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Sensitivity of Time Response to Characteristic Ratios

Youngchol Kim, Keunsik Kim and Shunji Manabe

***Abstract*— In recent works [1], [5], it has been shown that the damping of a linear time invariant system relates to the so-called *characteristic ratios* (α_k, $k = 1, \cdots, n-1$) which are defined by coefficients of the denominator of the transfer function. However, the exact relations are not yet fully understood. For the purpose of exploring the issue, this paper presents the analysis of time response sensitivity to the characteristic ratio change. We begin with the sensitivity of output to the perturbations of coefficients of the system denominator and then the first order approximation of the α_k perturbation effect is computed by an explicit transfer function. The results are also extended to the two kinds of all-pole systems. Finally, some illustrative examples are given.**

I. INTRODUCTION

In control system design, time response specifications, mainly overshoot and the speed of response, are the most popular ways of describing transient response. In [1], some classical studies regarding transient response are reviewed. The essence of these studies by Naslin [2] and Kessler [3] is to characterize the transient response in terms of coefficients of the characteristic polynomial rather than by its roots by defining the *characteristic ratios*. Naslin claimed that the characteristic ratios are closely related to the overshoot of the system step response. It was shown that all-pole systems with characteristic polynomials that share the same characteristic ratios give remarkably similar step responses. This led to the new design techniques called the characteristic ratio assignment (CRA) in [1] and the coefficient diagram method (CDM) [5]. These methods require properly choosing the characteristic ratios in order to build a target model that meets the desired time response. However, the analytical relationship between time response and α_k is not yet known. This paper presents some new results about the time response characteristics of α_k by analyzing the sensitivity of step response to the α_k perturbations. To consider the sensitivity of output $y(t, \alpha_k)$ of a system having a parameter of interest, α_k, we compute the effect of a perturbation $\Delta\alpha_k$ on the nominal response by using Taylor's series expansion,

$$y(t, \alpha_k + \Delta\alpha_k) = y(t, \alpha_k) + \frac{\partial y}{\partial \alpha_k}\Delta\alpha_k + \cdots, \quad \text{for } k = 1, 2, \cdots, n-1. \tag{1}$$

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Y.C. Kim is with the School of Electrical and Computer Engineering, Chungbuk National University, Cheongju, 361-763, Korea yckim@cbu.ac.kr

K.S. Kim is with the Division of Internet & Information, Daecheon College, Boryoung, 355-769, Korea kskim@www.dcc.ac.kr

S. Manabe is at 1-8-12, Kataseyama, Kanagawa, 251-0033, Japan manabes@cityfujisawa.ne.jp

The first order approximation of the α_k perturbation effect is

$$\Delta y(t) = \frac{\partial y}{\partial \alpha_k}\Delta\alpha_k \tag{2}$$

This function can be generated from the system itself as shown by Franklin at al. [4] and Perkins et al. [7]. The output sensitivity model in [7] is derived using superposition and several block diagram manipulations. However, the process of obtaining it is somewhat complicated. In the present paper, we provide a new method for directly generating the time response sensitivity relative to the individual characteristic ratio change. This sensitivity represents how effectively each characteristic ratio relates to the time response. In a later section, we will establish the fact that the overshoot of the step response of all-pole systems is largely affected by the characteristic ratios α_1, α_2 and α_3. This means that the rest of the $\alpha_k s$ have little effect on the step response but are crucial for maintaining stability. In [1], a method that obtains a special all-pole system with small or no overshoot has been proposed. This characteristic polynomial is completely characterized by the principal characteristic ratio α_1 and the remaining characteristic ratios are fixed functions of α_1. We call the polynomial $\mathcal{K}-$*polynomial* here. It was also shown that the stability of $\mathcal{K}-$*polynomial* is always preserved for any $\alpha_1 \geq 2$. This paper also deals with the sensitivity analysis of $\mathcal{K}-$*polynomial* to the α_1 perturbation. Some examples are given for illustration.

