A higher order sliding mode controller for a class of MIMO nonlinear systems: application to PM synchronous motor control

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***Abstract*— A new robust higher order sliding mode controller is proposed for a class of MIMO nonlinear systems. The controller synthesis takes three steps: a) the higher order sliding mode problem is formulated in input-output term; b) the problem is viewed in uncertain linear context by considering uncertain nonlinear functions as bounded non structured parametric uncertainties; c) following the optimal sliding-mode design for linear systems, a time varying manifold is designed through the minimization of a quadratic cost function over a finite time interval with a fixed final state. The control law which engenders the sliding on the time varying surface, yields the establishment of an r^{th} order sliding mode. In order to show that the designed controller is well-adapted for practical implementation and that all the features of linear quadratic control can be used to synthesize the controller's gain, a controller for a permanent magnet synchronous motor is designed and implemented on an experimental set-up.**

I. Introduction

The standard sliding mode features are high accuracy and robustness with respect to various internal and external disturbances. Let $\sigma(x,t)$ ($x \in I\!R^n$ is the state variable) the sliding variable, the basic idea is to force the state via a discontinuous feedback to move on a prescribed manifold $\mathcal{S} = \{x \in I\!R^n | \sigma(x,t) = 0\}$ (called the *sliding manifold*). Specific problem entailed by this technique is the chattering effect, *i.e.* dangerous high-frequency vibrations of the controlled system. To overcome this problem, a new approach called "higher order sliding mode" has been recently proposed [1], [5], [13]. Instead of influencing the first sliding manifold time derivative, the "sign" function is acting on its higher time derivative. Keeping the main advantages of the standard sliding mode control, the chattering effect is eliminated and higher order precision is provided [13]. In the case of r^{th} order sliding mode control, the objective is to keep the sliding variable σ and its $r-1$ first time derivatives to zero through a discontinuous function acting on the r^{th} time derivative of the sliding variable.

Several second order sliding mode algorithms are proposed in [5], [13] for SISO nonlinear systems. Among them are the well-known "twisting" and "super-twisting" algorithms. Another 2^{nd} order sliding mode control algorithm derived from the optimal bang-bang control is proposed for SISO nonlinear systems with uncertainties [1] and ensures a maximum convergence time. As only the second order sliding mode problem is studied, an algorithm is given, which does not need the knowledge of the first time derivative of σ. This procedure has been generalized in [2] to a class of MIMO systems with uncertainties, but only in the second order sliding mode case. Arbitrary-order sliding controller for SISO systems with finite time convergence has been proposed in [15], [16], [17]. The algorithm in [16] is inspired by the so-called "terminal sliding modes control" [22]. By tuning only one "gain" parameter and from the knowledge of the relative degree of the output, the controller allows the tracking of smooth signals. As the control algorithm needs the knowledge of high order time derivatives of the output, the author proposes to use the robust exact finite-time convergence differentiators based on second order sliding mode [14].

The aim of this paper is to present a new arbitrary-order sliding mode controller for a class of uncertain minimum-phase MIMO nonlinear systems. The main objective is to propose a controller for which the implementation is simple, the convergence time is finite and the robustness is ensured. The controller design is combining standard sliding mode control with linear quadratic (LQ) one over a finite time interval with a fixed final state [19]. The infinite-horizon linear quadratic control has been used by [20], [21] to synthesize sliding mode manifold for MIMO linear systems. Actually, the problem of the higher order sliding mode control of MIMO minimum-phase uncertain systems can be formulated in input-output terms only through the differentiation of the sliding vector σ, and is equivalent to the finite time stabilization of integrators chains with nonlinear uncertainties. These latter are considered as bounded non structured parametric uncertainties: in this case, the system can be viewed as an uncertain linear system. Then, following the optimal sliding mode formulation for linear systems [21], and considering the uncertain linear system, an optimal time varying switching manifold is determined by minimizing a quadratic cost function over a finite time interval $[0,\ t_f]$ with a fixed final state. The standard sliding mode over this manifold (which depends on the sliding vector σ and its $(r-1)$ first time derivatives) leads to the establishment of r^{th} sliding mode in finite time with respect to σ.

The algorithm needs the relative degree ρ_i with respect to the sliding variable σ_i and the bounds of uncertainties and has several advantages. First, the convergence time is fixed *a priori* via the parameter t_f and the control law can be adjusted via t_f and two weighting matrices P_f and Q. Furthermore, this strategy can be applied for all value of sliding mode order (greater or equal to the relative degree). Finally, the structure of the controller is well-adapted to

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a practical implementation: a robust second order sliding mode controller is designed to drive a permanent magnet synchronous motor and is implemented on an experimental setup [24] to reach an industrial benchmark defined in the framework of a CRAFT European project [9].

