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On Robust Stability with Nonlinear Parameter Dependence: Some Benchmark Problems Illustrating the Dilation Integral Method

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***Abstract*— In this paper, we consider the problem of robust stability analysis with uncertain parameters entering nonlinearly into the coefficients of the polynomial of interest. In this nonlinear setting, the few results which are available in the literature apply to special cases, or, due to computational complexity, are tractable for only a few uncertain parameters. The objective of this paper is to demonstrate the efficacy of the so-called "dilation integral" method of [1] and [2] in a robust stability context. This rather general method involves a "softening" of the robustness formulation to allow for an "acceptably small" volume of performance violation in parameter space.**

I. INTRODUCTION

In this paper, we consider the robust stability problem for systems whose characteristic polynomial has coefficients which depend nonlinearly on the uncertain parameters. We consider the polynomial

$$p(s,q) \doteq a_n(q)s^n + \cdots + a_1(q)s + a_0(q)$$

with coefficients $a_i(q)$ being multivariable polynomials in $q \doteq (q_1, q_2, \ldots, q_\ell)$ and associated compact bounding set $Q \subset \mathbf{R}^\ell$. Without loss of generality, this family of polynomials is assumed to have at least one stable member $p(s,q^0)$ and we further assume that $p(s,q)$ has invariant degree; i.e., without loss of generality, we assume that $a_n(q)$ is positive for all $q \in Q$.

While there is voluminous literature addressing the case when q enters linearly into the coefficients, for example, see [3]–[4], results obtained for nonlinear problems are limited in that they either apply to rather special cases, involve computations which are tractable only when q has low dimension or involve Monte Carlo sampling e.g., see [7]–[10]. In contrast, the approach taken here, the so-called *dilation integral method*, is based on exact arithmetic and symbolic computation. For the robust stability problem, we obtain a sequence of exact volume estimates for the set of performance violators

$$Q_{bad} \doteq \{q \in Q : p(s,q) \text{ is unstable}\}.$$

A. The Starting Point

In this paper, we work with the classical *Hurwitz matrix* $\mathcal{H}(q)$ associated with $p(s,q)$. The takeoff point for the dilation integral approach pursued here is the following lemma, a special case of the results given in [1] and [2].

B. Lemma

For all positive even integers k, it follows that

$$\mathrm{Vol}(Q_{bad}) \le \min_{\alpha \ge 0} \int_Q \Big(1 - \alpha \det \mathcal{H}(q)\Big)^k dq.$$

Moreover, as $k \to \infty$, the right hand side above converges to zero if and only if the given family of polynomials is robustly stable.

C. Remarks and Rate of Convergence

The lemma above enables us to obtain an upper bound on the volume of violation which can be expressed as a fraction of the total volume by minimizing the scalar convex function

$$\epsilon_k(\alpha) \doteq \frac{1}{\mathrm{Vol}(Q)} \int_Q \Big(1 - \alpha \det \mathcal{H}(q)\Big)^k dq$$

with respect to $\alpha \ge 0$ to obtain

$$\epsilon_k^* \doteq \min_{\alpha \ge 0} \epsilon_k(\alpha).$$

Two key points to note are as follows: First, while a system may fail to be robustly stable in a strict theoretical sense, we may choose to deem it *practically stable* if, for pre-specified $\epsilon > 0$, we certify, via solution of the optimization problem above, that $\epsilon_k^* < \epsilon$. Second, we note that the function $\epsilon_k(\alpha)$ can be obtained exactly because polynomials involved in $\det \mathcal{H}(q)$ are readily integrated in closed form when Q is a hypercube. In comparison with classical Monte Carlo solutions of nonlinear problems as in [7]–[9], the approach given here avoids the need to bring issues such as statistical measures of confidence and sample size into play. One fundamental concern, however, is that the symbolic expressions associated with the computation above may contain a large number of terms if a high k value is needed to obtain a low volume certification for Q_{bad}. To address this issue, a certain *underlying conditioner* θ is introduced in [1] and [2]. More specifically, with

$$f(q) \doteq -\det \mathcal{H}(q)$$

and its range $f(Q)$, the underlying conditioner θ is defined to be the percentage deviation of $f(Q)$, expressed as a fraction, about its midpoint. It is noted that small θ corresponds to a well-conditioned problem, a high θ corresponds to an ill-conditioned problem, and $\theta < 1$ is equivalent to robust

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stability. For example, if $f(Q) = [-2, -5]$, the midpoint is $f_0 = -3.5$ and the half-width is $\sigma = 1.5$. Hence, the maximum percentage deviation, expressed as a fraction is $\theta \approx 0.4286$.