II. DEFINITIONS AND PRELIMINARY RESULTS

In this section, we develop some preliminary results of time response sensitivity to coefficient changes. Consider a linear system whose transfer function is

$$T(s) = \frac{n(s)}{\delta(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}. \tag{3}$$

The *characteristic ratios* are defined as:

$$\alpha_1 = \frac{a_1^2}{a_0 a_2},\ \alpha_2 = \frac{a_2^2}{a_1 a_3}, \cdots,\ \alpha_{n-1} = \frac{a_{n-1}^2}{a_{n-2}a_n} \tag{4}$$

and the *generalized time constant* is defined to be

$$\tau := \frac{a_1}{a_0}. \tag{5}$$

Conversely, the coefficients a_i of $\delta(s)$ may also be represented in terms of α_i's and τ as follows:

$$a_1 = a_0\tau \tag{6}$$

$$a_i = \frac{a_0\tau^i}{\alpha_{i-1}\alpha_{i-2}^2\alpha_{i-3}^3\cdots\alpha_2^{i-2}\alpha_1^{i-1}},\ \text{for } i = 2, \cdots, n. \tag{7}$$

Although the analytical relationships between damping and characteristic ratio are not yet known, the dependency can be explained by using the Kessler's multiple loop structure [3]. For the purpose of this discussion, let us consider the 2-loop system shown in Fig. 1.

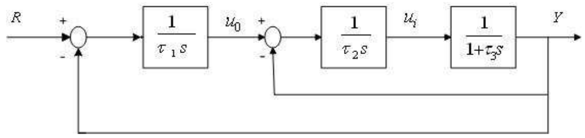

Fig. 1. Kessler's 2 loop structure

The transfer function of the overall system is

$$\begin{aligned} T(s) &= \frac{1}{1+\tau_1 s\,[1+\tau_2 s(1+\tau_3 s)]} \\ &= \frac{1}{\tau_1\tau_2\tau_3\ s^3+\tau_1\tau_2\ s^2+\tau_1\ s+1}. \end{aligned} \tag{8}$$

According to eq. (4), the characteristic ratios of $T(s)$ are

$$\alpha_1 = \frac{\tau_1}{\tau_2}, \qquad \alpha_2 = \frac{\tau_2}{\tau_3}. \tag{9}$$

Now, we first consider the dynamics of the inner loop system. Its transfer function and the characteristic ratio are as follows:

$$\begin{aligned} T_i(s) &= \frac{1}{1+\tau_2 s(1+\tau_3 s)} = \frac{1}{\tau_2\tau_3\ s^2+\tau_2\ s+1}, \\ \bar{\alpha_2} &= \frac{\tau_2}{\tau_3} = \alpha_2. \end{aligned} \tag{10}$$

Secondly, if we assume that with $\tau_1 > \tau_2 > \tau_3$, $T(s)$ can be approximated as follows:

$$T(s) \approx \frac{1}{1+\tau_1 s(1+\tau_2 s)} = \frac{1}{\tau_1\tau_2 s^2+\tau_1 s+1} := T_0(s).$$

Then the characteristic ratio of $T_0(s)$ becomes

$$\bar{\alpha_1} = \frac{\tau_1}{\tau_2} = \alpha_1. \tag{11}$$

Furthermore, it is easily seen that the damping ratio of an second-order system is identical to $\zeta = \frac{\sqrt{\alpha_1}}{2}$. Since the all-pole system of arbitrary order is also developed in the same manner, we can say from eqs. (10), (11) that the characteristic ratio of an all-pole system is closely related to the damping. It was shown in [1] that τ represents the speed of the response of a system with denominator $\delta(s)$. The speed of time response can be controlled by the generalized time constant independent on the characteristic ratios. As mentioned in the introduction, our concern of interest is to find out how the step response of the system in eq. (3) changes as each characteristic ratio changes. We now define the unnormalized function sensitivity to the perturbation of the j-th coefficient of $\delta(s)$ as

$$US^T_{a_j} := \frac{\partial T(s)}{\partial a_j/a_j} \tag{12}$$

Let the unit step response of the unnormalized function sensitivity be $y^s_{a_j}(t)$. That is, $Y^s_{a_j}(s) = US^T_{a_j}(s)\cdot R(s)$ where the input $R(s)=1/s$. As shown earlier, the effect of a perturbation Δa_j about a nominal a_j can be similarly evaluated by a Taylor's series:

$$y(t,a_j+\Delta a_j) = y(t,a_j) + \frac{\partial y}{\partial a_j}\Delta a_j + \cdots, \quad \text{for } j=0,1,2,\cdots,n. \tag{13}$$

$\Delta y(t,a_j) := \frac{\partial y}{\partial a_j}\Delta a_j$ can be determined by the following Lemma, which will be used for the proof of the main results in a later section.

