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Adaptive Extremum-Seeking Receding Horizon Control of Nonlinear Systems

V. Adetola, D.Dehaan and M. Guay
Department of Chemical Engineering, Queen's University
Kingston, Ontario, Canada K7L 3N6
{adetola,guaym}@chee.queensu.ca

***Abstract*— In this paper, we present a control algorithm that incorporates real time optimization and receding horizon control technique to solve an extremum seeking control problem for a class of nonlinear systems with parametric uncertainties. A Lyapunov-based technique is employed to develop a receding horizon controller that drives the system states to the desired unknown extremum points when it can be shown that a persistency of excitation condition is satisfied. A simulation example is provided to illustrate the effectiveness of the proposed method.**

I. INTRODUCTION

Optimization has become a key area in control theory due to the increasing need to optimize plant operation in order to reduce operating cost and meet product specifications. As better controllers are developed to adequately control a plant, the focus can be shifted to the solution of controller designs that guarantee optimal plant performance. If, for example, one can generate a reliable estimate of plant profitability, the purpose is shifted to the regulation of the process about conditions that provide maximum profitability. Such a task is usually tackled using a supervisory control technique. One such technique that has received considerable attention in the process industry is real-time optimization (RTO). One of the main challenges involved with the implementation of this technique is the difficulty associated with the integration of RTO with advanced process control (APC) applications. Despite the fact that these technologies are firmly established, their full integration remains troublesome in application.

In this paper, we propose a formal design technique that achieves the integrated task of RTO/APC supervisory systems. The control task is posed as an adaptive extremum-seeking control problem. Extremum seeking control has been proposed by a number of authors to handle optimization problems in nonlinear control systems ([3], [2], [1]). A number of applications of this method have been reported in the literature ([10], [9], [7]).

In this paper, we consider the approach proposed in [1] to solve a class of extremum-seeking problems which achieves the integrated task of an RTO/APC system where the APC consists of a nonlinear model predictive controller. Assuming that one can provide a suitable functional expression for the plant profit, an adaptive receding horizon controller design technique is developed that is able to steer the process states of the closed-loop system to an unknown optimum while ensuring transient performance and process regulation about the unknown optimum. Using the knowledge of a suitable input-to-state stable control Lyapunov function, the adaptive receding horizon control techniques can be shown to stabilize a nonlinear system with parametric uncertainties about an unknown optimum.

This paper is structured as follows. The problem description is given in section II and the design procedure is presented in section III. The proposed control and our main result is presented in section IV. Numerical simulation results are shown in section V and finally, conclusions are given in section VI.

II. PROBLEM DESCRIPTION

The system under study [1] is given by

$$\dot{x}_p = f(x) + F_p(x)\theta_p + F_q(x)\theta_q + G(x)u \quad (1)$$

$$\dot{x}_q = \phi(x) \quad (2)$$

where $x = [x_p^T \ \ x_q^T]^T \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the systems states and the control inputs respectively, $\theta_p \in \mathbb{R}^p$ and $\theta_q \in \mathbb{R}^q$ are vector of unknown constant parameters, $f(x) : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth vector function, $F_p(x) : \mathbb{R}^n \to \mathbb{R}^{m\times p}$, $F_q(x) : \mathbb{R}^n \to \mathbb{R}^{m\times q}$ and $G(x) : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ are smooth matrix-valued functions. The objective *(profit)* function is a smooth function of x_p and the unknown parameters θ_p. It is given by

$$y = p(x_p, \theta_p) \quad (3)$$

where $\theta_p \in \mathbb{R}^p$ is a parameter vector that satisfies $\theta_p \in \Omega_\theta$ where

$$\Omega_\theta = \left\{ \theta_p \in \mathbb{R}^p \left| \frac{\partial^2 p(x_p,\theta_p)}{\partial x_p \partial x_p} \leq c_0 I < 0, \ x_p \in \mathbb{R}^m \right. \right\} \quad (4)$$

This condition ensures that the performance function $p(x_p, \theta_p)$ is strictly convex. According to the theorem of Global Solutions of Convex Programs [6], there exists a unique constant vector x_p^* such that $\partial p(x_p,\theta_p)/\partial x_p \mid_{x_p = x_p^*} = 0$. This means that the objective function y achieves its maximum at x_p^*.

The objective function is assumed to depend on the state x_p and the parameter θ_p only. The remaining states $x_q \in \mathbb{R}^{n-m}$ represents the states that are not involved in the objective function. It is assumed that the x_q dynamic state belongs to a compact subset.