II. Problem formulation

Consider the nonlinear system

$$\begin{array}{rcl} \dot{x} & = & f(x) + g(x)u \\ y & = & \sigma(x,t) \end{array} \tag{1}$$

where $x \in I\!R^n$ is the state variable, $u \in I\!R^p$ is the input control and $\sigma \in I\!R^p$ is the output vector (sliding vector). $f(x)$, $g(x)$ and $\sigma(x,t)$ are uncertain sufficiently smooth functions. Assume that

H1. The relative degree ρ_i of each output σ_i of (1) with respect to u is assumed constant, known and such that $\rho_1 = \rho_2 = \cdots = \rho_p = \rho$. The associated zero dynamics are stable. The sliding mode order r is the same for all the outputs.

With these hypotheses, the context is more general, at our best knowledge, than previous works: in [2], [18], a solution has been given only to the second order sliding mode control for a less wide class of MIMO systems (relative degrees equal 1). Note also that it allows to deal with a number of relevant applications (for example, see [7], [3] for the control of induction motor, [4], [23], [11] for the control of synchronous motor).

Definition 1.[2] Given the sliding vector σ, and $r \in I\!N$ with $r \geq 1$. The"r^{th} order sliding set" of σ, denoted $\mathcal{S}^r$, is defined as $\mathcal{S}^r = \{x \mid \sigma(x,t) = \dot{\sigma}(x,t) = \cdots = \sigma^{(r-1)}(x,t) = 0\}$. r is called "sliding mode order". The behaviour of (1) satisfying $\mathcal{S}^r$ is called "r^{th} order sliding mode" with respect to the sliding vector σ. ■

The r^{th} order sliding mode control approach allows the finite time stabilization to zero of σ and its $r-1$ first time derivatives by defining a suitable discontinuous control function which is either the actual control if $\rho = r$, or its $(r-\rho)^{th}$ time derivative if $r > \rho$. Let us consider the case where $r > \rho$. Extend system (1) by introduction of successive time derivatives $u, \dot{u}, \cdots, u^{(r-\rho-1)}$ as new auxiliary state variables and $v = u^{(r-\rho)}$ as a new control; achieve a system with relative degree r. Denote $f_e = [[f(x) + g(x)\bar{x}_{n+1}]^T, \ \bar{x}_{n+2}^T, \cdots, \bar{x}_{n+r-\rho}^T, 0_{1\times p}]^T$, $g_e = [0, \ 0, \ \cdots, 0, {u^{(r-\rho)}}^T]^T$, $\bar{x}_{n+j} = [u_1^{(j-1)} \ \cdots \ u_p^{(j-1)}]^T$ $(1 \leq j \leq r-\rho)$. The output vector σ satisfies an equation of the form $\sigma^{(r)} = \varphi(x,t) + \gamma(x,t)v$, where $\gamma = L_{f_e}^r\sigma$ and $\varphi = L_{g_e}L_{f_e}^r\sigma$. Assume that

H2. $u \in \mathcal{U} = \{u : |u_i| < u_{Mi}, \ 1 \leq i \leq p\}$ where $u_M = [u_{M1}, \cdots, u_{Mp}]^T$ is a real constant vector; if $r = \rho$, then $u(t)$ is a bounded discontinuous function of time and the solution of the differential equation(1) with discontinuous input u admits a solution in Filippov sense [6] on $\mathcal{S}^r$ for all t. If $r > \rho$, the solution of (1) is well defined $\forall \ t \geq 0$, provided that $u(t)$ is continuous and $u(t) \in \mathcal{U}$ $\forall t$. Furthermore, $v = u^{(r-\rho)}$ is bounded by v_M.

H3. Functions $\varphi_i(z,t)$ and $\gamma_{ij}(z,t)$ are bounded uncertain functions: there exist $K_{ijm} \in I\!R$, $K_{ijM} \in I\!R$, $C_{0i} \in I\!R^+$ $(1 \leq i \leq p, \ 1 \leq j \leq p)$ such that

$$\begin{array}{l} |\varphi_i(z,t)| < C_{0i} \\ 0 < K_{iim} \leq |\gamma_{ii}(z,t)| \leq K_{iiM} \\ K_{ijm} \leq |\gamma_{ij}(z,t)| \leq K_{ijM} \quad \text{for } i \neq j. \end{array} \tag{2}$$