Lemma 1 *Consider a linear system in eq. (3). Then the unnormalized function sensitivity and the first order approximation of output response change to the perturbation of j-th coefficient of $\delta(s)$ are determined by*

$$\text{(i)} \qquad US^T_{a_j} = -\frac{a_j s^j}{\delta(s)}\cdot T(s), \ \text{for } j=0,1,\cdots,n \tag{14}$$

$$\text{(ii)} \qquad \Delta y(t,a_j) = \frac{\Delta a_j}{a_j}\cdot y^s_{a_j}(t) \tag{15}$$

Proof: It is straightforward to derive part (i). For the proof of part (ii), let us consider the Taylor's series of $T(s)$ when a_j perturbs.

$$T(s,a_j+\Delta a_j) = T(s,a_j) + \frac{\partial T}{\partial a_j}\Delta a_j + \cdots, \quad \text{for } j=0,1,2,\cdots,n.$$

The first order approximation of a_j perturbation effect, $\Delta T(s,a_j)$, becomes as follows:

$$\Delta T(s,a_j) := \frac{\partial T}{\partial a_j}\Delta a_j = \frac{\Delta a_j}{a_j}\cdot US^T_{a_j}.$$

Since $\Delta y(t,a_j)$ is the step response of $\Delta T(s,a_j)$, the proof is completed. ■

We now introduce another type of sensitivity that has the relative form. It is seen from eqs. (6) and (7) that the coefficients of $\delta(s)$ are nonlinear functions of the characteristic ratios. We define the coefficient sensitivity as

$$S^{a_j}_{\alpha_k} := \frac{\partial a_j/a_j}{\partial \alpha_k/\alpha_k} = \frac{\alpha_k}{a_j}\cdot\frac{\partial a_j}{\partial \alpha_k}. \tag{16}$$

The following relationship will be used in the derivation of the main result. We state the Lemma without the proof.

Lemma 2 *For the linear system in eq. (3), the coefficient sensitivity of a_j with respect to the perturbation of α_k is*

$$S^{a_j}_{\alpha_k} = \begin{cases} -(j-k), & \text{if } k<j, \\ 0, & \text{if } k \ge j, \end{cases} \quad \begin{aligned} &\text{for } k=1,2,\cdots,n-1, \\ &\quad j=0,1,2,\cdots,n. \end{aligned} \tag{17}$$

III. Time response sensitivity to characteristic ratio change

In this section, we will present how the step response changes as the characteristic ratio changes. Time response sensitivities are studied for three cases: (1) a general transfer function as in eq. (3), (2) all-pole systems of degree n, (3) a special class of all-pole system whose denominator shall be composed of $\mathcal{K}-$*polynomial*.

A. Time response sensitivity : a general case

We first define the other unnormalized function sensitivity relative to characteristic ratios as follows:

$$US^T_{\alpha_k} := \frac{\partial T(s)}{\partial \alpha_k/\alpha_k} \tag{18}$$

Recall eqs. (1) and (2). Let the unit step response of $US^T_{\alpha_k}$ be $Y^s_{\alpha_k}(s)$. That is,

$$Y^s_{\alpha_k}(s) = US^T_{\alpha_k} \cdot R(s), \tag{19}$$

where $R(s) = 1/s$. $y^s_{\alpha_k}(t)$ indicates the inverse Laplace transform of $Y^s_{\alpha_k}(s)$. Then the following Theorem 1 states that $\Delta y(t)$ in eq. (2) can be computed by an explicit transfer function.

Theorem 1 *Given a stable $T(s)$ as in eq. (3), the unnormalized function sensitivity and the first order approximation of step response perturbation to the α_k change are determined by*

$$\text{(i)} \quad US^T_{\alpha_k} := \sum_{j=k+1}^{n} \frac{(j-k)a_j s^j}{\delta(s)} \cdot T(s) \tag{20}$$

$$\text{(ii)} \quad \Delta y(t) = \frac{\Delta\alpha_k}{\alpha_k} \cdot y^s_{\alpha_k}(t), \text{ for } k = 1, \cdots, n-1. \tag{21}$$

Proof: Since $\delta(s)$ is stable, all a_j are non-zero and positive. Therefore, $T := T(s, a_0, a_1, \cdots, a_n)$ is continuously differentiable with respect to a_j and the coefficients a_j are also differentiable to α_k because they are functions of α_ks. Using the so-called chain rule we have

$$\frac{\partial T}{\partial \alpha_k} = \frac{\partial T}{\partial a_{k+1}} \cdot \frac{\partial a_{k+1}}{\partial \alpha_k} + \frac{\partial T}{\partial a_{k+2}} \cdot \frac{\partial a_{k+2}}{\partial \alpha_k} + \cdots + \frac{\partial T}{\partial a_n} \cdot \frac{\partial a_n}{\partial \alpha_k}. \tag{22}$$

Eq. (22)was derived using the fact that a_j is the function of α_k for only $k \le (j-1)$.