Assumption 1: G(x) is invertible $\forall x \in \mathbb{R}^n$

Assumption 2: The set Ω_θ is a convex subset of $\mathbb{R}^p$

III. Design Procedure

Let $\hat{\theta}_p$ and $\hat{\theta}_q$ denote the estimates of the true parameters θ_p and θ_q respectively. The predicted states, $\hat{x}_p$, are generated by

$$\dot{\hat{x}}_p = f(x) + F_p(x)\hat{\theta}_p + F_q(x)\hat{\theta}_q + G(x)u + Ke \qquad (5)$$

where $K = K^T > 0$ and $e = x_p - \hat{x}_p$ is the state prediction error. It follows from (1) and (5) that the dynamics of the prediction error e is

$$\dot{e} = F_p(x)\tilde{\theta}_p + F_q(x)\tilde{\theta}_q - Ke \qquad (6)$$

where $\tilde{\theta}_p = \theta_p - \hat{\theta}_p$ and $\tilde{\theta}_q = \theta_q - \hat{\theta}_q$ are the parameters estimation errors.

A. ISS Control Lyapunov Function

The concept of an ISS-CLF for input to state stabilizability was introduced in [4] for nonlinear systems of the form

$$\dot{x} = f(x) + P(x)d + g(x)u, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \qquad (7)$$

where $f(x)$, $P(x)$, and $g(x)$ are smooth, and $f(0) = 0$. The existence of an ISS-CLF guarantees that the nonlinear system (7) is input to state stable with respect to the disturbance input d.

Definition 1: [4]
A smooth positive definite radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called an ISS-control Lyapunov function (ISS-CLF) for (7) if there exist class $\mathscr{K}$ functions α_1, α_2 and a class $\mathscr{K}_\infty$ function ρ such that $\alpha_1 \le V \le \alpha_2$ and the following holds for all $x \neq 0$ and all $d \in \mathbb{R}^r$:

$$\begin{gathered} \|x\| \ge \rho(\|d\|) \\ \Downarrow \\ \inf_{u \in \mathbb{R}^m} \{\frac{\partial V}{\partial x}[f(x) + P(x)d + g(x)u]\} < 0 \end{gathered}$$

B. ISS CLF for the Extremum seeking problem

Define

$$y = p(x_p + d(t), \theta_p) \triangleq p(\bar{x}_p, \theta_p)$$

where $d(t) \in \mathrm{C}^1$ is a bounded dither signal vector that will be assigned later. Consider a Lyapunov function candidate

$$V = \frac{1}{2}\left\|\frac{\partial p(\bar{x}_p, \hat{\theta}_p)}{\partial \bar{x}_p}\right\|^2 \qquad (8)$$

Taking the time derivative of V, we have

$$\dot{V} = \frac{\partial p(\bar{x}_p, \hat{\theta}_p)}{\partial \bar{x}_p}\frac{d}{dt}\left(\frac{\partial p(\bar{x}_p, \hat{\theta}_p)}{\partial \bar{x}_p^T}\right)$$

The purpose of the ISS-controller design is to provide a benchmark controller for the RHC controller design. Since the controller is meant to represent the anticipated action of an ISS-controller at time t over the interval $[t, T]$, it must be assumed that the parameter estimates are known and constant. In the ISS-controller design, we let $\hat{\theta}_p = \bar{\theta}_p =$ constant estimate, which implies that $\dot{\hat{\theta}}_p = 0$.
Therefore,

$$\begin{aligned} \dot{V} &= \frac{\partial p}{\partial \bar{x}_p}\left[\frac{\partial^2 p}{\partial \bar{x}_p \partial \bar{x}_p^T}\left(f(x) + F_p(x)\hat{\theta}_p + F_q(x)\hat{\theta}_q + \right.\right. \\ &\quad \left.\left. G(x)u + \dot{d}(t)\right)\right] + \frac{\partial p}{\partial \bar{x}_p}\left[\frac{\partial^2 p}{\partial \bar{x}_p \partial \bar{x}_p^T}\left(F_p(x)\tilde{\theta}_p\right.\right. \\ &\quad \left.\left. + F_q(x)\tilde{\theta}_q\right)\right] \end{aligned}$$

Considering the control law

$$u = -G(x)^{-1}\Big[f(x) + F_p(x)\hat{\theta}_p + F_q(x)\hat{\theta}_q \qquad (9)$$

$$+\dot{d}(t) + k_1 z^T F_p F_p^T + k_2 z^T F_q F_q^T \qquad (10)$$

$$+\left(\frac{\partial^2 p}{\partial \bar{x}_p \partial \bar{x}_p^T}\right)^{-1}\left(\frac{\partial p}{\partial \bar{x}_p^T}\right)\Big], \qquad (11)$$

where $z = \frac{\partial p}{\partial \bar{x}_p}\left(\frac{\partial^2 p}{\partial \bar{x}_p \partial \bar{x}_p^T}\right)$ and k_1, k_2 are positive constants.
We obtain

$$\begin{aligned} \dot{V} &= z^T\Big[F_p(x)\tilde{\theta}_p + F_q(x)\tilde{\theta}_q - k_1 F_p F_p^T z - k_2 F_q F_q^2 z \\ &\quad - \left(\frac{\partial^2 p}{\partial \bar{x}_p \partial \bar{x}_p^T}\right)^{-1}\left(\frac{\partial p}{\partial \bar{x}_p^T}\right)\Big] \end{aligned}$$

