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H_2-optimal decoupling of previewed signals in the continuous-time domain

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***Abstract*— The synthesis of a feedforward unit for H_2-optimal decoupling of previewed signals in continuous time-invariant linear systems is considered. The H_2-optimal compensator herein presented consists of a finite impulse response system working in connection with a standard finite-dimensional dynamic unit. An explicit expression of the compensator transfer function matrix is derived through a simple procedure based on invariance properties of subspaces related to the autonomous Hamiltonian system.**

I. INTRODUCTION

This paper deals with the synthesis of a non-conventional feedforward control scheme for H_2-optimal decoupling of previewed signals in continuous time-invariant linear systems. Disturbance decoupling is a well-known problem in control theory, deeply investigated mainly in the geometric approach context. The unaccessible disturbance localization was first tackled in [1], [2] and, independently, in [3]. A few years later, the localization of measurable signals was treated without stability conditions in [4], and with stability conditions in [5], [6]. However, the question of how to take advantage of some preview of the signal to be decoupled — and, by extension, tracked —, although it dates back to the middle seventies [7] and has been widely discussed during the following decades [8], [9], [10], [11], [12], [13], is still an open problem. Many valuable contributions on this subject can be found in the most recent literature, focusing on a variety of different optimization techniques, see e.g. [14], [15], [16], [17], [18], [19], [20]. The controller herein presented consists of a finite impulse response (FIR) system working in connection with a standard dynamic unit having the structure of a linear quadratic (LQ) regulator. Actually, the use of a FIR system cooperating with a usual dynamic system has proved to be a particularly efficient means to exploit preview, whatever the strategies adopted to design the FIR part and the dynamic unit are. For instance, in the discrete-time case, this scheme has been used to achieve almost perfect tracking of previewed signals by means of steering along zeros techniques, [21]. In the discrete-time case, it has also been assumed to obtain H_2-optimal decoupling with preview: in [22], the FIR convolution profiles are such that they produce the optimal (in the LQ sense) transition between two given states during the preview time interval, while the dynamic unit has the structure of the Kalman regulator. As far as continuous-time systems are concerned, a very recent article can be found in the literature, sharing a similar layout in the dual setting of fixed-lag smoothing: in fact, in [20], a FIR system is in parallel connection with an H_∞ filter. In this framework of linear continuous-time systems, the contribution of this paper consists in providing an explicit expression for the transfer function matrix of the composite controller, based on a simple time-domain interpretation of H_2-optimal decoupling of previewed signals as a compound optimal control problem, i.e., as a problem consisting of the optimized connection of a finite-horizon LQ control problem with constraints on the final state and a standard infinite-horizon LQ control problem. The former is efficiently solved by resorting to invariance properties of suitably defined subspaces of the associated autonomous Hamiltonian system. This approach is inspired by a geometric view of the Hamiltonian system structure which has already resulted in a straightforward treatment of singular and cheap discrete-time control problems [23] and can lead to similar results in the continuous-time case.

Notation: $\mathbb{R}$ stands for the set of real numbers. The following symbols are assumed for the most frequently used subsets of the complex plane $\mathbb{C}$: $\mathbb{C}^-$, $\mathbb{C}^+$, and $\mathbb{C}^\circ$ respectively stand for the open left-half complex plane, the open right-half complex plane, and the imaginary axis. Sets, vector spaces and subspaces are denoted by script capitals like $\mathcal{X}$, matrices and linear maps by slanted capitals like A. The image, the null space, and the set of eigenvalues of A are denoted by $\mathrm{im}\, A$, $\ker A$, and $\sigma(A)$, respectively. The symbols $\mathrm{rank}\,(A)$, $\mathrm{tr}\,(A)$, $A^\top$, and A^H are respectively used for the rank, the trace, the transpose, and the complex conjugate transpose of A. Moreover, the definitions $A^{-\top} := (A^\top)^{-1}$ and $A^{-\mathrm{H}} := (A^\mathrm{H})^{-1}$ are set. The matrix A is said to be $\mathbb{C}^-$-stable if all its eigenvalues are in $\mathbb{C}^-$. The pair (A, B) is said to be $\mathbb{C}^-$-stabilizable if all the uncontrollable modes of (A, B) are in $\mathbb{C}^-$. The symbol I is used to denote an identity matrix, and I_n is used to denote the identity matrix of dimension n. For a continuous-time signal $y(t)$, we denote by $\|y(t)\|_{l_2}$ the l_2 norm, and, for a stable real transfer function matrix $G(s)$, we denote by $\|G(s)\|_2$ the H_2 norm. The symbol $\nabla f(x)$ stands for the gradient of function $f(x)$. The symbol $\mathcal{L}\,[\,f(t)\,]$ stands for the Laplace transform of function $f(t)$.