Now rewrite eqs. (12) and (16) as

$$\frac{\partial T}{\partial a_j} = \frac{1}{a_j} \cdot US^T_{a_j}, \tag{23}$$

$$\frac{\partial a_j}{\partial \alpha_k} = \frac{a_j}{\alpha_k} \cdot S^{a_j}_{\alpha_k}. \tag{24}$$

From eqs. (18) and (22)-(24), we have

$$US^T_{\alpha_k} = \alpha_k \cdot \frac{\partial T}{\partial \alpha_k} = \sum_{j=k+1}^{n} US^T_{a_j} \cdot S^{a_j}_{\alpha_k}. \tag{25}$$

Using Lemma 1 and Lemma 2 in this sequence, eq. (25) becomes

$$\begin{aligned} US^T_{\alpha_k} &= \sum_{j=k+1}^{n} \frac{(j-k)a_j s^j n(s)}{\delta^2(s)} \\ &= \sum_{j=k+1}^{n} \frac{(j-k)a_j s^j}{\delta(s)} \cdot T(s). \end{aligned}$$

Therefore, part (i) is proven. The proof of part (ii) is similar to the one of Lemma 1. Thus, from eq.(18), it is easily seen that

$$\Delta T(s) := \Delta T(s, \alpha_k) = \frac{\partial T}{\partial \alpha_k}\Delta\alpha_k = \frac{\Delta\alpha_k}{\alpha_k} \cdot US^T_{\alpha_k}.$$

Since $\Delta y(t)$ is identical to the step response of $\Delta T(s)$, the proof is completed. ■

Remark 1 (**The effect of τ on the output sensitivity function**): So far, we have only dealt with the time response sensitivity for characteristic ratios under the assumption that the generalized time constant is constant. In [1], it was shown that if two systems, whose generalized time constants are τ_1 and τ_2 respectively, share the same characteristic ratios, then the step response is exactly time-scaled by factor $\beta = \tau_1/\tau_2$. From this result, it is easy to derive that

$$y^s_{\tau_2}(t) = y^s_{\tau_1}(\frac{1}{\beta} \cdot t), \tag{26}$$

where $y^s_{\tau_k}(t)$ for $k = 1, 2$ indicate the output sensitivity functions of the systems having τ_k while their characteristic ratios share the same values.

Example 1 (**Time response sensitivities when α_1 and α_3 perturb**): Consider the system in eq. (3) of which

$$\begin{aligned} n(s) &= s^3 + 5s^2 + 11s + 15 \\ \delta(s) &= 2.47 \times 10^{-5}s^7 + 1.11 \times 10^{-3}s^6 + 2.18 \times 10^{-2}s^5 \\ &\quad +0.241s^4 + 1.617s^3 + 6.565s^2 + 15s + 15. \end{aligned}$$

The generalized time constant is $\tau = 1$ and the characteristic ratios of the nominal model are

$$[\alpha_1, \cdots, \alpha_6] = [2.285, 1.777, 1.651, 1.651, 1.777, 2.285].$$

Let the step response of the nominal system be $y^o(t)$. Consider that α_k for $k = 1, 3$ are changed by $\pm 10\%$ individually while the rest of α_ks are fixed at the same values as the nominal system. The $y^p(t)$ denotes the step response of the corresponding perturbed model. According to Theorem 1, let the estimated output be $y^s(t)$ in the form

$$y^s(t) := y^o(t) + \Delta y(t).$$

Fig. 2 and Fig. 3 show $y^s_{\alpha_1}$ and step responses for the cases where α_1 is changed by 10%, respectively. The estimated output $y^s(t)$ is shown closely along with the true response $y^p(t)$. As predicted by part (ii) of Theorem 1, the parameter perturbation effect either increases or decreases according to the sign of $\Delta y(t)$. Fig. 4 and Fig. 5 show the cases where α_3 is changed by $\pm 10\%$. The results are almost the same. However, it is noted that the profile of $y^s_{\alpha_3}$ has the largest value at the second extremum, and the shape is quite different from that of $y^s_{\alpha_1}$. ∇∇∇

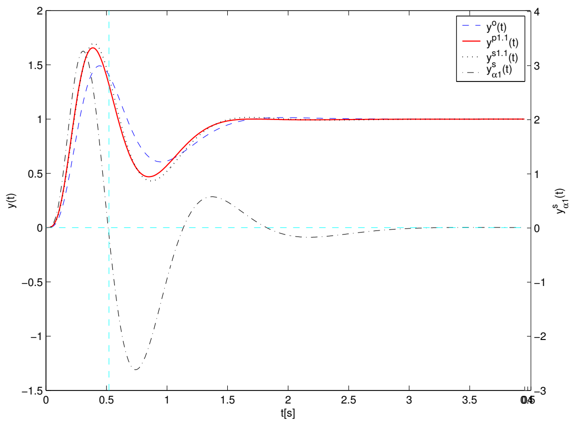

Fig. 2. Step responses and output sensitivity function with $1.1\alpha_1$ (Example 1)