Using the fact that

$$\begin{aligned} & z^T F_p(x)\tilde{\theta}_p - k_1 z^T F_p F_p^T z \\ &= -k_1\left\|\left(z^T F_p(x) - \frac{1}{2}\tilde{\theta}_p\right)\right\|^2 + \frac{1}{4k_1}\|\tilde{\theta}_p\|^2 \\ &\le \frac{1}{4k_1}\left\|\tilde{\theta}_p\right\|^2 \end{aligned}$$

and similarly,

$$z^T F_q(x)\tilde{\theta}_q - k_2 z^T F_q F_q^T z \le \frac{1}{4k_2}\left\|\tilde{\theta}_q\right\|^2$$

Then,

$$\dot{V} \le -\left\|\frac{\partial p}{\partial \bar{x}_p}\right\|^2 + \frac{1}{4}\left(\frac{\left\|\tilde{\theta}_p\right\|^2}{k_1} + \frac{\left\|\tilde{\theta}_q\right\|^2}{k_2}\right)$$

This implies that $\partial p/\partial \bar{x}_p$ is bounded whenever $\tilde{\theta}_p$ and $\tilde{\theta}_q$ are bounded. Since the Hessian matrix is assumed to be positive definite for all $\hat{\theta}$, it follows that the point $\bar{x}_p$ at which the gradient of p vanishes constitutes a minimum of y, as required. Thus the perturbed state variables entire a neighbourhood of the optimum of $p(\bar{x}_p, \hat{\theta}_p)$. Hence, eq.(12) demonstrates that eq.(8) is an ISS-Lyapunov function candidate for the extremum seeking problem under consideration. A suitable ISS-controller is given by eq.(11).

IV. EXTREMUM SEEKING RHC FORMULATION

The goal of the ESRHC scheme is to minimize a given cost while ensuring that the performance function y achieves its maximum value. The formulation is such that a finite horizon optimal control problem is solved subject to the system dynamics and terminal state inequality constraints at any time t with the measured plant states $x(t)$ as initial condition.

The proposed ESRHC scheme is given by:

$$\min_{u} J = \int_t^{t+T_p} \left(\left\| \frac{\partial p}{\partial \bar{x}_p} \right\|_R^2 + \|u(\tau)\|_Q \right) d\tau \quad (12)$$

subject to

$$\dot{x}_q = \phi(x) \quad (13)$$

$$\dot{x}_p = f(x) + F_p(x)\bar{\theta}_p + F_q(x)\bar{\theta}_q + G(x)u \quad (14)$$

$$\bar{\theta}_p = \hat{\theta}_p(t), \quad \bar{\theta}_q = \hat{\theta}_q(t) \quad (15)$$

$$V(t+T_p) \leq V^{iss}(t+T_p) \quad (16)$$

where R and Q are positive definite weighting matrices, T_p is the length of the prediction horizon, the function V is the value of the CLF resulting from the application of the ESRHC and V^{iss} is the value of the CLF that results from the application of the iss controller. Constraint (16) guarantees that the states under the ESRHC are brought within the level set of the iss-controller at the end of the prediction horizon, thereby ensuring that the states under the ESRHC remain bounded. By (15), the unknown parameters $\bar{\theta}_p$ and $\bar{\theta}_q$ in (12) and (14) are replaced with the estimated parameters values. The optimizer computes the required control moves over the control horizon $[t, t+T_c]$. The input $u(t)$ on the plant time t and an estimate of the unknown parameter $\hat{\theta}(t)$ is obtained via a parameter update law. The prediction and the control horizons are shifted forward and a new optimization problem is solved at next time step $t+\varepsilon$ with the new $\bar{\theta} = \hat{\theta}(t+\varepsilon)$. The control $u(t+\varepsilon)$ is applied at time $t+\varepsilon$ and the process is repeated. In general, it is assumed that the time step length ε can be chosen to be arbitrarily small.

A. Stability Analysis for the Extremum seeking RHC Scheme

The stability and performance of the proposed scheme is demonstrated in the following.

