $\Gamma(\cdot) = \Gamma(\cdot)'$ is an affine matrix function of σ, can be very conservative if we only consider an LMI version of (27) as follows

$$\Gamma(\sigma) > 0,\ \forall\ \sigma \in \mathcal{V}(\Sigma). \tag{28}$$

Notice that the above implies $\rho'\Gamma(\sigma)\rho > 0$ for all $\sigma \in \Sigma$ and $\rho \in \mathbb{R}^{n_\sigma}$. Trofino in [13] introduced the following linear annihilator[2]

$$\mathcal{N}(\sigma) = \begin{bmatrix} \sigma_2 & -\sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_3 & -\sigma_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \sigma_{n_\sigma} & -\sigma_{(n_\sigma - 1)} \end{bmatrix} \tag{29}$$

where σ_i are i-th elements of σ and $\mathcal{N}(\sigma) \in \mathbb{R}^{(n_\sigma - 1)\times n_\sigma}$.

Applying the Finsler's lemma to (28) with the constraint $\mathcal{N}(\sigma)\sigma = 0$, leads to the following LMI condition

$$\Gamma(\sigma) + L\mathcal{N}(\sigma) + \mathcal{N}(\sigma)'L' > 0,\ \forall\ \sigma \in \mathcal{V}(\Sigma),$$

where L is a free multiplier.

For state-dependent LMIs, the idea is to incorporate the constraint $\mathcal{N}(x)x = 0$ into the stability conditions involving the matrices $A_1, A_2, \ldots, F_1, F_2$. As a consequence, we are

[2] A matrix $\mathcal{N}(x)$ is a linear annihilator of x if is a linear function of x and $\mathcal{N}(x)x = 0$.

in part reducing the problem of choosing these matrices since more degrees of freedom are added to the problem.

Now, we are ready to state sufficient LMI conditions that analyze the regional stability of system (5) providing an upper-bound on its $\mathcal{L}_2$-gain.

Theorem 1: Consider system (1) with **A1-A3**, and its representation as defined in (10) with **A4**. Let A_c, B_c, C_c be given constant matrices such that the unforced system (5) is regionally stable. Let $\Theta(x,\delta)$ be a given affine matrix function of (x,δ). Consider the notation previously defined and $\Omega_{a1} = \left[\begin{array}{cc} 0_{m\times n_\theta} & \Omega_1 \end{array}\right]$, $E_{1a} = \left[\begin{array}{cc} 0_{n_z\times n_\theta} & E_1 \end{array}\right]$,

$$\Omega_\zeta = \left[\begin{array}{cc} \mathcal{N}(x) & 0 \\ I_{n_\theta} & -\Theta(x,\delta) \end{array}\right]. \tag{30}$$

Suppose that P_0, P_1, P_2, P_3, P_4, L_{ij} and N_{ik} (for $i = 1,2,3,\ j = 1,2$ and $k = 1,\ldots,n_e$) are a solution to the following optimization problem where the LMIs are constructed at $\mathcal{V}(\mathcal{X}\times\Delta)$.

$$\min\ \gamma \quad \text{subject to (26) and } \left[\begin{array}{c} \Psi_{ij} \end{array}\right] < 0 \tag{31}$$

where $\Psi_{ij} = \Psi_{ji}$, $\Psi_{11} = v_{11} + L_{11}\Omega_\zeta + \Omega_\zeta' L_{11}' + L_{12}\Omega_{a1} + \Omega_{a1}' L_{12}'$, $\Psi_{21} = v_{21} + L_{21}\Omega_\zeta + L_{22}\Omega_{a1} + \Omega_2' L_{12}'$, $\Psi_{22} = L_{22}\Omega_2 + \Omega_2' L_{22}'$, $\Psi_{31} = v_{31} + L_{31}\Omega_\zeta + L_{32}\Omega_{a1}$, $\Psi_{32} = v_{32} + L_{32}\Omega_2$, $\Psi_{33} = v_{33}$, $\Psi_{41} = v_{41} + L_{41}\Omega_\zeta + L_{42}\Omega_{a1} + \Phi_1' L_{13}'$, $\Psi_{42} = L_{42}\Omega_2 + \Phi_1' L_{23}'$, $\Psi_{43} = v_{43} + \Phi_1' L_{33}'$, $\Psi_{44} = -\gamma I_{n_w} + L_{43}\Phi_1 + \Phi_1' L_{43}'$, $\Psi_{51} = v_{51} + L_{51}\Omega_\zeta + L_{52}\Omega_{a1} + \Phi_2' L_{13}'$, $\Psi_{52} = L_{52}\Omega_2 + \Phi_2' L_{23}'$, $\Psi_{53} = v_{53} + \Phi_2' L_{33}'$, $\Psi_{54} = L_{53}\Phi_1 + \Phi_2' L_{43}'$, $\Psi_{55} = L_{53}\Phi_2 + \Phi_2' L_{53}'$, $\Psi_{61} = E_{1a}$, $\Psi_{62} = E_2$, $\Psi_{63} = F_u C_c$, $\Psi_{64} = F_1$, $\Psi_{65} = F_2$ and $\Psi_{66} = -\gamma I_{n_z}$.