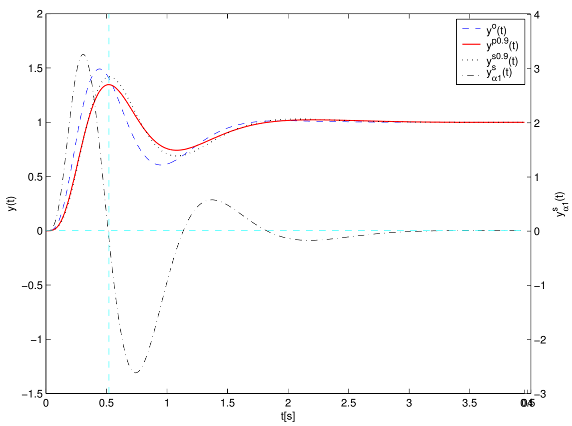

Fig. 3. Step responses and output sensitivity function with $0.9\alpha_1$ (Example 1)

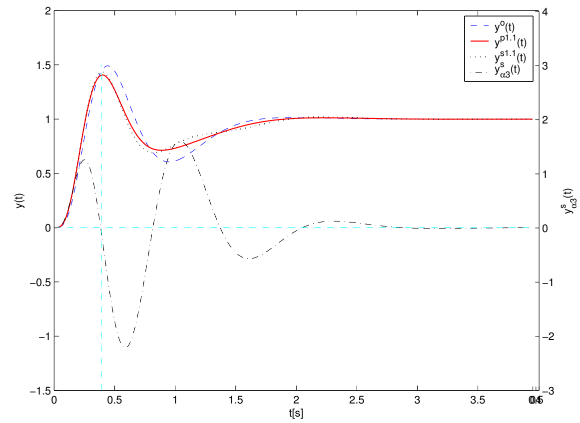

Fig. 4. Step responses and output sensitivity function with $1.1\alpha_3$ (Example 1)

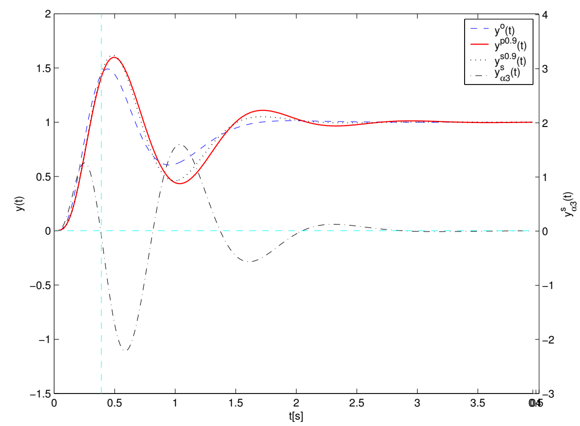

Fig. 5. Step responses and output sensitivity function with $0.9\alpha_3$ (Example 1)

B. Dominant characteristic ratios of all-pole systems

Since the transient response is generally affected by zeros as well as poles, it is more appropriate to consider the all-pole systems in order to study the pure time response sensitivity of the characteristic ratio. This section presents the issue. As a result, we will establish the fact that the transient response of all-pole systems is dominantly affected by only the principal characteristic ratios α_1, α_2 and α_3. Consider the stable all-pole system

$$T_A(s) = \frac{a_0}{\delta(s)} = \frac{a_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}. \quad (27)$$

Then from eq. (20) in Theorem 1, the unnormalized function sensitivity for $T_A(s)$ is given by

$$US^{T_A}_{\alpha_k} := \frac{\partial T_A(s)}{\partial \alpha_k/\alpha_k} = \sum_{j=k+1}^{n} \frac{(j-k)a_j s^j}{\delta(s)} \cdot T_A(s). \quad (28)$$

For the sake of convenience, we describe the output sensitivity functions $Y^s_{\alpha_k} = US^{T_A}_{\alpha_k} \cdot R(s)$ when the degree of $\delta(s)$ is $n = 7$.