Let $V(x)$ be a global CLF for the system. Consider the function

$$W(\bar{x}(t)) = \frac{1}{T}\int_t^{t+T_p} V(\bar{x}(\tau)) d\tau \quad (17)$$

where $\bar{x}(.)$ is the state trajectory resulting from the extremum seeking RHC control and $\bar{x}^{iss}(.)$ is the trajectory resulting from the implementation of the iss-controller eq.(11) starting at state $\bar{x}(t)$. This function is positive definite and it is radially unbounded if V is radially unbounded and positive definite. Differentiating W with respect to t, we get

$$\dot{W}(\bar{x}(t)) = \frac{1}{T}(V(\bar{x}(t+T_p)) - V(\bar{x}(t)))$$

By the formulation of the RHC controller, it follows from eq.(16) that

$$\dot{W}(\bar{x}(t)) \leq \frac{1}{T}(V(\bar{x}^{iss}(t+T_p)) - V(\bar{x}(t)))$$

or

$$\dot{W}(x(t)) \leq \frac{1}{T}\int_t^{t+T} \dot{V}^{iss}(\tau) d\tau$$

where V^{iss} indicates that the rate of change of V is taken along the predicted closed-loop trajectories starting at $\bar{x}(t)$ subject to the ISS-controller and parameter estimate, θ. From eq.(9),

$$\begin{aligned}\dot{W}(\bar{x}(t)) \leq& \frac{1}{T}\int_t^{t+T} z(\tau)^T F(\bar{x}^{iss}(\tau)) d\tau \tilde{\theta}(t) \\ &- \frac{1}{T}\int_t^{t+T} \Bigg[k_1 z(\tau) F_p(\bar{x}^{iss}(\tau)) F_p(\bar{x}^{iss}(\tau))^T z(\tau) \\ &+ k_2 z(\tau) F_q(\bar{x}^{iss}(\tau)) F_q(\bar{x}^{iss}(\tau))^T z(\tau) \\ &+ \left\| \frac{\partial p}{\partial \bar{x}_p^T} \right\|^2 \Bigg] d\tau \end{aligned} \quad (18)$$

Next an estimation algorithm is proposed. The estimation routine consists of a state prediction and a parameter update law. The predicted states, x_p, using $\hat{\theta}$, are generated by the dynamical system

$$\dot{\hat{x}} = f(x) + F(x)\hat{\theta} + G(x)u + K(x - x_p), \quad (19)$$

Denoting the prediction error by $e = x - \hat{x}$, and the parameter estimation error by $\tilde{\theta} = \theta - \hat{\theta}$, the prediction error dynamic are described by

$$\dot{e} = F(x)\tilde{\theta} - Ke. \quad (20)$$

Consider a Lyapunov function

$$V_1 = W(x) + \frac{1}{2}e^T e + \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \quad (21)$$

Taking its derivative along the solutions of (20), we have

$$\begin{aligned}\dot{V}_1(\bar{x}(t)) \leq& (\Psi - \dot{\hat{\theta}}^T \Gamma^{-1})\tilde{\theta}(t) \\ &- \frac{1}{T}\int_t^{t+T} \Bigg[k_1 z(\tau)^T F_p(\bar{x}^{iss}(\tau)) F_p^T(\bar{x}^{iss}(\tau)) z(\tau) \\ &+ k_2 z(\tau)^T F_p(\bar{x}^{iss}(\tau)) F_p^T(\bar{x}^{iss}(\tau)) z(\tau) \\ &+ \left\| \frac{\partial p}{\partial \bar{x}_p^T} \right\|^2 \Bigg] d\tau - e^T K e \end{aligned} \quad (22)$$

where

$$\Psi = \Gamma F(x)^T e + \Gamma \left(\frac{1}{T}\int_t^{t+T} z(\tau)^T F(\bar{x}^{iss}(\tau)) d\tau \right)^T$$

and $\Gamma = \Gamma^T > 0$. In order to produce bounded parameter estimates, and also to account for the fact that parameters often have a physical meaning, it would be desirable to ensure that the parameter estimates remain in some given set. For this reason, a parameter projection law [5] is used. It is given by

$$\dot{\hat{\theta}} = Proj\left\{\hat{\theta}, \Psi\right\} \tag{23}$$
$$= \begin{cases} \Psi, & \text{if } \left\|\hat{\theta}\right\| < w_m \\ & \text{or } \left(\left\|\hat{\theta}\right\| = w_m \text{ and } \nabla\mathscr{P}(\hat{\theta})\Psi \leq 0\right) \\ \Psi - \Psi\frac{\gamma\nabla\mathscr{P}(\hat{\theta})\nabla\mathscr{P}(\hat{\theta})^T}{\left\|\nabla\mathscr{P}(\hat{\theta})\right\|_\gamma^2}, & \text{otherwise} \end{cases}$$

where $\mathscr{P}(\hat{\theta}) = \hat{\theta}^T\hat{\theta} - w_m \leq 0$, $\hat{\theta}$ is the vector of parameter estimates, γ is a positive definite symmetric matrix and w_m is chosen such that $\left\|\hat{\theta}\right\| \leq w_m$.