II. PROBLEM STATEMENT

Consider the continuous time-invariant linear system Σ,

$$\dot{x}(t) = A\,x(t) + B\,u(t) + H\,h(t),\ \ x(0) = 0,\ \ t \geq 0, \quad (1)$$

$$y(t) = C\,x(t) + D\,u(t), \quad (2)$$

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where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $h \in \mathbb{R}^s$, and $y \in \mathbb{R}^q$ (with $q \geq p$) respectively denote the state, the control input, the previewed exogenous input, and the controlled output. The set of admissible control functions is defined as the set $\mathcal{U}_f$ of all bounded piecewise-continuous functions with values in $\mathbb{R}^p$. Assume that i) (A, B) is $\mathbb{C}^-$-stabilizable; ii) $\operatorname{rank}(D) = p$; iii) (A, B, C, D) has no invariant zeros on $\mathbb{C}^\circ$. These assumptions guarantee that the associated Hamiltonian matrix has no eigenvalues on $\mathbb{C}^\circ$. Henceforth, A is assumed to be $\mathbb{C}^-$-stable. This latter assumption causes no loss of generality, since stabilization by state feedback is allowed by $\mathbb{C}^-$-stabilizability of (A, B).

The problem of minimizing the effect of the input signal $h(t)$, supposed to be known with a preview time T, obviously reduces to a causal problem if a delay T is inserted in the input h signal flow and included in a new plant Σ_P having $h_P(t) := h(t+T)$ as exogenous input (Fig. 1).

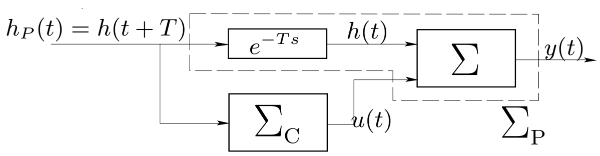

Fig. 1. Block diagram for previewed signal decoupling.

Problem 1: H_2-optimal decoupling with preview – frequency-domain formulation. Refer to Fig. 1. Denote by $G(s)$ the transfer function matrix of the compensated system, from the exogenous input h_P to the output y. Find a feedforward linear dynamic compensator Σ_C such that i) $G(s)$ is $\mathbb{C}^-$-stable; ii) $\|G(s)\|_2$ is minimal.

Parseval theorem yields the following time-domain formulation of Problem 1, functional to the later developments. In fact,

$$\|G(s)\|_2^2 := \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \operatorname{tr}\left[G(j\,\omega)\,G^\mathrm{H}(j\,\omega)\right] d\omega =$$
$$= \int_0^\infty \operatorname{tr}\left[g(t) g^\top(t)\right]\, dt = \int_0^\infty \Big[\, \sum_{j=1}^{s} \sum_{i=1}^{q} g_{ij}^2(t) \,\Big]\, dt,$$

holds, where $g(t) = [g_{ij}(t)]_{i=1,\ldots,q;\ j=1,\ldots,s}$ denotes the impulse response matrix matching $G(s)$.

Problem 2: H_2-optimal decoupling with preview – time-domain formulation. Refer to Fig. 1. Let $g_j(t) := [\, g_{1j}(t) \ \ldots \ g_{qj}(t) \,]^\top$, with $j = 1, \ldots, s$, i.e., $g_j(t)$ is the response of the compensated system (with initial zero state) to input $h_{Pj}(t) := e_j \delta(t)$, where e_j and $\delta(t)$ are the j-th vector of the main basis of $\mathbb{R}^s$ and the Dirac impulse, respectively. Find a feedforward linear dynamic compensator Σ_C such that

$$\sum_{j=1}^{s} \int_0^\infty g_j^\top(t)\, g_j(t)\, dt = \sum_{j=1}^{s} \|g_j(t)\|_{l_2}^2$$

is bounded and minimal.

III. REDUCTION TO A COMPOUND OPTIMAL CONTROL PROBLEM

To avoid notation clutter in the outline of the compensator design, we will first focus on optimal decoupling of the exogenous input $h_{Pj}(t) := e_j \delta(t)$, in terms of the corresponding output l_2-norm. Symbols are used with the meaning introduced in previous sections, wherever not explicitly defined.

Theorem 1: Assume that the exogenous input $h_{Pj}(t) := e_j \delta(t)$ is applied to system Σ_P, with initial zero state. The problem of finding the control law $u_j(t)$, $t \geq 0$, which minimizes the l_2 norm of the corresponding output $y_j(t) = g_j(t)$ is a compound optimal control problem which refers to the quadruple (A, B, C, D) and consists of

a. the finite-horizon LQ control problem defined in $[0, T)$, with zero initial state, parameterized final state x_{1j}, and cost functional
$$J_1(x_{1j}) := \int_0^T y^\top(t)\, y(t)\, dt;$$

b. the infinite-horizon LQ control problem defined in $[T, \infty)$, with parameterized initial state $x_{2j} = x_{1j} + H_j$ (where x_{1j} is the parameterized final state introduced in item *a.* and H_j is the j-th column of the exogenous input matrix H) and with cost functional
$$J_2(x_{1j}) := \int_T^\infty y^\top(t)\, y(t)\, dt;$$

c. the problem of finding x_{1j} so as to minimize the global cost functional
$$J(x_{1j}) := J_1(x_{1j}) + J_2(x_{1j}).$$