D. Lemma

For all positive even integers k, it follows that $\epsilon_k^* \leq \theta^k$ and $\theta_k \doteq \epsilon_k^{*\frac{1}{k}}$ defines a non-decreasing sequence which converges to θ in the strictly robust case and unity otherwise.

E. Implications and Objectives

Combining the above lemmas, we obtain the inequality

$$\frac{\text{Vol}(Q_{bad})}{\text{Vol}(Q)} \leq \min_{\alpha \geq 0} \epsilon_k(\alpha) \leq \theta^k$$

as the basis for computations to follow. To illustrate, for a robustly stable family of polynomials with $\theta \approx 0.4286$ as in the example above, we are guaranteed that the volume of Q_{bad} will be less than one percent of Vol(Q) with $k = 6$. As stated earlier, for cases when a suitably low volume certification is obtained, say Vol$(Q_{bad}) < \epsilon$Vol(Q) for some small user-defined $\epsilon > 0$ is deemed acceptable, we say that *practical robust stability* has been certified. In this regard, for prescribed $\delta > 0$, we can also consider the case of *practical instability* obtained when $\theta > 1 - \delta$. While a family of polynomials may be robustly stable in a strict mathematical sense, we adopt the point of view that when θ is high, the large percentage spread about the midpoint of the uncertainty range suggests that the system is liable to lose its stability when uncertainty bounds differ slightly from those assumed in our analysis.

In the remainder of this paper, our first objective is to illustrate, via four benchmark examples, the type of computations involved when the dilation integral method is specialized to the robust stability setting. Our second objective is to provide a plausible explanation why certain well-known examples from the literature are known to be notoriously difficult.

II. ACKERMANN'S TRACK-GUIDED BUS

Ackermann's track-guided bus problem, as given in [5], is a benchmark example which includes nonlinear parameter dependence and has been studied by many authors in the robustness literature. For this system with fifth order plant and third order compensator, the lightly damped poles of the system make classical robust stability analysis quite challenging to perform. Our objective in this section is to show that difficulties in robust stability analysis for this example are reflected in the rate of convergence of ϵ_k^* to zero and the closeness of the conditioner θ to one. We consider the characteristic polynomial

$$p(s, q) = \sum_{i=1}^{8} a_i(q) s^i,$$

with uncertain parameters bounding set $Q = Q_\mu$ described by $11.5 - 8.5\mu \leq q_1 \leq 11.5 + 8.5\mu$ and $21 - 11\mu \leq q_2 \leq 21 + 11\mu$ with $0 \leq \mu \leq 1$ defining the radius of uncertainty, and coefficients $a_i(q)$ as given in [5]. For this system, it is well known that as $\mu \to 1$, we obtain an uncertainty domain $Q = Q_\mu$ which nearly touches the instability boundary in parameter space. This fact suggests that the robustness analysis may be difficult in this case. To this end, we seek to demonstrate that our analysis based on Lemmas B and D is consistent with this observation; i.e., the ill conditioning of this problem is manifested by a high θ value and slow rate of convergence of ϵ_k^* to zero.

Using the dilation integral method, we consider the "difficult" case $\mu = 1$ and seek to estimate an upper bound for the volume of violation with various k values. In order to apply lemmas B and D, we first verify that the family of polynomials above has a stable member. Indeed, with $q = q^0 = (11.5, 21)$, we obtain stable roots. Using the formula for ϵ_k, we now illustrate computation using lemma B. Carrying out the requisite integrations, we obtain

$$\begin{aligned} \epsilon_2(\alpha) &= 1 + 2.7566 \times 10^{23}\alpha + 3.6459 \times 10^{47}\alpha^2; \\ \epsilon_4(\alpha) &= 1 - 5.5132 \times 10^{23}\alpha + 2.1875 \times 10^{48}\alpha^2 \\ &\quad - 6.4471 \times 10^{72}\alpha^3 + 8.8731 \times 10^{96}\alpha^4 \end{aligned}$$

and note that finding the minima of these convex functions results in an upper bound on the fractional volume of violation in parameter space and estimates of the underlying conditioner. For example with $k = 4$, the minimum value of $\epsilon_4(\alpha)$ is 0.9292 with corresponding estimate $\theta_4 \approx 0.9818$.

Based on these results, since the true value of θ is at least equal to θ_4, we conclude that this problem is highly ill-conditioned. While a classical analysis indicates that the system is robustly stable, our point of view, based on the extremely high θ certification given, is that this system should be viewed as "practically unstable." It is also interesting to note that a high θ value will result for levels of uncertainty far below those considered here. For example, with $\mu = 0.2$, it can readily be verified that $\theta \geq 0.95$.