$$\begin{aligned} Y^s_{\alpha_1} &= \frac{a_0(a_2 s + 2a_3 s^2 + 3a_4 s^3 + \cdots + 6a_7 s^6)}{\delta^2(s)} \\ Y^s_{\alpha_2} &= \frac{a_0(a_3 s^2 + 2a_4 s^3 \cdots + 5a_7 s^6)}{\delta^2(s)} \\ &\ \vdots \\ Y^s_{\alpha_5} &= \frac{a_0(a_6 s^5 + 2a_7 s^6)}{\delta^2(s)} \\ Y^s_{\alpha_6} &= \frac{a_0(a_7 s^6)}{\delta^2(s)}. \end{aligned} \quad (29)$$

Before going on to eq. (29), we introduce two important properties regarding the relationships between τ, α_k and coefficients of $\delta(s)$. As mentioned in Remark 1 (see also [1]), the change of τ in eqs. (6), (7), make the output sensitivity function in the time domain merely time-scaled. In other words, the minimum and the maximum values of $y^s_{\tau_k}$ remain the same regardless of the τ change. Thus, we may set $\tau = 1$ without loss of generality as long as we deal with the sensitivity problems of α_k. The other property comes from results developed in [8]and [5]. Several sufficient conditions for stability in [8] are in terms of the coefficients of the characteristic polynomial. Rewriting the conditions in terms of α_k, a real polynomial is stable if one of the following is satisfied.

(i) $\sqrt{\alpha_i \alpha_{i+1}} > 1.4656$ for $i = 1, 2, \cdots, n-2$,

(ii) $\alpha_i \geq 1.12374(\frac{1}{\alpha_{i-1}} + \frac{1}{\alpha_{i+1}})$ for $i = 1, 2, \cdots .n-2$

Furthermore, Lipatov and Sokolov discovered the fact that if $\alpha_k \geq 4$ for all $k = 1, 2, \cdots, n-1$, then every root of $\delta(s)$ is distinctively located on the negative real axis. Manabe [5] has investigated the problem of obtaining good transient response of control systems by means of the characteristic ratio and the generalized time constant (which he calls the *stability index* and the *equivalent time constant*). According to his observations, the all-pole system of any degree of which its characteristic ratios are $\alpha_1 = 2.5, \alpha_i = 2$ for $i = 2, 3, \cdots, n-1$ gives good damping. Now, recalling that $\tau = 1$, and if we substitute the conditions of Lipatov and Sokolov above into eq. (7), it is obvious that higher degree coefficients of $\delta(s)$ should be much smaller than those of lower degree. In other words, the following inequality holds:

$$a_0 = a_1 > a_2 > a_3 > \cdots \gg a_n, \quad \text{if} \quad \tau \leq 1.$$

Applying this relationship to eq. (29) results in

$$|y^s_{\alpha_1}|_{max} > |y^s_{\alpha_2}|_{max} > \cdots \gg |y^s_{\alpha_6}|_{max}. \qquad (30)$$

where $|y^s_{\alpha_k}|_{max}$ indicates the maximum value of the impulse response of $Y^s_{\alpha_k}$.

Example 2 Consider a stable all-pole system that is of Type I, $n = 7$ and $\tau = 1$. The values of the characteristic ratios for the nominal model are chosen to be the same values as those in Example 1. The coefficients of corresponding $\delta(s)$ are given as follows:

$$[a_0,\ a_1,\ \cdots,\ a_7] = [15,\ 15,\ 6.565,\ 1.617,\ 0.241,\ 2.18\times10^{-2},\ 1.11\times10^{-3},\ 2.47\times10^{-5}].$$

Comparing the values of a_2, a_3, a_4 with a_6, a_7, it is verified immediately from eq. (29) that eq. (30) holds. The problem of interest is to show by which values of α_k for $k = 1, 2, \cdots, n-1$ the transient response is largely affected. To do this, we make one characteristic ratio at a time either increased by 2.5 times or decreased by 0.8 times while the rest of α_ks are fixed at nominal values. Since every pole is real and negative if $\alpha_k \geq 4$ for all k, increasing the value more than 2.5 times is of no use. The decreasing factor was determined near the marginal value for which the stability condition of Lipatov and Sokolov is not lost. For all these cases, step responses and output sensitivity functions have been computed. Fig. 6 - Fig. 9 show the results for four cases. The effects of α_k for $k \geq 5$ are vanishingly small and can be neglected. In the figures, $y^{2.5}(t)$ and $y^{0.8}(t)$ indicate the step response of $T_A(s)$ for which only one α_k is changed to $2.5\alpha_k$ and $0.8\alpha_k$, respectively. It is shown that α_1, α_2 and α_3 have much greater influence than the rest. ∇∇∇

It has been observed in many other examples that indeed these three principal characteristic ratios are the most dominant factors dictating the transient response.