Then, system (5) is exponentially stable in $\mathcal{R}$ for all $\delta \in \Delta$, $w \in \mathcal{W}$ with $\mu \geq 1/\gamma$ and zero initial condition. Moreover, the $\mathcal{L}_2$-gain of system (5) satisfies (9).

V. Output Feedback Synthesis

A straight application of Theorem 1 for control design leads to bilinear matrix inequalities (BMIs) [17]. However, we can transform the conditions of Theorem 1 into convex ones using the parameterization proposed in [18] for filter design.

The idea is to pre- and post-multiply the matrix inequalities (26) and (31) respectively by

$$\begin{array}{l} Q_1 = \mathsf{diag}\{1, I_{n_\theta+n_x}, P_3 P_4^{-1}\}, \\ Q_2 = \mathsf{diag}\{I_{n_x+n_\theta+n_\xi}, P_3 P_4^{-1}, I_{n_w+n_\phi}\}. \end{array} \tag{32}$$

and then redefine some multipliers.

Note from Theorem 1 that P_4 is nonsingular and then the matrices Q_1 and Q_2 are well defined. In the following, we state the main result of this paper where is proposed sufficient LMI conditions for nonlinear $\mathcal{H}_\infty$ control design via measurement feedback.

Theorem 2: Consider system (1) with **A1-A3**, and its representation as defined in (10) with **A4**. Consider the notation used in Theorem 1. Let $\Theta(x,\delta)$ be a given affine matrix function of (x,δ). Let $M \in \mathbb{R}^{n_u\times n_x}$ be a given constant matrix. Suppose that P_0, P_1, P_2, L_{ij}, N_{jk}, and M_j (for $i = 1,\ldots,5; j = 1,2,3;$ and $k = 1,\ldots,n_e$) are a solution to the following optimization problem where the LMIs are constructed at $\mathcal{V}(\mathcal{X}\times\Delta)$.

$$\min\ \gamma \text{ subject to } \left[\begin{array}{c} \tilde{\Psi}_{ij} \end{array}\right] < 0, \tag{33}$$

$$\left[\begin{array}{cc} 1 & \left[\begin{array}{ccc} 0 & a_k' & 0 \end{array}\right] \\ \left[\begin{array}{c} 0 \\ a_k \\ 0 \end{array}\right] & \left[\begin{array}{c} \tilde{\Pi}_{ij} \end{array}\right] \end{array}\right] > 0,\ \forall\ k \tag{34}$$