Proof: First, note that the state trajectory $x(t)$, $t \geq 0$, of Σ is discontinuous at time T, since the application of $h_{Pj}(t) := e_j\, \delta(t)$ to the exogenous input of Σ_P corresponds to the application of $h_j(t) := e_j\, \delta(t-T)$ to the exogenous input of Σ. Thus, if x_{1j} is the state of Σ at time T^- (due to the sole forcing action $u(t)$, $0 \leq t < T$) and x_{2j} denotes the state of Σ at T^+, we have

$$x_{2j} = \int_0^T e^{A(T-\tau)} B\, u(\tau)\, d\tau +$$
$$\int_0^T e^{A(T-\tau)} H\, e_j\, \delta(\tau - T)\, d\tau = x_{1j} + H_j.$$

Then, the minimization of $\|g_j(t)\|_{l_2}$ follows from the minimization of $J(x_{1j})$ by definition of l_2 norm. ■

Remark 1: On the assumption of $\mathbb{C}^-$-stabilizability of (A, B), the presence in system (1), (2) of a subsystem which is not controllable by input u must not be neglected. Henceforth, the items listed in the statement of Theorem 1 will be referred to the quadruple (A, B, C, D) partitioned as shown in Appendix. In particular, this implies that the sole controllable part $x_c(t)$ of the state is arbitrarily assignable at time T^-, while the uncontrollable part $x_u(t)$, starting from zero, remains equal to zero until T^-. Therefore, we

will introduce the new parameter x_{cfj}, related to x_{1j} by $x_{1j} = [\,x_{cfj}^\top \;\; 0\,]^\top$.
The following propositions provide the respective solutions to the problems listed in the statement of Theorem 1.

Proposition 1: Solution of the finite-horizon LQ control problem – item a. in Theorem 1. The optimal control law is

$$u_j(t) = U_c(t)\,\Gamma_{cf}^{-1} x_{cfj}, \quad 0 \le t < T, \tag{3}$$

and the optimal value of the cost functional is

$$J_1(x_{cfj}) = -x_{cfj}^\top \,\Lambda_{cf}\,\Gamma_{cf}^{-1}\,x_{cfj}, \tag{4}$$

where $U_c(t)$, Γ_{cf}, and Λ_{cf} are defined as in Appendix, equations (37), (34), (35), (39).

Proof: Equations (3), (4) directly follows from Propositions 9, 10 in Appendix, where t_f corresponds to T and x_{cf} to x_{cfj}. ■

Proposition 2: Solution of the infinite-horizon LQ control problem – item b. in Theorem 1. The optimal control law is

$$u_j(t) = K\,x_j(t), \quad t \ge T, \tag{5}$$

where $K := -\,(D^\top D)^{-1}(B^\top X + D^\top C)$ with X denoting the symmetric stabilizing solution of (19). The optimal value of the cost functional is

$$\begin{aligned} J_2(x_{cfj}) \;=\; & x_{cfj}^\top\,X_c\,x_{cfj} + 2\,(H_{cj}^\top\,X_c + H_{uj}^\top\,X_{cu}^\top)\,x_{cfj} + \\ & H_{cj}^\top\,X_c H_{cj} + 2\,H_{cj}^\top\,X_{cu}\,H_{uj} + H_{uj}^\top\,X_u\,H_{uj}, \end{aligned} \tag{6}$$

where H_{cj} and H_{uj} are such that $[\,H_{cj}^\top \;\; H_{uj}^\top\,]^\top$ is H_j, i.e. the j-th column of the exogenous input matrix H partitioned according to the state, and X_c, X_u, and X_{cu} are blocks of the solution X of (19), partioned as in (23).

Proof: Standard results of LQ control theory directly provide the control law (5) and the expression $J_2(x_{1j}) = (x_{1j}+H_j)^\top\,X\,(x_{1j}+H_j)$ for the optimal value of the cost functional. Equation (6) is then derived taking into account the correspondences $x_{1j} = [\,x_{cfj}^\top \;\; 0\,]^\top$, $H_j = [\,H_{cj}^\top \;\; H_{uj}^\top\,]^\top$, and the consistent partition of X. ■

Proposition 3: Minimization of the global cost functional – item c. in Theorem 1. The cost functional $J(x_{cfj})$ is minimal with

$$x_{cfj} = -\Delta^{-1}\left(I - e^{-A_2^\top t_f} e^{A_1^\top t_f}\right)(X_c H_{cj} + X_{cu} H_{uj}), \tag{7}$$

where $\Delta := X_c - X_c^-$, with X_c and X_c^- respectively denoting the stabilizing and antistabilizing symmetric solution of (24).