The properties of the projection operator, as defined in [5], ensures that the parameters are bounded and that

$$\dot{V}_1 \leq -\frac{1}{T}\int_t^{t+T}\left[\left\|\frac{\partial p}{\partial x_p^T}\right\|^2\right]d\tau - \frac{1}{2}e^TKe \tag{24}$$

Since the last inequality is negative semi-definite with respect to e, $\frac{\partial p}{\partial x_p^T}$ and $\tilde{\theta}$, we can conclude by the LaSalle-Yoshizawa's theorem [5] that e, $\tilde{\theta}$ and $\frac{\partial p}{\partial x_p^T}$ are bounded. Furthermore, both $\frac{\partial p}{\partial x_p^T}$ and e converge to the origin. Consequently, it is guaranteed from the adaptive law (23) that

$$\lim_{t\to\infty}\dot{\hat{\theta}}(t) = 0 \tag{25}$$

Since e converges to zero, we know that $\int_0^\infty \dot{e}(\sigma)d\sigma = e(\infty) - e(0) = -e(0)$ exists and is finite. Also, from (20), we know that $\dot{e}$ is a function of bounded signals $x, \tilde{\theta}$ and e which means that $\ddot{e}$ is bounded. Hence, $\dot{e}$, is uniformly continuous. By Barbalat's lemma [5], we conclude that $\dot{e} \to 0$ as $t \to \infty$. This implies that

$$\lim_{t\to\infty} F(x)\tilde{\theta} = 0 \tag{26}$$

or

$$\lim_{t\to\infty}\tilde{\theta}^TF^T(x)F(x)\tilde{\theta} = 0 \tag{27}$$

where $F(x) = [F_p(x) \quad F_q(x)]$ and $\theta = [\theta_p^T \quad \theta_q^T]^T$.
If $F^T(x)F(x)$ is positive definite, then the parameter error $\tilde{\theta}$ converges to zero asymptotically. However, this condition is not always true because $F(x)^TF(x)$ can be singular at any given time. We consider the integral of $F(x)^TF(x)$ for $t \to \infty$. It then follows that over any bounded interval of length $0 < T_0 < \infty$, we have

$$\lim_{t\to\infty}\frac{1}{T_0}\int_t^{t+T_0}\left(\tilde{\theta}(\tau)^TF(\tau)^TF(\tau)\tilde{\theta}(\tau)\right)d\tau = 0 \tag{28}$$

In order to prove the convergence of $\tilde{\theta}$ to zero, we will require a condition on the richness of the dither signal $d(t)$.

Definition 2 (Persistence of Excitation): The closed-loop dynamics exhibit *persistency of excitation* (PE) if there exists constants $T_0 > 0$, $c_{PE} > 0$ and a sequence $\{t_i\}$ with $t_i \to \infty$ as $i \to \infty$ such that the following is true

$$\frac{1}{T_0}\int_{t_i}^{t_i+T_0}F(\tau)^TF(\tau)d\tau \geq c_{PE}I \tag{29}$$

Lemma 1: Consider the nonlinear system, eq.(1), with receding horizon controller eqs.(12)-(16), the adaptive laws (23) and state estimation dynamics eq.(19). If the dither signal $d(t)$ is chosen such that the PE condition (29) is satisfied, then the parameter estimation error $\tilde{\theta}$ converges to zero asymptotically.

Proof: Let $z = \frac{\partial p}{\partial x_p}$. From eq.(24), it follows from the Lasalle's invariance principle that $(z, e) \to 0$ as $t \to \infty$. If Lemma 1 is true, for every compact neighbourhood of $(z, e, \tilde{\theta}) = 0$, there must exist a finite time from which the neighbourhood of the origin of the closed-loop system is positively invariant. Since (24) ensures V_1 is non-increasing, we know that level curves of V_1 are rendered positively invariant. To prove the Lemma, it is therefore sufficient to prove that $(z, e, \tilde{\theta})$ will enter every level curve of V_1.