where $\tilde{\Psi}_{ij} = \tilde{\Psi_j}i$, $\tilde{\Psi}_{11} = A_{a1}' P + P A_{a1} + U'(M_1 C_1 + C_1' M_1')U + L_{11}\Omega_\zeta + \Omega_\zeta' L_{11}' + L_{12}\Omega_{a1} + \Omega_{a1}' L_{12}'$, $\tilde{\Psi}_{21} = A_{a2}' P + C_2' M_1' U + L_{21}\Omega_\zeta + L_{22}\Omega_{a1} + \Omega_2' L_{12}'$, $\tilde{\Psi}_{22} = \Psi_{22}$, $\tilde{\Psi}_{31} = M' B_a' P + (M_3' + M_2 A_1 + M_1 C_1)U + L_{31}\Omega_\zeta + L_{32}\Omega_{a1}$, $\tilde{\Psi}_{32} = M_2 A_2 + M_1 C_2 + L_{32}\Omega_2$, $\tilde{\Psi}_{33} = M_3 + M_3' + M_2 B_u M + M' B_u' M_2$, $\tilde{\Psi}_{41} = B_{a1}' P + D_1' M_1' U + L_{41}\Omega_\zeta + L_{42}\Omega_{a1} + \Phi_1' L_{13}'$, $\tilde{\Psi}_{42} = \Psi_{42}$, $\Psi_{43} = B_1' M_2 + D_1' M_1' + \Phi_1' L_{33}'$, $\tilde{\Psi}_{44} = -\gamma I_{n_w} + L_{43}\Phi_1 + \Phi_1' L_{43}'$, $\tilde{\Psi}_{51} = B_{a2}' P + D_2' M_1' U + L_{51}\Omega_\zeta + L_{52}\Omega_{a1} + \Phi_2' L_{13}'$, $\tilde{\Psi}_{52} = \Psi_{52}$, $\tilde{\Psi}_{53} = B_2' M_2 + D_2' M_1' + \Phi_2' L_{33}'$, $\tilde{\Psi}_{54} = \Psi_{54}$, $\tilde{\Psi}_{55} = \Psi_{55}$, $\tilde{\Psi}_{61} = E_{1a}$, $\tilde{\Psi}_{62} = E_2$, $\tilde{\Psi}_{63} = F_u M$, $\tilde{\Psi}_{64} = F_1$, $\tilde{\Psi}_{65} = F_2$, $\tilde{\Psi}_{66} = -\gamma I_{n_z}$, $\tilde{\Pi}_{ij} = \tilde{\Pi}_{ji}$, $\tilde{\Pi}_{11} = P_2 + N_{1k} + N_{1k}'$, $\tilde{\Pi}_{21} = P_1 - \Theta' N_{1k}' + N_{2k}$, $\tilde{\Pi}_{22} = P_0 - N_{2k}\Theta - \Theta' N_{2k}'$, $\tilde{\Pi}_{31} = N_{3k}$, $\tilde{\Pi}_{32} = M_2 - N_{3k}\Theta$ and $\tilde{\Pi}_{33} = M_2$.

Then, system (5) with the control matrices

$$A_c = M_3 M_2^{-1},\ B_c = M_1,\ C_c = M M_2^{-1} \tag{35}$$

is exponentially stable in $\mathcal{R} = \{(x,0) : V(x,x_c,\delta) \leq 1\}$, where $P_3 = I_{n_x}$ and $P_4 = M_2^{-1}$, for all $\delta \in \Delta$ and $w(t) \in \mathcal{W}$ with $\mu \geq 1/\gamma$. Moreover, the $\mathcal{L}_2$-gain of system (5) satisfies (9). □

The conservatism of Theorem 2 depends on the choice of M which defines the control gain C_c. In other words, a bad guess of M in Theorem 2 may lead to a poor performance and even fail to provide a stabilizing controller. To overcome this problem, we propose in the sequel a simple procedure for choosing the matrix M.

To this end, assume that the triple (A_1, B_1, C_1) is stabilizable and detectable. In the sequel, an observer based algorithm is proposed to determine the matrix M in Theorem 2 taking into account the following sub-system

$$\left\{\begin{array}{rcl} \dot{\eta} & = & A_1\eta + B_1 w + B_u u, \\ y & = & C_1\eta + D_1 w,\ z = E_1 x + F_1 w + F_u u. \end{array}\right. \tag{36}$$

Algorithm 1: (Initial guess of M) Consider sub-system (36) and the following steps

Step 1 Solve the optimization problem

$$\min_{X,Y}\ \gamma : X > 0, \left[\begin{array}{c} \Xi_{ij} \end{array}\right] < 0,$$

where $\Xi_{ij} = \Xi_{ji}$, $\Xi_{11} = AX + X'A + B_u Y + Y' B_u'$, $\Xi_{21} = B_1'$, $\Xi_{22} = -\gamma I_{n_w}$, $\Xi_{31} = E_1 X + F_u Y$, $\Xi_{32} = F_1$ and $\Xi_{33} = -\gamma I_{n_z}$. In addition, define $K = Y X^{-1}$.

Step 2 Determine matrices $W = W'$ and Z such that $W > 0 : A_1' W + W A_1 + Z C_1 + C_1' Z' < 0$ and define $G = W^{-1} Z$.