PROBLEM STATEMENT. The r^{th} order sliding mode control problem of (1) is equivalent to stabilize to zero, in finite time, the following MIMO uncertain linear system

$$\begin{array}{rcl} \dot{Z}_1 & = & \widehat{A}_{11}Z_1 + \widehat{A}_{12}Z_2 \\ \dot{Z}_2 & = & \widehat{\varphi} + \widehat{\gamma}v \end{array} \tag{3}$$

where $\widehat{A}_{11} = diag[A_{11} \cdots A_{p1}] \in I\!R^{p\cdot(r-1)\times p\cdot(r-1)}$, $\widehat{A}_{12} = diag[A_{12} \cdots A_{p2}] \in I\!R^{p\cdot(r-1)\times p}$ and

$$\begin{array}{rcl} Z_1 & = & \left[\begin{array}{ccccccc} z_1^1 & \cdots & z_{r-1}^1 & \cdots & z_1^p & \cdots & z_{r-1}^p \end{array}\right]^T, \\ Z_2 & = & \left[\begin{array}{ccc} z_r^1 & \cdots & z_r^p \end{array}\right]^T, \\ A_{i1} & = & \left[\begin{array}{ccccc} 0 & 1 & \ldots & 0 & \ldots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ldots & 1 \\ 0 & \ddots & \ddots & \ddots & 0 \end{array}\right]_{(r-1)\times(r-1)}, \\ A_{i2} & = & \left[\begin{array}{cccc} 0 & \cdots & 0 & 1 \end{array}\right]_{(r-1)\times 1} \\ v & = & \left[\begin{array}{ccc} u_1^{(r-\rho)} & \cdots & u_p^{(r-\rho)} \end{array}\right], \\ \widehat{\gamma} & = & \left[\begin{array}{cccc} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1p} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{p1} & \gamma_{p2} & \cdots & \gamma_{pp} \end{array}\right]_{p\times p}, \\ \widehat{\varphi} & = & \left[\begin{array}{c} \varphi_1 \vdots \varphi_p \end{array}\right]_{p\times 1} \end{array} \tag{4}$$

for $1 \leq i \leq p$ and $z = [z_1^1, \cdots, z_r^1, \cdots, z_1^p, \cdots, z_r^p] = [\sigma_1, \cdots, \sigma_1^{(r-1)}, \cdots, \sigma_p, \cdots, \sigma_p^{(r-1)}]$.

III. Synthesis of an higher order sliding mode controller

The synthesis of an higher order sliding mode controller for (1) is made through the following idea: an optimal time varying switching manifold is designed by minimizing a linear quadratic criterion over a finite time interval $[t_0, \ t_0+t_f]$ with a fixed final state on (3). Let S_0 (resp. S_f) denote the optimal switching manifold at the time t_0 (resp. at t_0+t_f) with t_0 the time for which the sliding mode begins on the optimal manifold. An higher order sliding mode behavior occurs at $t = t_0 + t_f$. On the interval $[t_0, \ t_0+t_f]$, the coefficients of the optimal switching manifold depend on time and can be computed *off line*. In general at $t = 0$, the system trajectories are not on $S_0 = 0$. Thus, sum up the control strategy by the three following stages

- $t \in [0,\ t_0[$. At $t = 0$, the system is generally not on the switching manifold S_0. Then, the control task is to drive the system trajectories of (3) to reach $S_0 = 0$. t_0 is the time necessary to reach the switching manifold $S_0 = 0$
- $t \in [t_0,\ t_0 + t_f[$. From t_0, the control task is to maintain the system trajectories of (3) on the time varying switching manifold $S(t)$ ($t \in [t_0,\ t_0 + t_f]$) which permits to reach $S_f = 0$.
- $t \in [t_0 + t_f,\ \infty[$. At $t = t_0 + t_f$, all the components of S_f equal 0, and from $t_0 + t_f$ to ∞, the control task is to maintain the system trajectories on $S_f = 0$.

A. Optimal switching manifold design

First, note that $\tau = t - t_0$ with $t \in [t_0,\ t_0 + t_f]$. We want to stabilize (3) in finite time while minimizing the following linear quadratic cost over a finite time interval $[0,\ t_f]$,

$$J = \frac{1}{2}\{Z(t_f)^T P_f Z(t_f) \mathrm{d}\tau\},\ t_f < +\infty \tag{5}$$

under the fixed final states constraint $Z(t_f) = 0$ with $Z = [Z_1^T \quad Z_2^T]^T$, where

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \tag{6}$$

is a positive symmetric matrix, such that Q_{11}, Q_{12} and Q_{22} are $(p\cdot(r-1)\times p\cdot(r-1))$-, $(p\cdot(r-1)\times(p))$- and $(p\times p)$-dimensional matrices respectively. Criterion (5) becomes

$$J = \frac{1}{2}\int_0^{t_f} Z_1^T Q_{11} Z_1 + 2Z_1^T Q_{12} Z_2 + Z_2^T Q_{22} Z_2 \,\mathrm{d}\tau \tag{7}$$