The proof will proceed by contradiction, with the contradictory assumption that $\exists\varepsilon_V > 0$ such that $V_1 \geq \varepsilon_V$, $\forall t \geq 0$. Therefore, $\lim_{t\to\infty}\left(\frac{1}{2}\left\|z\right\|^2 + \frac{1}{2}\left\|e\right\|^2 + \tilde{\theta}^T\Gamma^{-1}\tilde{\theta}\right) \geq \varepsilon_V$. However from (24) we know that $(z, e) \to 0$, from which we can conclude that $\exists t_{ze}^* = t_{ze}^*(\varepsilon_V, k) < \infty$ such that $\max(\left\|z\right\|, \left\|e\right\|) \leq \sqrt{k\varepsilon_V}$, $\forall t \geq t_{ze}^*$, for any $0 < k < \frac{1}{2}$. It then follows from (21)

$$\left\|\tilde{\theta}\right\| \geq \sqrt{(1-2k)\lambda_{min}\{\Gamma\}\varepsilon_V} \qquad \forall t \geq t_{ze}^* \tag{30}$$

From (28), we can conclude that for any $\varepsilon > 0$ (independent of ε_V), $\exists t_{F\tilde{\theta}}^* = t_{F\tilde{\theta}}^*(\varepsilon) < \infty$ such that

$$\frac{1}{T_0}\int_t^{t+T_0}\left(\tilde{\theta}(\tau)^TF(\tau)^TF(\tau)\tilde{\theta}(\tau)\right)d\tau \leq \varepsilon \qquad \forall t \geq t_{F\tilde{\theta}}^* \tag{31}$$

Substituting $\tilde{\theta}(\tau) = \tilde{\theta}_t + \int_t^\tau\dot{\hat{\theta}}(\sigma)d\sigma$, where $\tilde{\theta}_t = \tilde{\theta}(t)$ is constant over the interval of integration,

$$\begin{aligned}&\frac{1}{T_0}\tilde{\theta}_t\int_t^{t+T_0}F(\tau)^TF(\tau)d\tau\,\tilde{\theta}_t \\ &\quad + \frac{2}{T_0}\tilde{\theta}_t\int_t^{t+T_0}F(\tau)^TF(\tau)\left(\int_t^\tau\dot{\hat{\theta}}d\sigma\right)d\tau \\ &+ \frac{1}{T_0}\int_t^{t+T_0}\left(\int_t^\tau\dot{\hat{\theta}}d\sigma\right)^TF(\tau)^TF(\tau)\left(\int_t^\tau\dot{\hat{\theta}}d\sigma\right)d\tau \leq \varepsilon \\ &\qquad \forall t \geq t_{F\tilde{\theta}}^*\end{aligned} \tag{32}$$

From (23) and the properties of the Projection algorithm, we can deduce that

$$\begin{aligned}\left\|\int_t^\tau\dot{\hat{\theta}}d\sigma\right\| &= \left\|\int_t^\tau \text{Proj}\{\Gamma F(\tau)^T[z_t^T\,e^T]^T\}d\tau\right\| \\ &\leq \sqrt{\frac{\lambda_{max}\{\Gamma\}}{\lambda_{min}\{\Gamma\}}}\left\|\int_t^\tau\Gamma F(\tau)^T[z_t^T\,e^T]^Td\tau\right\| \\ &\leq T_0M_{\dot{\hat{\theta}}}\sqrt{2k\varepsilon_V} \qquad \forall t \geq t_{ze}^*\end{aligned} \tag{33}$$

$$M_{\dot{\hat{\theta}}} \triangleq \left(\sqrt{\frac{\lambda_{max}\{\Gamma\}}{\lambda_{min}\{\Gamma\}}}\right)\sup_{x\in B_x}\left\|\Gamma F(x)^T\right\| < \infty \tag{34}$$

By the uniform boundedness of z, and because of the continuity with respect to x_p, the uniform boundedness of x_p is guaranteed while x_q is bounded by assumption; hence the supremum in (34) exists independently of ε or ε_V.

From the smoothness of $F(x)$ and the uniform boundedness of all closed loop dynamics, it follows that there exists a constant $\bar{c}_{PE} < \infty$ such that the PE condition can be rewritten

$$c_{PE} I \leq \frac{1}{T_0} \int_{t_i}^{t_i+T_0} F(\tau)^T F(\tau) d\tau \leq \bar{c}_{PE} I \qquad (35)$$

Furthermore, since $t_i \to \infty$ and $i \to \infty$, we define the nonempty set $i^* \triangleq \{i \in \{1,2,...\} \,|\, t_i \geq \max(t_{ze}^*, t_{F\tilde{\theta}}^*)\}$. Substituting into (32), noting the semi-positive definiteness of the third term on the LHS, yields

$$c_{PE} \left\|\tilde{\theta}(t_i)\right\|^2 - 2\bar{c}_{PE} T_0 M_{\dot{\hat{\theta}}} \sqrt{2k\varepsilon_V} \left\|\tilde{\theta}(t_i)\right\|^2 - \varepsilon \leq 0 \quad \forall i \in i^* \qquad (36)$$

from which it follows

$$\left\|\tilde{\theta}(t_i)\right\| \leq \frac{\bar{c}_{PE}}{c_{PE}} T_0 M_{\dot{\hat{\theta}}} \sqrt{2k\varepsilon_V} + \frac{1}{c_{PE}} \sqrt{2\bar{c}_{PE}^2 T_0^2 M_{\dot{\hat{\theta}}}^2 k\varepsilon_V + \varepsilon c_{PE}} \quad \forall i \in i^* \qquad (37)$$