Proof: Equations (4) and (6) imply

$$\begin{aligned} J(x_{cfj}) \;=\; & x_{cfj}^\top\,(X_c - \Lambda_{cf}\Gamma_{cf}^{-1})\,x_{cfj} + \\ & 2(H_{cj}^\top\,X_c + H_{uj}^\top\,X_{cu}^\top)\,x_{cfj} + \\ & H_{cj}^\top\,X_c H_{cj} + 2\,H_{cj}^\top\,X_{cu}\,H_{uj} + H_{uj}^\top\,X_u\,H_{uj}. \end{aligned}$$

Then, (7) follows from

$$\begin{aligned} & \nabla J(x_{cfj}) = \\ & 2\,x_{cfj}^\top\,(X_c - \Lambda_{cf}\Gamma_{cf}^{-1}) + 2\,(H_{cj}^\top\,X_c + H_{uj}^\top\,X_{cu}^\top) = 0. \end{aligned}$$

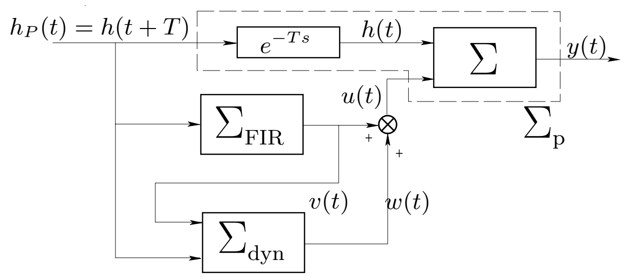

Fig. 2. Structure of the feedforward regulator.

In fact, $X_c - \Lambda_{cf}\Gamma_{cf}^{-1} = \Delta\left(I - e^{A_1 t_f} e^{-A_2 t_f}\right)^{-1}$, where Δ is symmetric positive definite. ■

Remark 2: If, with a slight abuse of notation, the matrix input $H_{\mathrm{P}}(t) := I\,\delta(t)$ is considered to be applied to input h_P of system Σ_{P} with initial zero state, then (3), (4), (5), (6), and (7) still hold in a modified form where the state is an $n \times s$ matrix state, the control law is a $p \times s$ matrix control law and the cost functional is an $s \times s$ matrix cost functional, provided that x_{cfj}, $x_j(t)$, H_{cj} and H_{uj} are respectively replaced by $X_{cf} := [\,x_{cfj}\,]_{j=1,\ldots,s}$, $X(t) := [\,x_j(t)\,]_{j=1,\ldots,s}$, H_c, and H_u.

IV. THE FEEDFORWARD REGULATOR AND ITS TRANSFER FUNCTION MATRIX

In this section, the feedforward regulator Σ_{C} is specified in its inner structure and its transfer function matrix is derived. Refer to Fig. 2. The control input $u(t)$, $t \ge 0$, is obtained as $u(t) = v(t) + w(t)$, where $v(t)$ is the output of a finite impulse response system Σ_{FIR} whose impulse response matrix is

$$V(t) = \begin{cases} U_c(t)\,\Gamma_{cf}^{-1} X_{cf}, & \text{if } 0 \le t < T, \\ 0, & \text{otherwise,} \end{cases} \tag{8}$$

and $w(t)$ is the output of a standard dynamic unit Σ_{dyn} having the structure of the LQ regulator, i.e. ruled by

$$\dot{\tilde{x}}(t) \;=\; (A + BK)\,\tilde{x}(t) + B\,v(t) + H\,h_P(t-T), \tag{9}$$

$$w(t) \;=\; K\,\tilde{x}(t), \tag{10}$$

with $t \ge 0$ and $\tilde{x}(0) = 0$.

Remark 3: Refer to Fig. 2. The FIR system performs its action on a system which is subject to the forcing input $w(t) = K\,\tilde{x}(t)$ from time $t = 0$ (not $t = T$ as was considered in Theorem 1). Nevertheless, the FIR system impulse response has the expression (8), where all the symbols are used exactly with the same meaning introduced in Appendix, due to the fact that the H_2-CARE associated to the quadruple $(A+BK, B, C+DK, D)$, where K is the optimal state feedback matrix, exactly matches the H_2-CARE associated to the original quadruple (A, B, C, D). Hence, the superimposed feedback is zero.

Proposition 4: Transfer function matrix of Σ_{FIR}. The transfer function matrix of the FIR system is

$$G_{\mathrm{FIR}}(s) = -(D^\top D)^{-1}\,(\,\Theta(s) - \Omega(s)\,)\,\Gamma_{cf}^{-1} X_{cf}, \tag{11}$$

where

$$\Theta(s) :=$$
$$(B_c^\top X_c^- + D^\top C_c)\, e^{-A_2 T}(sI - A_2)^{-1} \left(I - e^{-(sI-A_2)T}\right), \tag{12}$$

$$\Omega(s) :=$$
$$(B_c^\top X_c + D^\top C_c)(sI - A_1)^{-1} \left(I - e^{-(sI-A_1)T}\right) e^{-A_2 T}. \tag{13}$$

Proof: Equation (11) with definitions (12) and (13) is obtained by means of trivial algebraic manipulations from

$$\mathcal{L}\,[\,V(t)\,] = \left(\int_0^T U_c(t)\, e^{-st}\, dt\right) \Gamma_{cf}^{-1} X_{cf}.$$

■

Proposition 5: Transfer function matrix of Σ_{dyn}. Let $A_K := A + BK$. The transfer function matrix of the dynamic system is

$$G_{\mathrm{dyn}}(s) = K(sI - A_K)^{-1} \begin{bmatrix} B & e^{-Ts} H \end{bmatrix}, \tag{14}$$

where inputs $v(t)$ and $h_p(t)$ have been considered in order.