The idea is to determine the switching manifold resulting in the minimum of the criterion (7); the sliding mode occurs on this manifold. $\tau = 0$ (*i.e.* $t = t_0$) is the instant for which the sliding mode begins, it is regarded as the initial point in function (7). In the first equation of (3), consider Z_1 as the state variable, and Z_2 as a fictive control input. Then, the problem leads back to the resolution of the LQ problem (7) for the dynamics of Z_1, under the constraint $Z_1(t_f) = 0$. A fictive control Z_2, stabilizing Z_1 to $Z_1(t_f) = 0$ in finite time and minimizing the quadratic cost function (7), is given by [19]

$$\begin{aligned} Z_2 = & -(Q_{22}^{-1}\widehat{A}_{12}^T P - Q_{22}^{-1}\widehat{A}_{12}^T V H^{-1} V^T \\ & + Q_{22}^{-1} Q_{12}^T) Z_1. \end{aligned} \tag{8}$$

where $P(t) \in I\!R^{(r-1)\times(r-1)}$ is the unique solution to the differential Riccati equation (with a stated $P(t_f) = P_f$)

$$\begin{aligned} -\dot{P} = & P(\widehat{A}_{11} - \widehat{A}_{12}Q_{22}^{-1}Q_{12}^T) + (\widehat{A}_{11} - \widehat{A}_{12}Q_{22}^{-1} \\ & Q_{12}^T)^T P - P\widehat{A}_{12}Q_{22}^{-1}\widehat{A}_{12}^T P + (Q_{11} \\ & - Q_{12}Q_{22}^{-1}Q_{12}^T) \end{aligned} \tag{9}$$

$V \in I\!R^{(r-1)\times(r-1)}$ and $H \in I\!R^{(r-1)\times(r-1)}$ are the solutions to two linear differential equations ($t \le t_f$, $V(t_f) = I$, $H(t_f) = 0$)

$$-\dot{V} = (\widehat{A}_{11} - \widehat{A}_{12}Q_{22}^{-1}Q_{12}^T - \widehat{A}_{12}Q_{22}^{-1}\widehat{A}_{12}^T P)^T V, \tag{10}$$

$$\dot{H} = V^T \widehat{A}_{12} Q_{22}^{-1} \widehat{A}_{12}^T V. \tag{11}$$

From (8), let $S(Z,\tau)$ defined by

$$\begin{aligned} S(Z,\tau) = & \begin{bmatrix} S_1 \\ \vdots \\ S_p \end{bmatrix} \\ = & Z_2 + (Q_{22}^{-1}\widehat{A}_{12}^T P(\tau) - Q_{22}^{-1}\widehat{A}_{12}^T \\ & V(\tau)H(\tau)^{-1}V(\tau)^T + Q_{22}^{-1}Q_{12}^T) Z_1 \end{aligned} \tag{12}$$

Equation $S(Z,\tau) = 0$ describes the desired dynamics which satisfy the finite time stabilization of vector $[Z_1^T \ Z_2^T]^T$ to zero and minimize the quadratic cost function (7). Then, the *optimal switching manifold*, on which system (3) is forced to slide on via the discontinuous control v, is defined as

$$\mathcal{S} = \{(Z_1, Z_2) \ : \ S(Z_1, Z_2, \tau) = 0\} \tag{13}$$

B. Controller design

The attention is now focused on the design of the discontinuous vector control law v which drives the system trajectory of (3) to lie on $\mathcal{S}$ in a finite time and which maintains it on the origin. Consider only the second stage control, from the reaching of S_0 at $t = t_0$ (i.e. $\tau = 0$) to the reaching of $S_f = 0$ (i.e. $Z = 0$) at $t = t_0 + t_f$ (i.e. $\tau = t_f$).

H4. The matrix $\widehat{\gamma}$ is positive definite with $\gamma_{ii} > 0$ ($1 \le i \le p$) and is dominant diagonal.

H5. At $\tau = 0$ (i.e. $t = t_0$), $S(Z_1, Z_2, 0) = 0$.

Theorem 1. Consider the nonlinear system (3). Suppose that it is minimum phase and that hypotheses H_1, H_2, H_3, H_4 and H_5 are fulfilled. Let $S \in I\!R^p$ defined by (12) with $\widehat{A}_{12}$ defined by (4), $P(\tau)$ the unique non-negative definite solution of the differential matrix Riccati equation (9) (with a given $P(t_f) = P_f$), V and H the solutions of equations (10) and (11) and Q is a symmetrical and positive matrix defined by (6). The control input u whose the $(r-\rho)^{th}$ time derivative is defined as

$$v = \begin{bmatrix} u_1^{(r-\rho)} \\ \vdots \\ u_p^{(r-\rho)} \end{bmatrix} := -\alpha \cdot \begin{bmatrix} \mathrm{sign}(S_1) \\ \vdots \\ \mathrm{sign}(S_p) \end{bmatrix} \tag{14}$$