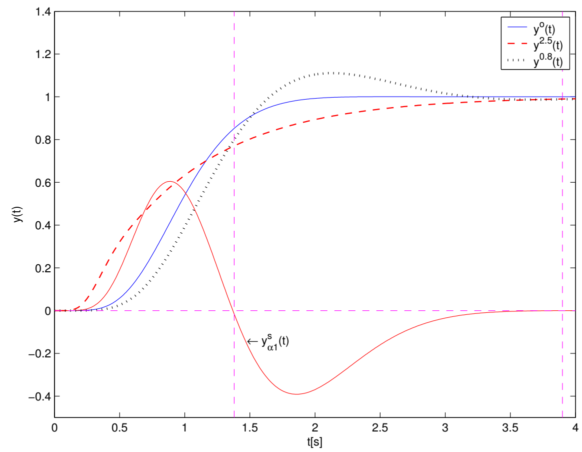

Fig. 6. Step responses and output sensitivity function when α_1 changes(Example 2)

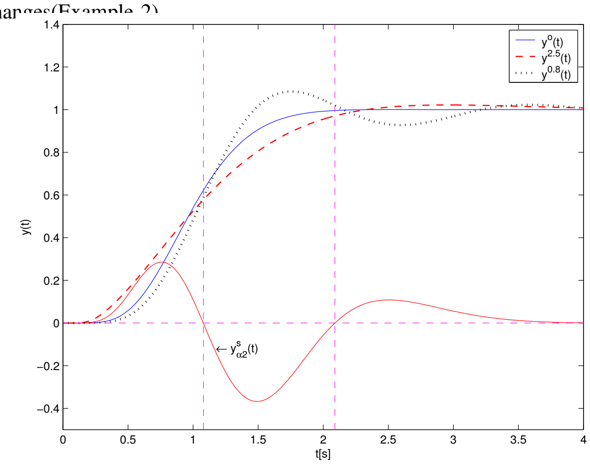

Fig. 7. Step responses and output sensitivity function when α_2 changes(Example 2)

C. Time response sensitivity for the all-pole system with $\mathcal{K}-$polynomial

In the present section, we consider a special class of all-pole systems whose denominator shall be composed of $\mathcal{K}-$*polynomial* [1]. The $\mathcal{K}-$*polynomial* is defined as the polynomial whose coefficients are generated using eqs. (6),(7) and the following function:

$$\text{(i)} \qquad \alpha_1 > 2 \qquad (31)$$

$$\text{(ii)} \qquad \alpha_k = \frac{\sin\left(\frac{k\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right)}{2\sin\left(\frac{k\pi}{n}\right)} \cdot \alpha_1, \quad \text{for } k = 2, \cdots, n-1. \qquad (32)$$

In [1], it was shown that this all-pole system guarantees the stability and its frequency magnitude function is monotonically decreasing. Furthermore, it is important to note that the $\mathcal{K}-$*polynomial* is generated by only α_1 for a given τ. Let the all-pole system be $T_K(s)$. Now, we are going to examine the time response sensitivity when α_1 perturbs.

Lemma 3 *The coefficient sensitivity of $T_K(s)$ relative to the α_1 change is determined by*

$$S^{a_j}_{\alpha_1} := \frac{\partial a_j / a_j}{\partial \alpha_1 / \alpha_1} = -\sum_{k=1}^{j}(k-1). \qquad (33)$$

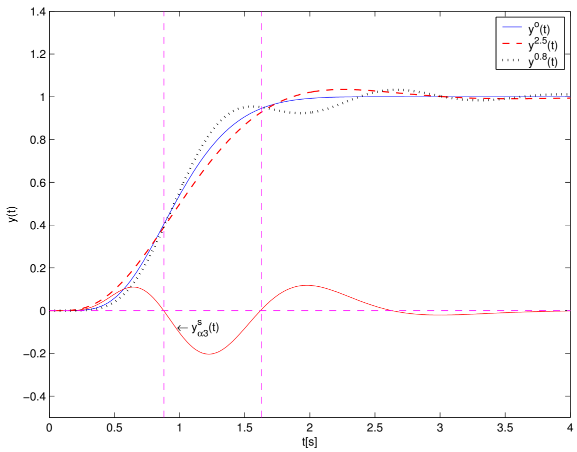

Fig. 8. Step responses and output sensitivity function when α_3 changes(Example 2)