Proof: Equation (14) directly follows from (9), (10). ■

Proposition 6: Transfer function matrix of Σ_{C}. Let $A_K := A + BK$. The transfer function matrix of the feedforward regulator is

$$\begin{aligned} G_{\mathrm{C}}(s) \;=\; & -(D^\top D)^{-1} \left(\, \Theta(s) - \Omega(s)\,\right) \Gamma_{cf}^{-1} X_{cf} \\ & + K\,(sI - A_K)^{-1} e^{-Ts} H \\ & - K\,(sI - A_K)^{-1} B (D^\top D)^{-1} (\Theta(s) - \Omega(s)) \Gamma_{cf}^{-1} X_{cf}. \end{aligned} \tag{15}$$

Proof: Equation (15) follows from (11), (12), (13), (14) taking into account the connections shown in Fig. 2. ■

V. CONCLUSIONS

The design of a non-conventional feedforward control unit for H_2-optimal decoupling with preview has been presented in the continuous-time case. The design strategy is based on a simple interpretation of the time-domain formulation of the H_2-optimal decoupling problem as a compound optimal control problem consisting of a finite-horizon LQ control problem with constraints on the final state, an infinite-horizon LQ control problem, and a problem of minimization of the global cost functional. The transfer function matrix of the regulator has been derived by exploiting an effective treatment of the finite-horizon LQ control problem with final state constraints, based on invariance properties of suitably defined subspaces of the autonomous Hamiltonian system. The discussion has been carried out on standard assumptions. However, this geometric view of the Hamiltonian system structure is expected to foster extensions to singular and cheap control problems.

APPENDIX

Formulas used in Section III, concerning the finite-horizon LQ control problem with constraints on the final state, are derived, under standard assumptions, by exploiting invariance properties of suitably defined subspaces. Consider the continuous time-invariant linear system

$$\dot{x}(t) \;=\; A\,x(t) + B\,u(t), \tag{16}$$
$$y(t) \;=\; C\,x(t) + D\,u(t), \tag{17}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, and $y \in \mathbb{R}^q$, with $p \leq q$. Assume that i) (A, B) is $\mathbb{C}^-$-stabilizable; ii) $\operatorname{rank}(D) = p$; iii) (A, B, C, D) has no invariant zeros on $\mathbb{C}^{\circ}$. Denote by H the corresponding Hamiltonian matrix:

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & -H_{11}^\top \end{bmatrix}, \tag{18}$$

where

$$\begin{aligned} H_{11} &:= A - B(D^\top D)^{-1} D^\top C, \\ H_{12} &:= -B\,(D^\top D)^{-1}\, B^\top, \\ H_{21} &:= -C^\top C + C^\top D (D^\top D)^{-1} D^\top C, \end{aligned}$$

and by X the symmetric stabilizing solution of the H_2-CARE:

$$A^\top P + PA - (PB + C^\top D)\,(D^\top D)^{-1} (B^\top P + D^\top C) + C^\top C = 0. \tag{19}$$

Henceforth, the state is denoted by $\begin{bmatrix} x_c^\top(t) & x_u^\top(t) \end{bmatrix}^\top$, where $x_c \in \mathbb{R}^{n_c}$, with $n_c \leq n$, and $x_u \in \mathbb{R}^{n_u}$, with $n_u := n - n_c$, are the controllable and the uncontrollable part of x, respectively. Thus, system (16), (17) is written as

$$\begin{bmatrix} \dot{x}_c(t) \\ \dot{x}_u(t) \end{bmatrix} = \begin{bmatrix} A_c & A_{cu} \\ 0 & A_u \end{bmatrix} \begin{bmatrix} x_c(t) \\ x_u(t) \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u(t), \tag{20}$$

$$y(t) \;=\; \begin{bmatrix} C_c & C_u \end{bmatrix} \begin{bmatrix} x_c(t) \\ x_u(t) \end{bmatrix} + D\,u(t). \tag{21}$$