with

$$\begin{aligned} \alpha & \ge \mathrm{Max}_{1\le j\le p} \left[\frac{C_{0j} + \Theta_j}{K_{jjm} - \sum_{i=1, i\ne j}^{p} K_{jiM}} \right], \\ \begin{bmatrix} \Theta_1 \\ \vdots \\ \Theta_p \end{bmatrix} & > \mathrm{Max}(|\psi \cdot \dot{\Sigma} + \Delta \cdot \Sigma|) \end{aligned} \tag{15}$$

where

$$\begin{array}{rcl}\Sigma & = & [\sigma^T \ \dot{\sigma}^T \ \cdots \ \sigma^{(r-2)^T}]^T \\ \Psi & = & Q_{22}^{-1}\widehat{A}_{12}^T P - Q_{22}^{-1}\widehat{A}_{12}^T V H^{-1} V^T + Q_{22}^{-1} Q_{12}^T \\ \Delta & = & Q_{22}^{-1}\widehat{A}_{12}^T \cdot (\dot{P} - \dot{V} H^{-1} V^T - V(\dot{H^{-1}}) V^T \\ & & -VH^{-1}(\dot{V^T}))\end{array} \tag{16}$$

with $\dot{P}$, $\dot{V}$ and $\dot{H}$ defined respectively by (9)-(10)-(11), leads to the establishment of r^{th} order sliding mode with respect to σ by attracting each trajectory in finite time. The convergence time is t_f. ■

Sketch of proof. By the same way as [12], and knowing that the outputs are sufficiently decoupled (Hypothesis H4), *i.e.* it is possible to *independently* choose each component of the control vector v in accordance to the control law (14) which guarantees that each component of $S(Z_1, Z_2, t)$ reachs zero in finite time, it is easily proved that (15) is sufficient to ensure $S_i \cdot \dot{S}_i < 0$ for $1 \leq i \leq p$ for the under consideration uncertainties. ■

The instant $\tau = 0$ (i.e. $t = t_0$), which is the initial time in (5), is the instant for which the sliding mode begins [20]. In general, before $t = t_0$ (i.e. $t \in [0, \ t_0]$) the system is not on the optimal switching manifold, i.e. not on (from 12)

$$S_0 \quad = \quad S(Z, 0) = [S_1(Z, 0) \ \cdots \ S_p(Z, 0)] = 0. \tag{17}$$

$t = t_0$ is the time necessary to reach $S_0 = 0$ by the control law $v_i = -\alpha \text{ sign } (S_i(Z, 0))$ $(1 \leq i \leq p)$. At $t = t_0$ (i.e. $\tau = 0$), the state variables are on the optimal manifold. Over the time interval $[t_0, \ t_0 + t_f]$ (i.e. $\tau \in [0, \ t_f]$), the control law $v_i = -\alpha \text{ sign } S_i(t)$ maintains

$$S(Z, \tau) = 0. \tag{18}$$

Consequently, the equality (8) minimizing (5) under the constraint $Z(t_f) = 0$, holds. Then, higher order sliding mode occurs. The convergence time is $t_0 + t_f$. From $t = t_0 + t_f$ the control task is to maintain the system trajectory on the origin. This objective is fulfilled by the control law $v_i = -\alpha \text{ sign } (S_{fi})$ which allows the continuation of the sliding on

$$S_f \quad = \quad S(Z, t_f) = 0. \tag{19}$$

The proposed algorithm can be expressed through the following sequence of steps.

Algorithm. After the determination of the equation of the optimal switching manifold (13), the control is described by

(i) At $t = 0$, if $S_0 \neq 0$, apply $v_i = -\alpha \text{ sign } (S_{0i})$,

(ii) If $S_0 = 0$, then $t = t_0$ (i.e. $\tau = 0$). Apply for any $t \in [t_0, \ t_0 + t_f]$ (i.e. $\tau \in [0, \ t_f]$) $v_i = -\alpha \text{ sign } (S_i(\tau))$.

(iii) If $t \in [t_0 + t_f, \ \infty[$, apply $v_i = -\alpha \text{ sign } (S_{fi})$. ■