Step 3 Determine a symmetric positive definite matrix $\hat{P}$ such that

$$\min_{\hat{P}} \ \gamma \ : \left[\begin{array}{ccc} \hat{A}'\hat{P} + \hat{P}\hat{A} & \hat{P}\hat{B} & \hat{C}' \\ \hat{B}'\hat{P} & -\gamma I_{n_w} & F_1' \\ \hat{C} & F_1 & -\gamma I_{n_z} \end{array} \right] < 0$$

where $\hat{C} = [\ E_1 \quad F_u K\ ]$, and

$$\hat{A} = \left[\begin{array}{cc} A_1 & B_u K \\ -GC_1 & A_1 + GC_1 + B_u K \end{array} \right], \hat{B} = \left[\begin{array}{c} B_1 \\ -GD_1 \end{array} \right]$$

Consider the following partition of the Lyapunov matrix

$$\hat{P} = \left[\begin{array}{cc} P_a & P_b \\ P_b' & P_c \end{array} \right], \ P_a, P_b, P_c \in \mathbb{R}^{n_x \times n_x}.$$

Finally, define $M = K P_c^{-1} P_b'$. □

To illustrate the proposed approach, we give the following example.

Example 2: Consider a controlled pendulum described by $\dot{x}_1 = x_2$, $\dot{x}_2 = 2\sin(x_1) - x_2 + u + 0.1w_1$, $y = x_1 + w_2$, $z = x_1$, where $x = [\ x_1 \quad x_2\ ]'$ is the state vector bounded by

$$x \in \mathcal{X} = \{x : |x_1| < \pi/2, \ |x_2| < 1\} \tag{37}$$

The above system is non-rational and then cannot be modelled in the differential-algebraic representation as defined in (10). To avoid this restriction, consider the following (2nd order Taylor) approximation of $\sin(x_1)$

$$\sin(x_1) = x_1 - \frac{x_1^3}{9} + \delta x_1 \tag{38}$$

where δ represents the mismatch between $\sin(\cdot)$ and its approximation. Taking into account (37) and above, δ and $\dot{\delta}$ are bounded by

$$(\delta, \dot{\delta}) \in \Delta = \left\{ (\delta, \dot{\delta}) : |\delta| \leq 0.05, \ |\dot{\delta}| < 3\pi/2 \right\} \tag{39}$$

From above, we get the following

$$\dot{x}_1 = x_2, \ \dot{x}_2 = 2x_1 - \frac{2}{9}x_1^3 - x_2 + 2\delta x_1 + u + 0.1w. \tag{40}$$

As the above nonlinearities are rational in x, we can represent it by the differential-algebraic representation defined in (10). In addition, applying Algorithm 1 we get $M = [\ 3.25454 \quad -0.41021\ ]$. From Theorem 2, we obtain an upper-bound $\gamma = 17.14$ for all $\delta \in \Delta$ and $w \in \mathcal{W}$ with $\mu \geq 1/\gamma$. Figure 1 shows the closed-loop trajectory of $x_1(t)$ for the disturbance signals $w_1(t) = 0.06$ for $0 \leq t \leq 1$ and $w_2(t) = 0.05 * sin(100 * t)$ for $0 \leq t \leq 10$.

VI. Concluding Remarks

This paper has proposed an alternative approach to the nonlinear $\mathcal{H}_\infty$ output feedback problem for a class of uncertain nonlinear systems in terms of linear matrix inequalities. The proposed LMI conditions assure the regional stability of the closed-loop system for bounded disturbance signals and zero initial conditions while providing an upper-bound on its (induced) $\mathcal{L}_2$-gain. As future topic of research, the authors are studying the nonlinear dynamic output feedback case.

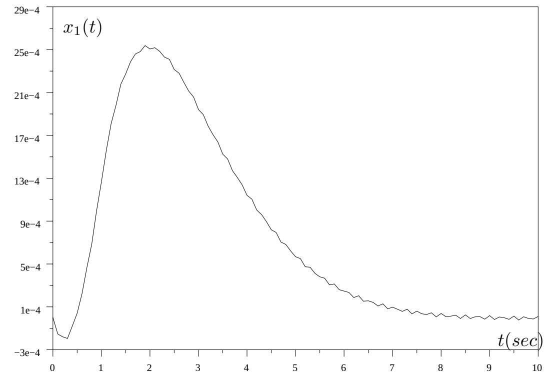

Fig. 1. Closed-loop $x_1(t)$ trajectory.

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