The matrix H is accordingly partitioned as

$$H = \begin{bmatrix} H_{11} & H_{12} & H_{13} & 0 \\ 0 & A_u & 0 & 0 \\ H_{31} & H_{32} & -H_{11}^\top & 0 \\ H_{32}^\top & H_{42} & -H_{12}^\top & -A_u^\top \end{bmatrix} \tag{22}$$

where

$$\begin{aligned} H_{11} &= A_c - B_c (D^\top D)^{-1} D^\top C_c, \\ H_{12} &= A_{cu} - B_c (D^\top D)^{-1} D^\top C_u, \\ H_{13} &= -B_c (D^\top D)^{-1} B_c^\top, \\ H_{31} &= -C_c^\top C_c + C_c^\top D (D^\top D)^{-1} D^\top C_c, \\ H_{32} &= -C_c^\top C_u + C_c^\top D (D^\top D)^{-1} D^\top C_u, \\ H_{42} &= -C_u^\top C_u + C_u^\top D (D^\top D)^{-1} D^\top C_u, \end{aligned}$$

and the matrix X is partitioned as

$$X = \begin{bmatrix} X_c & X_{cu} \\ X_{cu}^\top & X_u \end{bmatrix}. \tag{23}$$

The H_2-CARE (19) is equivalent to

$$A_c^\top P_c + P_c A_c + C_c^\top C_c - (P_c B_c + C_c^\top D)(D^\top D)^{-1} (B_c^\top P_c + D^\top C_c) = 0, \tag{24}$$

$$A_c^\top P_{cu} + P_c A_{cu} + P_{cu} A_u + C_c^\top C_u - (P_c B_c + C_c^\top D)(D^\top D)^{-1} (B_c^\top P_{cu} + D^\top C_u) = 0, \tag{25}$$

$$A_{cu}^\top\, P_{cu} + A_u^\top\, P_u + P_{cu}^\top\, A_{cu} + P_u A_u + C_u^\top C_u - (P_{cu}^\top\, B_c + C_u^\top D)(D^\top D)^{-1} (B_c^\top P_{cu} + D^\top C_u) = 0. \tag{26}$$

Note that (24) is the H_2-CARE restricted to the controllable part of system (20), (21). Since (A_c, B_c) is controllable, the symmetric solutions of (24) form a lattice with a common largest element, which coincides with X_c introduced in (23), and a common smallest element, henceforth denoted by X_c^-, positive and negative semidefinite, respectively (see e.g. [24], [25]).

Property 1: The subspace

$$\mathcal{S}_X := \mathrm{im} \begin{bmatrix} I & 0 \\ 0 & I \\ X_c & X_{cu} \\ X_{cu}^\top & X_u \end{bmatrix}$$

is an n-dimensional H-invariant subspace, complementary to $\mathrm{im}\,\{[\,0\ I_n\,]^\top\}$. The restriction of H to $\mathcal{S}_X$ is

$$H|_{\mathcal{S}_X} = \begin{bmatrix} A_1 & M_1 \\ 0 & A_u \end{bmatrix},$$

where

$$A_1 := A_c - B_c\,(D^\top D)^{-1}(B_c^\top X_c + D^\top C_c),$$
$$M_1 := A_{cu} - B_c\,(D^\top D)^{-1}(B_c^\top X_{cu} + D^\top C_u).$$

Proof: It is a matter of simple algebraic manipulations to verify that

$$H \begin{bmatrix} I & 0 \\ 0 & I \\ X_c & X_{cu} \\ X_{cu}^\top & X_u \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \\ X_c & X_{cu} \\ X_{cu}^\top & X_u \end{bmatrix} L_1$$

holds with

$$L_1 = \begin{bmatrix} L_{11} & L_{12} \\ 0 & A_u \end{bmatrix},$$

where

$$L_{11} = A_c - B_c(D^\top D)^{-1}(B_c^\top X_c + D^\top C_c),$$
$$L_{12} = A_{cu} - B_c(D^\top D)^{-1}(B_c^\top X_{cu} + D^\top C_u).$$

Complementarity of $\mathcal{S}_X$ to $\mathrm{im}\,\{[\,0\ I_n\,]^\top\}$ is shown e.g. in [25]. ∎

Property 2: The subspace

$$\mathcal{S}_{X^-} := \mathrm{im} \begin{bmatrix} I & 0 \\ 0 & 0 \\ X_c^- & 0 \\ 0 & I \end{bmatrix},$$

is an n-dimensional H-invariant subspace, complementary to $\mathcal{S}_X$. The restriction of H to $\mathcal{S}_{X^-}$ is

$$H|_{\mathcal{S}_{X^-}} = \begin{bmatrix} A_2 & 0 \\ M_2 & -A_u^\top \end{bmatrix},$$

where

$$A_2 := A_c - B_c(D^\top D)^{-1}(B_c^\top X_c^- + D^\top C_c),$$
$$M_2 := -C_u^\top C_c + C_u^\top D(D^\top D)^{-1} D^\top C_c - A_{cu}^\top X_c^- + C_u^\top D(D^\top D)^{-1} B_c^\top X_c^-.$$