IV. Control of synchronous motor

A. Model and uncertainties

The electrical and mechanical equations of a 3-phase permanent magnet synchronous motor can be expressed in the so-called (d, q)-frame by application of the Park transformation and described in [8]

$$\left\{\begin{array}{rcl}\dfrac{d\theta}{dt} & = & \omega \\ \dfrac{d\omega}{dt} & = & \dfrac{P}{J}[(L_d - L_q)i_d + \phi_f]i_q - \dfrac{f_v}{J}\omega - \dfrac{C_l}{J} \\ \dfrac{di_d}{dt} & = & -\dfrac{R_s}{L_d}i_d + P\dfrac{L_q}{L_d}\omega i_q + \dfrac{1}{L_d}v_d \\ \dfrac{di_q}{dt} & = & -P\dfrac{\Phi_f}{L_q}\omega - P\dfrac{L_d}{L_q}\omega i_d - \dfrac{R_s}{L_q}i_q + \dfrac{1}{L_q}v_q\end{array}\right. \tag{20}$$

where θ is the angular position of the motor shaft, ω the angular velocity of the motor shaft, i_d the direct current and i_q the quadrature current. ϕ_f is the flux of the permanent magnet, P the number of pole pairs, R_s the stator windings resistance, L_d and L_q the direct and quadrature stator inductances respectively. J is the rotor moment of inertia, f_v the viscous damping coefficient and C_l the load torque. v_d is the direct voltage and v_q is the quadrature voltage. The parameters R_s, L_d, L_q and f_v are supposed to vary with respect to their nominal values R_{s0}, L_{d0}, L_{q0} and f_{v0} (for instance, R_s has high variations due to the temperature). The formalization of these variations is stated through

$$\begin{array}{lll}\frac{P}{J}(L_d - L_q) = k_1 & \frac{P\phi_f}{J} = k_2 & -\frac{f_v}{J} = k_3 \\ -\frac{R_s}{L_d} = k_4 & P\frac{L_q}{L_d} = k_5 & \frac{1}{L_d} = k_6 \\ -\frac{P\phi_f}{L_q} = k_7 & -\frac{PL_d}{L_q} = k_8 & -\frac{R_s}{L_q} = k_9 \\ \frac{1}{L_q} = k_{10} & & \end{array} \tag{21}$$

where $k_i = k_{0i} + \delta k_i$ $(1 \leq i \leq 10)$ with k_{0i} the nominal value of the concerned parameter and δk_i is uncertainty on the concerned parameter such that $|\delta k_i| \leq \delta k_{0i} < |k_i|$, with δk_{0i} a known positive bound. Let x denote the state $x = [x_1 \ x_2 \ x_3 \ x_4]^T = [\theta \ \omega \ i_d \ i_q]^T$ and u the input $u = [u_1 \ u_2]^T = [v_d \ v_q]^T$. Then, a state space representation of the synchronous motor can be written as the following nonlinear system

$$\left\{\begin{array}{rcl}\dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & (k_1 x_3 + k_2)x_4 + k_3 x_2 - \frac{C_l}{J} \\ \dot{x}_3 & = & k_4 x_3 + k_5 x_2 x_4 + k_6 u_1 \\ \dot{x}_4 & = & k_7 x_2 + k_8 x_2 x_3 + k_9 x_4 + k_{10} u_2\end{array}\right. \tag{22}$$

with $x \in \mathcal{X} \subset I\!R^4$ and $u \in \mathcal{U} \subset I\!R^2$ such that $\mathcal{X} = \{x \in I\!R^4 \mid x_1 \in I\!R, |x_i| \leq x_{iMAX}, \ 2 \leq i \leq 4\}$ and $\mathcal{U} = \{u \in I\!R^2 \mid |u_i| \leq u_{iMAX}, \ 1 \leq i \leq 2\}$, x_{2MAX} the maximum value of the angular velocity, x_{3MAX} and x_{4MAX} the maximum values of the currents, and u_{1MAX} and u_{2MAX} the maximum values of the voltage inputs.

B. Problem statement

The aim is to design an appropriate control which guarantees robust performance in presence of parameters and load torque variations. The control objective is double. First, the rotor angular position $x_1 = \theta$ must track a reference trajectory x_{1ref}. Secondly, the nonlinear electromagnetic torque can be linearized to avoid reluctance effects and torque ripple. This objective is equivalent to constrain $x_3 = i_d$ to track a constant direct current reference $x_{3ref} = 0$.

C. Control design

The problem under interest in this section is to design a MIMO second order sliding mode controller for a permanent magnet synchronous motor. It is assumed that all state variables are available for measurement. The control goal is to steer to zero, in finite time, the sliding vector $\sigma = [\sigma_1 \ \sigma_2]^T$ defined as (with $e_1 = x_1 - x_{1ref}$ and $e_3 = x_3 - x_{3ref}$)

$$\sigma = \begin{bmatrix} e_3 \\ \ddot{e}_1 + \lambda_1 \dot{e}_1 + \lambda_2 e_1 \end{bmatrix} \tag{23}$$