III. ACKERMANN'S UNSTABLE ENCLAVE

We consider the polynomial, given in [6], described by

$$\begin{aligned} p(s, q) &= s^3 + (q_1 + q_2 + 1)s^2 + (q_1 + q_2 + 3)s \\ &\quad + (1 + d^2 + 6q_1 + 6q_2 + 2q_1 q_2), \end{aligned}$$

where $d > 0$ is a parameter which can be viewed as part of the given data. The range for the uncertainty is $0.3 \leq q_1 \leq 2.5$ and $0 \leq q_2 \leq 1.7$. With this setup, it is easy to see that this family of polynomials has an "unstable enclave" described by $(q_1 - 1)^2 + (q_2 - 1)^2 - d^2 > 0$. That is, for $q \in Q$, stability is guaranteed if and only if q is outside the circle with radius d, centered at $(1, 1)$. Since the size of the unstable enclave can be made arbitrarily small by choice of $d > 0$, this example provides a good benchmark against which one can demonstrate the efficacy of a formal theory aimed at "detection" of the instability. To this end, we applied lemmas B and D with increasingly smaller values of $d > 0$. For a first computation we took $d = 0.5$ and $k = 6$ and found the minimum of $\epsilon_6(\alpha)$ to be $\epsilon_6^* \approx 0.5536$, corresponding to at most a 55.36% volume of violation. This finding, via lemma D, gives us a lower bound on the underlying conditioner $\theta \geq \theta_6 \approx 0.9061$. Therefore, even at value as high as $d = 0.5$, we already deem this system ill-conditioned and it is arguable that this system is to be categorized as "practically unstable". Next, we carried out a similar analysis for the smaller radii $d = 0.1$ and $d = 0.01$. For the case $d = 0.1$, we obtained $\epsilon_6^* \approx 0.3356$ with corresponding lower bound $\theta \geq \theta_6 \approx 0.8336$, and for the case $d = 0.01$, we obtained $\epsilon_6^* \approx 0.3294$ and $\theta_6 \approx 0.8310$. For both of these cases, due to the high θ_k value, it is arguable that the system should be deemed ill-conditioned and practically unstable.

Finally we carried out the same analysis for the extreme case $d = 0$ with $(q_1, q_2) = (1, 1)$ being the only unstable point in the uncertainty domain. In this case, based on the computed value $\epsilon_6^* \approx 0.3293$ with corresponding lower bound $\theta \geq \theta_6 \approx 0.8310$, we continue to deem this system practically unstable even though an exact analysis indicates that the instability set has measure zero.

IV. PROBLEM OF SAYDY, TITS AND ABED

This problem, based on the theory in [10] and taken from an exercise in [4], is addressed here via the dilation integral method. Indeed, we consider the robustly stable family of matrices described by $A(q) = A_0 + A_1 q + A_2 q^2$ with

$$A_0 = \begin{bmatrix} -5 & 6 \\ 2 & -4 \end{bmatrix}; A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; A_2 = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

and uncertainty bound $q \in [0, 1]$. Now, we form the Hurwitz determinant

$$\det \mathcal{H}(q) = 34q^4 - 68q^3 + 203q^2 - 169q + 72.$$

Subsequently, we computed ϵ_k^* for various values of k. For example, for $k = 8$, we obtained $\epsilon_8^* \approx 1.03 \times 10^{-4}$ and for $k = 10$, we have $\epsilon_{10}^* \approx 1.343 \times 10^{-5}$. With these small values for ϵ_k^*, upper bounds on the fractional volume of violation, we deem this system to be practically stable.

V. AN INTERVAL MATRIX STABILITY PROBLEM

As a final example, we analyze the robust stability of a 3×3 interval matrix described by $A(q) = A_0 + \triangle A(q)$, with

$$A_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}; \quad \triangle A(q) = \begin{bmatrix} q_1 & q_2 & q_3 \\ q_4 & q_5 & q_6 \\ q_7 & q_8 & q_9 \end{bmatrix}$$

and the uncertainty bounds $|q_k| \leq 0.21 + 0.01k$ for $k = 1, 2, \ldots, 8, 9$. Beginning with the characteristic polynomial $p(s, q) = \det(sI - A(q))$, we minimized $\epsilon_k(\alpha)$ for various values of k and rapidly certified a low volume of violation. For example, for $k = 6$, a symbolic computation yields $\epsilon_6^* \approx 0.19 \times 10^{-3}$.

VI. CONCLUSION

In this paper, we used four benchmark problems to demonstrate the efficacy of Lemmas B and D for problems with nonlinear parameter dependence. Most notably, in some of the examples, while the polynomial was robustly stable in the strict theoretical sense, we deemed it to be "practically unstable" based on considerations of the conditioner θ. By way of future research, it would be of interest to investigate efficient computational methods which would enable the integrand in Lemma B to raised to higher k powers, while still remaining calculable. It would also be of interest to investigate the possibility of exploiting the structure of the Hurwitz matrix in lessening the computational burden.

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