Proof: It is straightforward to verify that

$$H \begin{bmatrix} I & 0 \\ 0 & 0 \\ X_c^- & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \\ X_c^- & 0 \\ 0 & I \end{bmatrix} L_2$$

holds with

$$L_2 = \begin{bmatrix} L_{11} & 0 \\ L_{21} & -A_u^\top \end{bmatrix},$$

where

$$L_{11} = A_c - B_c(D^\top D)^{-1}(B_c^\top X_c^- + D^\top C_c),$$
$$L_{21} = -C_u^\top C_c + C_u^\top D(D^\top D)^{-1} D^\top C_c - A_{cu}^\top X_c^- + C_u^\top D(D^\top D)^{-1} B_c^\top X_c^-.$$

Complementarity of $\mathcal{S}_{X^-}$ to $\mathcal{S}_X$ follows from complementarity of $\mathcal{S}_X$ to $\mathrm{im}\,\{[\,0\ I_n\,]^\top\}$ in $\mathbb{R}^{2n}$ and from complementarity of $\mathcal{S}_{X_c} := \mathrm{im}\{[\,I\ X_c^\top\,]^\top\}$ to $\mathcal{S}_{X_c^-} := \mathrm{im}\,\{[\,I\ (X_c^-)^\top\,]^\top\}$ in $\mathbb{R}^{2n_c}$. ∎

Remark 4: The H-invariant subspaces $\mathcal{S}_X$ and $\mathcal{S}_{X^-}$ are internally stable and antistable, respectively. In fact, $\sigma(H|_{\mathcal{S}_X}) \subset \mathbb{C}^-$ and $\sigma(H|_{\mathcal{S}_{X^-}}) \subset \mathbb{C}^+$.

Problem 3: Finite-horizon LQ control with constraints on the final state Consider system (20), (21) with initial conditions $x_c(0)=0$ and $x_u(0)=0$. Find the control law $u(t)$, $0 \le t \le t_f$, which minimizes the cost functional

$$J = \frac{1}{2}\int_0^{t_f} y^\top(t) y(t)\, dt,$$

under the constraint $x_c(t_f)=x_{cf}$, with $x_{cf} \in \mathbb{R}^{n_c}$ given.
The standard procedure for solving Problem 3 leads to the autonomous Hamiltonian system

$$\begin{bmatrix} \dot{x}_c(t) \\ \dot{x}_u(t) \\ \dot{p}_c(t) \\ \dot{p}_u(t) \end{bmatrix} = H \begin{bmatrix} x_c(t) \\ x_u(t) \\ p_c(t) \\ p_u(t) \end{bmatrix}, \tag{27}$$

where H is partitioned as in (22) and $\left[p_c^\top(t)\ p_u^\top(t)\right]^\top$, with $p_c \in \mathbb{R}^{n_c}$ and $p_u \in \mathbb{R}^{n_u}$, denotes the costate: (27) is derived from Euler-Lagrange equations of Problem 3 by eliminating

$$u(t) = -(D^\top D)^{-1}\left(D^\top C_c x_c(t) + D^\top C_u x_u(t) + B_c^\top p_c(t)\right). \tag{28}$$

Theorem 2: A trajectory $\left[x_c^\top(t)\ x_u^\top(t)\ p_c^\top(t)\ p_u^\top(t)\right]^\top$ is a solution of the autonomous Hamiltonian system (27) if and only if

$$\begin{bmatrix} x_c(t) \\ x_u(t) \\ p_c(t) \\ p_u(t) \end{bmatrix} = V_1 e^{L_1\, t} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + V_2 e^{-L_2\,(t_f - t)} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \tag{29}$$

where

$$V_1 := \begin{bmatrix} I & 0 \\ 0 & I \\ X_c & X_{cu} \\ X_{cu}^\top & X_u \end{bmatrix}, \quad V_2 := \begin{bmatrix} I & 0 \\ 0 & 0 \\ X_c^- & 0 \\ 0 & I \end{bmatrix},$$
$$L_1 := \begin{bmatrix} A_1 & M_1 \\ 0 & A_u \end{bmatrix}, \qquad L_2 := \begin{bmatrix} A_2 & 0 \\ M_2 & -A_u^\top \end{bmatrix},$$

$\alpha_1,\ \beta_1 \in \mathbb{R}^{n_c}$, and $\alpha_2,\ \beta_2 \in \mathbb{R}^{n_u}$.

Proof: If. It is easily verified by substitution.

Only if. It is a direct consequence of Properties 1 and 2, namely of complementarity of $\mathcal{S}_X$ and $\mathcal{S}_{X^-}$ as n-dimensional H-invariant subspaces, and expressions of the respective restrictions of H, $H|_{\mathcal{S}_X}$ and $H|_{\mathcal{S}_{X^-}}$. ∎

The following propositions provide compact formulas to express the state trajectory, the control law, and the cost functional solving Problem 3.