where λ_1 and λ_2 are positive parameters such that $P(z) = \ddot{z} + \lambda_1 \dot{z} + \lambda_2 z$ is Hurwitz polynomial. Note that the relative degree of σ equals 1. In order to eliminate chattering phenomena, the second order sliding mode strategy exposed in Section 2 is applied. So, the control $\acute{u} = \dot{u}$ is used instead of the actual control $u(t)$. It turns out that $\acute{u}$ acts directly on $\ddot{\sigma}$. Then, the problem is to steer σ to zero by acting on its second derivative. Consider the second time derivative of σ, $\ddot{\sigma} = A + B\acute{u}$ where $\acute{u} = [\dot{u}_1 \ \dot{u}_2]^T$,

$$\begin{aligned} A &= \begin{bmatrix} A_{10} \\ A_{20} \end{bmatrix} + \begin{bmatrix} \delta A_1 \\ \delta A_2 \end{bmatrix} =: A_0 + \delta A, \\ B &= \begin{bmatrix} B_{110} & 0 \\ B_{210} & B_{220} \end{bmatrix} + \begin{bmatrix} \delta B_{11} & 0 \\ \delta B_{21} & \delta B_{22} \end{bmatrix} \\ &=: B_0 + \delta B. \end{aligned} \tag{24}$$

Expressions $A_{10},\ A_{20},\ B_{110},\ B_{210}$ and B_{220} are the well known nominal expressions, whereas the expressions δA_1, δA_2, δB_{11}, δB_{21} and δB_{22} contain all the uncertainties due to parameters and load torque variations (see the formal expressions in [11]). Now, using the static feedback

$$\acute{u} = B_0^{-1} \cdot [-A_0 + v] \tag{25}$$

where $[v_1, v_2]^T$ is the new control vector, it leads to [11]

$$\begin{bmatrix} \ddot{\sigma}_1 \\ \ddot{\sigma}_2 \end{bmatrix} = \begin{bmatrix} \widehat{A}_1 \\ \widehat{A}_2 \end{bmatrix} + \begin{bmatrix} \widehat{B}_{11} & 0 \\ \widehat{B}_{21} & \widehat{B}_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{26}$$

where $\begin{bmatrix} \widehat{B}_{11} & 0 \\ \widehat{B}_{21} & \widehat{B}_{22} \end{bmatrix} > 0$. In fact, $\bar{B}^{-1}A_0$, the first part of control (25) which is also the first derivative of the so-called *equivalent control* in the sliding mode context [21], allows to cancel partially the non-linearities and guaranteed that $\min(\widehat{B}_{22}) > \max(|\widehat{B}_{21}|)$. In this case, decoupling the MIMO problem into a set of single-input problems can be done. As x_2, x_3, x_4, and δk_i are assumed to be bounded , and under assumption that $|\delta B_{11}| < |B_{11}|$, $|\delta B_{21}| < |B_{21}|$ and $|\delta B_{22}| < |B_{22}|$, there exist positives constants C_1, C_2, $K_{11m},\ K_{11M},\ K_{22m},\ K_{22M}$ and K_{21} such that $|\widehat{A}_1| < C_1$, $0 < K_{11m} < \widehat{B}_{11} < K_{11M}$, $|\widehat{A}_2| < C_2$,$0 < K_{22m} < \widehat{B}_{22} < K_{22M}$ and $|\widehat{B}_{21}| < K_{21}$. Then, one can apply the higher order algorithm previously presented: the optimal sliding manifold and the control law are defined by (for $i = [1, 2]$)

$$\begin{aligned} S_i &= \dot{\sigma}_i + (Q_{22}^{-1} A_{12}^T P(t) + Q_{22}^{-1} Q_{12}^T \\ &\quad - Q_{22}^{-1} A_{12}^T V(t) H(t)^{-1} V(t)^T) \sigma_i \\ v_i &= -\alpha_i \cdot \mathrm{sign}(S_i). \end{aligned} \tag{27}$$

Then, it is possible to choose each component of the control v in accordance to Theorem 1, such that the sliding mode occurs on the intersection of $\sigma = 0$ and $\dot{\sigma} = 0$.