The constants $k > 0$ and $\varepsilon > 0$ may be chosen arbitrarily small, independent of ε_V. As $(k, \varepsilon) \to 0$, (37) approaches $\left\|\tilde{\theta}(t_i)\right\| \leq 0$, which is a violation of (30). ■

Therefore, if the dither signal $d(t)$ is designed to satisfy the PE condition eq.(29) then the parameter error converges to zero asymptotically. This implies that

$$\lim_{t\to\infty} \frac{\partial p(\bar{x}_p(t), \bar{\theta}_p)}{\partial \bar{x}_p} = \frac{\partial p(\bar{x}_p(t), \theta_p)}{\partial \bar{x}_p} \qquad (38)$$

and, as a result,

$$\lim_{t\to\infty} \frac{\partial p(\bar{x}_p(t), \theta_p)}{\partial x_p} = 0 \qquad \Rightarrow \qquad \lim_{t\to\infty} \bar{x}_p(t) = x_p^*$$

Hence x_p converges to $x_p^* - d(t)$ as $t \to \infty$, leading to the following theorem.

Theorem 4.1: Suppose the system dynamics (1) satisfies Assumptions 1 and 2 and the dither signal satisfies the persistence of excitation condition (29), then the ESRHC (12)-(16) and the parameter estimation scheme (19) and (23) solves the extremum seeking problem.

V. Simulation Examples

A. Example 1

Consider the plant

$$\begin{aligned} \dot{x}_1 &= \theta_1 x_1^2 + \theta_2 x_2 + u \\ \dot{x}_2 &= -x_2 + \theta_2 x_1^2 \\ y &= p(x_1, \theta_1) = 1 + x_1 - \theta_1 x_1^2 \end{aligned}$$

where θ_1 and θ_2 are constant and unknown parameters. The control objective is to maximize the objective function $p(x_1, \theta_1)$. The above system can be expressed in the form (1) by defining $f(x) = 0$, $\theta_p = \theta_1$, $\theta_q = \theta_2$, $F_p(x) = x_1^2$, $F_q(x) = x_2$ and $G(x) = 1$. The parameter θ_1 is assumed to lie within the compact set $\Omega_\theta = \{\theta_1 | \theta_1 > 0\}$ in order to ensure that the objective function is convex.

Since

$$\frac{\partial p(x_1, \theta_1)}{\partial x_1} = 1 - 2\theta_1 x_1 \quad \text{and} \quad \frac{\partial^2 p(x_1, \theta_1)}{\partial x_1^2} = -2\theta_1$$

it follows that the performance function reaches its maximum at $x_1 = x_1^* = 1/2\theta_1$. Following the design procedure, the predicted state is generated by

$$\dot{\hat{x}}_1 = \hat{\theta}_1 x_1^2 + \hat{\theta}_2 x_2 + u + k(x_1 - \hat{x}_1),$$

and the adaptive laws are designed as

$$\dot{\hat{\theta}}_1 = \begin{cases} \Gamma_1 x_1^2 (x_1 - \hat{x}_1), & \text{if } \hat{\theta}_1 > \varepsilon \quad \text{or} \\ & \hat{\theta}_1 = \varepsilon \text{ and } x_1^2(\hat{x}_1 - x_1) \leq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\dot{\hat{\theta}}_2 = \begin{cases} \Gamma_2 x_2 (x_1 - \hat{x}_1), & \text{if } \hat{\theta}_2 > \varepsilon \quad \text{or} \\ & \hat{\theta}_2 = \varepsilon \text{ and } x_2(x_1 - \hat{x}_1) \leq 0 \\ 0, & \text{otherwise} \end{cases}.$$

The formulation of the optimization is as follows

$$\begin{aligned} \min_{u} J &= \int_t^{t+T_p} \left(1 - 2\bar{\theta}_1 (x_1(\tau) + d(\tau))\right)^2 + u(\tau)^2 d\tau \\ \text{s.t. } \dot{x}_1 &= \bar{\theta}_1 x_1^2 + \bar{\theta}_2 x_2 + u \\ \dot{x}_2 &= -x_2 + \bar{\theta}_2 x_1^2 \\ \bar{\theta}_1 = \hat{\theta}_1(t), &\quad \bar{\theta}_2 = \hat{\theta}_2(t) \\ V(t+T_p) &\leq V^{iss}(t+T_p) \end{aligned}$$

where, in this case, the function

$$V(.) = \frac{1}{2}\left(1 - 2\bar{\theta}_1 (x_1(.) + d(.))\right)^2$$

and

$$V^{iss}(.) = \frac{1}{2}\left(1 - 2\bar{\theta}_1 (x_1^{iss}(.) + d(.))\right)^2.$$

are obtained.