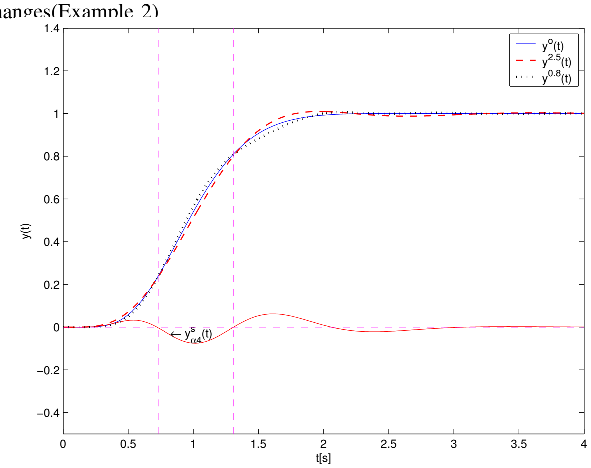

Fig. 9. Step responses and output sensitivity function when α_4 changes (Example 2)

Proof: The proof is omitted here. ■

Let the unit step response of $US^{T_K}_{\alpha_1}$ be $Y^{ks}_{\alpha_1}(s)$. That is, $Y^{ks}_{\alpha_1}(s) = US^{T_K}_{\alpha_1} \cdot R(s)$, where $R(s) = 1/s$. $y^{ks}_{\alpha_1}(t)$ indicates the inverse Laplace transform of $Y^{ks}_{\alpha_1}(s)$. Then, similarly to Section III-A, we have the following result.

Theorem 2 *Given a $T_K(s)$ as in eq. (27), the unnormalized function sensitivity and the first order approximation of step response perturbation to the α_1 change are determined by*

$$\text{(i)} \qquad US^{T_K}_{\alpha_1} := \sum_{j=1}^{n} \frac{a_j s^j}{\delta(s)} \cdot T_K(s) \cdot \sum_{k=1}^{j} (k-1), \quad (34)$$

$$\text{(ii)} \qquad \Delta y(t, \alpha_1) = \frac{\Delta \alpha_1}{\alpha_1} \cdot y^{ks}_{\alpha_1}(t). \quad (35)$$

Proof: Since $T_K(s)$ is in the form of eq.(27), by eq. (14) we have

$$US^{T_K}_{a_j} := \frac{\partial T_K(s)}{\partial a_j / a_j} = -\frac{a_0 a_j s^j}{\delta^2(s)}, \text{ for } j = 1, 2, \cdots, n. \quad (36)$$

From eq. (23),

$$\frac{\partial T_K}{\partial a_j} = \frac{1}{a_j} \cdot US^{T_K}_{a_j}. \quad (37)$$

Since $T_K(s) = T_K(s, a_0, a_1, \cdots, a_n)$ is stable because $\mathcal{K}-$*polynomial* is always stable and its coefficients are continuously differentiable with respect to α_1, by using the chain rule, we have

$$\frac{\partial T_K}{\partial \alpha_1} = \frac{\partial T_K}{\partial a_2} \cdot \frac{\partial a_2}{\partial \alpha_1} + \frac{\partial T_K}{\partial a_3} \cdot \frac{\partial a_3}{\partial \alpha_1} + \cdots + \frac{\partial T_K}{\partial a_n} \cdot \frac{\partial a_n}{\partial \alpha_1}. \quad (38)$$

By eqs. (16) and (37), the unnormalized function sensitivity of $T_K(s)$ is given as

$$US^{T_K}_{\alpha_1} := \frac{\partial T_k}{\partial \alpha_1 / \alpha_1} = \sum_{j=1}^{n} US^{T_K}_{a_j} \cdot S^{a_j}_{\alpha_1}. \quad (39)$$

Finally, by substituting eq. (36) and Lemma 3 into eq. (39), part (i) is proven. The proof of part (ii) is trivial because it is derived in the same manner as Theorem 1. ■

IV. Conclusions

When we deal with the problem of time response control of a linear system, it has been mainly carried out in root space. Recently, instead of pole and zero, a different method that uses the relationships between coefficients of characteristic polynomial and time response such as overshoot and response rate has started to attract attention. The idea was initially provided by Naslin in the mid of 1960s [2]. The present paper presented some new results in this regard. Through the analysis of time response sensitivities to the characteristic ratios, we have shown how the parameters relate to the step response. In particular, if it is the all-pole system, the transient behaviors of step response is dominantly characterized by only α_1, α_2 and α_3. We extended this approach to a special all-pole system in which the denominator is composed of $\mathcal{K}-$*polynomial* [1]. These results can be useful for constructing a reference model by means of the characteristic ratio, as shown in CRA [1], [9] and CDM [5].

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