D. Implementation Results

The designed second order sliding mode controller is implemented on the experimental set-up located at IRCCyN's laboratory (Nantes, France) [24]. To implement the feedback controller, a real time controller board dSPACE DS1103 drives the PM synchronous motor. Another synchronous motor is coupled to the shaft of the PM motor in order to apply a load torque. Four sensors give measurements of phase currents and voltages. An optical encoder is used to measure the position of the motor. The PM motor is a DutymAx 95DSC060300 (Leroy Somer Co.) drive. Its parameters are $R_s = 3.3\Omega$, $L_d = 0.027H$, $L_q = 0.0339H$, $\phi_f = 0.341Wb$, $J = 0.00037kg.m^2$, $f_v = 0.0034N.m.s$ and $P = 3$. The maximum accepted values are a phase current equal to $= 6.0A$, load torque equal to $6N.m$, and angular velocity equal to $3000rpm$. The rotor inertia of the load synchronous motor is $J = 0.00223kg.m^2$. The angular position reference and the behavior of the load torque Cl applied to the synchronous motor are represented in Figure 1. The controller is synthesized so that the performances are kept despite the load torque applying and parameters variations ($\pm 50\%$ with respect to R_0, $\pm 25\%$ with respect to L_{d0} and L_{q0}, $\pm 20\%$ with respect to f_{v0}). The controller parameters are $\alpha_1 = -9\ e8$, $\alpha_2 = -3\ e5$, $t_f = 0.3s$ $Q_{11} = 2$, $Q_{12} = 0$ and $Q_{22} = 25\ e-7$. The experimental results are given in Figures 2-3. An excellent tracking is observed in Figure 2-a that shows the position tracking error does not exceed 0.01 rad in spite of the load torque and parameters variations. Figure 2-b displays the good convergence of the current i_d towards its desired value (0.0A). Figure 3 display the voltages v_d and v_q.

V. Conclusion

A methodology for the design of a robust arbitrary order sliding mode controller with a simple structure for a class of MIMO nonlinear uncertain systems has been proposed. The controller design combines standard sliding mode control with linear quadratic LQ one over a finite time interval with a fixed final state. The controller is able to steer to zero in finite time the outputs of any uncertain smooth MIMO minimum-phase dynamic system for which the outputs have the same relative degree, and for which the sliding mode order is the same for all the outputs. The effectiveness of the method is shown through experimental results of a permanent magnet synchronous motor control.

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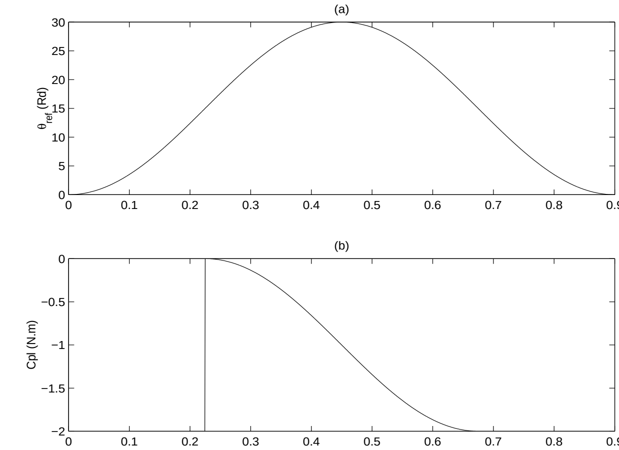

Fig. 1. (a) Benchmark angular position reference θ_{ref}, (b) Load torque versus time (sec.)

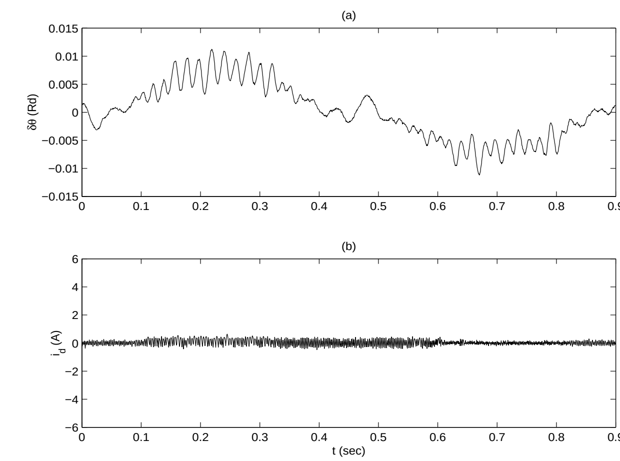

Fig. 2. (a) Position tracking error, (b) Current i_d versus time (sec.)

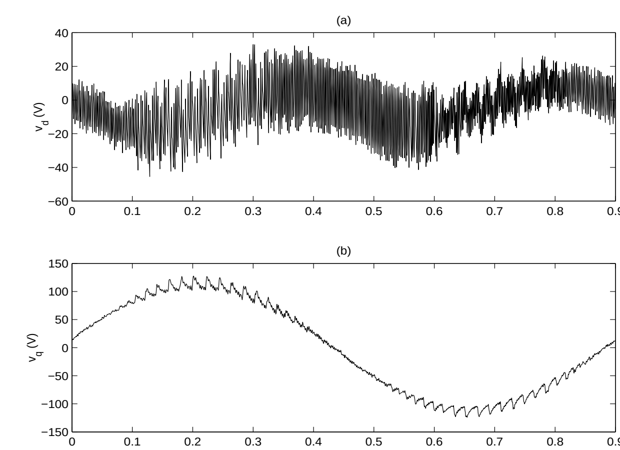

Fig. 3. (a) Voltage v_d, (b) Voltage v_q versus time (sec.)