Proposition 7: Refer to Problem 3. The optimal state trajectory is

$$\begin{bmatrix} x_c(t) \\ x_u(t) \end{bmatrix} = e^{L_1\, t} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} e^{-L_2\,(t_f - t)} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \tag{30}$$

with

$$\alpha_1 = -\left(I - e^{-A_2\, t_f} e^{A_1\, t_f}\right)^{-1} e^{-A_2\, t_f} x_{cf}, \quad \alpha_2 = 0, \tag{31}$$

$$\beta_1 = \left(I - e^{A_1\, t_f} e^{-A_2\, t_f}\right)^{-1} x_{cf}, \quad \forall\, \beta_2 \in \mathbb{R}^{n_u}. \tag{32}$$

Proof: Optimality of trajectories of the type (30) is a direct consequence of Theorem 2. As for $x_u(t)$, it follows that $x_u(t) = e^{A_u\, t}\alpha_2$. By imposing $x_u(0)=0$, one gets $\alpha_2 = 0$, hence

$x_u(t) = 0$, $0 \le t \le t_f$. As for $x_c(t)$, taking into account $\alpha_2 = 0$, and the boundary conditions $x_c(0) = 0$, $x_c(t_f) = x_{cf}$, one obtains

$$\begin{bmatrix} 0 \\ x_{cf} \end{bmatrix} = \begin{bmatrix} I & e^{-A_2 t_f} \\ e^{A_1 t_f} & I \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}.$$

Hence, α_1 and β_1 follow by matrix inversion. ■

Proposition 8: The optimal state and costate trajectories, $x_c(t)$ and $p_c(t)$ respectively, are

$$x_c(t) = \Gamma_c(t)\,\Gamma_{cf}^{-1} x_{cf}, \quad p_c(t) = \Lambda_c(t)\,\Gamma_{cf}^{-1} x_{cf}, \tag{33}$$

where

$$\Gamma_c(t) := e^{-A_2(t_f - t)} - e^{A_1 t} e^{-A_2 t_f}, \quad \Gamma_{cf} := \Gamma_c(t_f), \tag{34}$$

$$\Lambda_c(t) := X_c^- e^{-A_2\,(t_f - t)} - X_c\, e^{A_1\, t} e^{-A_2\, t_f}. \tag{35}$$

Proof: From (29), with (31), (32), it follows that

$$x_c(t) = (e^{-A_2(t_f-t)} - e^{A_1 t} e^{-A_2 t_f})(I - e^{A_1 t_f} e^{-A_2 t_f})^{-1} x_{cf},$$

$$p_c(t) = (X_c^- e^{-A_2(t_f-t)} - X_c e^{A_1 t} e^{-A_2 t_f})(I - e^{A_1 t_f} e^{-A_2 t_f})^{-1} x_{cf}.$$

Hence, (33) are obtained with definitions (34), (35). ■

Proposition 9: The optimal control law is

$$u(t) = U_c(t)\,\Gamma_{cf}^{-1} x_{cf}, \tag{36}$$

where

$$U_c(t) := -(D^\top D)^{-1}\left(D^\top C_c\,\Gamma_c(t) + B_c^\top \Lambda_c(t)\right). \tag{37}$$

Proof: Equation (36) is derived from (28) by replacing $x_c(t)$ and $p_c(t)$ with (33) and taking into account (37). ■

Proposition 10: The optimal value of the cost functional is

$$J = -\frac{1}{2} x_{cf}^\top\,\Lambda_{cf}\,\Gamma_{cf}^{-1}\,x_{cf}, \tag{38}$$

where

$$\Lambda_{cf} := \Lambda_c(t_f). \tag{39}$$

Proof: According to the system partition, the cost functional can be written as

$$J = \frac{1}{2}\int_0^{t_f} y^\top(t) y(t)\,dt,$$

where

$$y^\top(t)y(t) = x_c^\top(t) C_c^\top C_c x_c(t) + 2 x_c^\top(t) C_c^\top D u(t) + u^\top(t) D^\top D u(t).$$

Hence, by (28), it follows that

$$J = \frac{1}{2}\int_0^{t_f} \left[x_c^\top(t)\,\Psi\,x_c(t) + p_c^\top(t)\,\Phi\,p_c(t)\right] dt,$$

with

$$\Psi := C_c^\top C_c - C_c^\top D(D^\top D)^{-1} D^\top C_c, \quad \Phi := B_c(D^\top D)^{-1} B_c^\top,$$

and, finally, by (33), it follows that

$$J = \frac{1}{2}\,x_{cf}^\top\,\Gamma_{cf}^{-\top}\Big(\int_0^{t_f} \Gamma_c^\top(t)\,\Psi\,\Gamma_c(t) + \Lambda_c^\top(t)\,\Phi\,\Lambda_c(t)\,dt\Big)\Gamma_{cf}^{-1}\,x_{cf}.$$

Equation (38) follows by noting that

$$\Gamma_c^\top(t)\,\Psi\,\Gamma_c(t) + \Lambda_c^\top(t)\,\Phi\,\Lambda_c(t) = -\frac{d}{dt}\left(\Gamma_c^\top(t)\Lambda_c(t)\right).$$

■

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