The parameters used in the simulation were selected as $k = 5.0$, $\Gamma_1 = \Gamma_2 = 250$, $x_1(0) = \hat{x}_1(0) = x_2(0) = 2.0$, $\hat{\theta}_1(0) = 0.5$ and $\hat{\theta}_2(0) = 0$. The dither signal was chosen to be $d(t) = 0.1\sin(3t)\exp(-0.1t)$. The exponential term appearing in the dither signal ensures that the excitation signal $d(t)$ disappears as t increases. The prediction and control horizons length are chosen to be $T_p = 0.2$ and $T_c = 0.12$ respectively. A sampling time of 0.02 is used for the simulation experiment.

Figures 1 and 2 show the states, performance function, parameter estimates and control input from the simulation. From the above discussion, it is clear that the the optimum occurs when the state $x_1 = 0.5$ and Figure 1(a) showed that the state x_1 oscillates about this optimum value. Also, it is seen that the performance function converges to the maximum value 1.25 in about $t = 2$.

The parameter estimates, shown in Figure 2(a) and 2(b), converge to the true parameter values of $\theta_1 = \theta_2 = 1.0$. This suggests that the proposed control action Figure 2(c) provides sufficient excitation for the system.

VI. Conclusion

A method is proposed to solve a class of extremum seeking control problems for nonlinear systems with unknown parameters. The method is based on a receding horizon technique that employs a control Lyapunov function to ensure stability. An input-to-state stabilizing controller is used to guarantee stability of the proposed scheme by requiring the satisfaction of a terminal state constraint dependent on the Lyapunov function. A parameter update law is implemented on the plant to provide estimates of the unknown parameters which are used, at each iteration step, to update the unknown parameters in the optimization scheme. It is shown that the proposed scheme is able to drive the system states to unknown desired states that optimize the value of an objective function.

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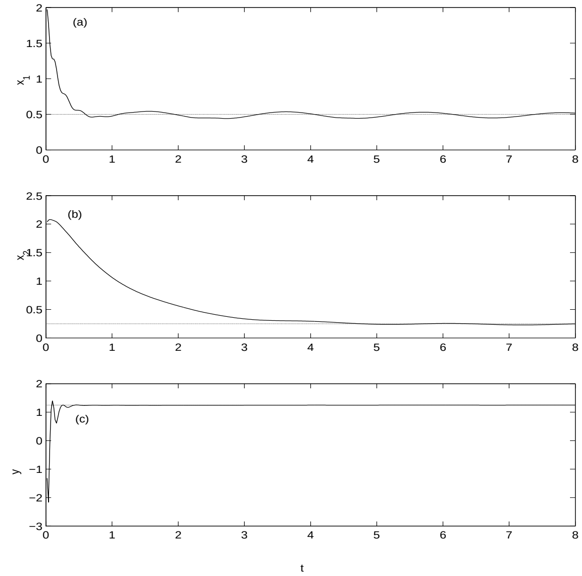

Fig. 1. ESRHC: (a) state x_1, (b) state x_2, (c) performance function

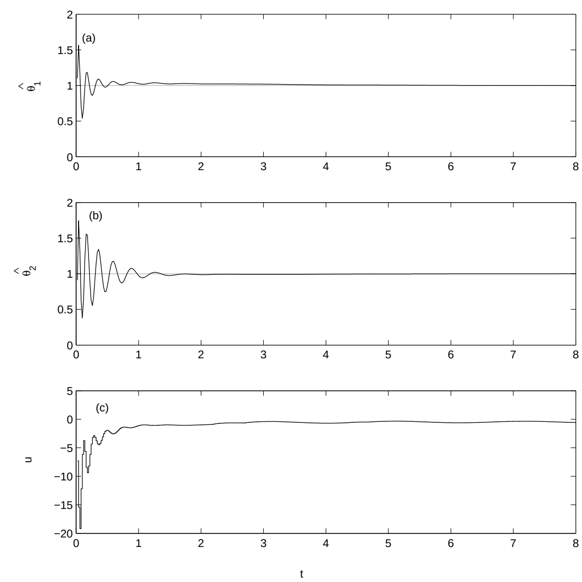

Fig. 2. ESRHC: (a) parameter estimate $\hat{\theta}_1$, (b) parameter estimate $\hat{\theta}_2$, (c) control input