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On existence of Maximal Solution for Infinite Dimensional Perturbed Algebraic Riccati Equations Associated to Markov Jump Linear Systems

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Abstract— Under the structural assumption of stochastic stability, we prove existence of maximal solution for a certain perturbed algebraic Riccati equation in infinite dimensional Banach space. The positive perturbation operator is as it appears in control problems involving Markov jump linear systems with infinite countable state space.

I. INTRODUCTION

An extensive literature can be found on the study of Markov jump linear systems (MJLS) for the case in which the state space of the Markov chain is finite (see, e.g., [1], [7]-[9], [16], [17], [20] and references therein). Recently, a research thrust on MJLS has been in the infinite countable case (see, [2], [3], [10], [11]).

In this paper, we contribute to this subject proving existence (and uniqueness) of maximal solution for an algebraic Riccati equation in infinite dimensional Banach space, where a positive operator acting as a perturbation appears. The space under concern is that of all infinite norm bounded sequences of complex matrices. This sort of problem has been treated in [6] in the finite dimensional case and using a contraction assumption (assumption 2.1 of this reference), originally introduced in [20]. We have discarded such condition. Instead, we use a certain structural concept of stabilizability. On the other hand, the positive operator, instead of being arbitrary as in [6], is as it appears in control problems involving MJLS with infinite countable state space. This has allowed us to use an equivalence result involving stabilizability, Lyapunov equations and the spectrum of a certain operator in infinite dimension (see [10] and [12]) to achieve the result of Section III.

Other previous results on maximal solution for MJLS with finite state space Markov chain are in [14] and [15].

II. NOTATIONS AND CONCEPTUAL PRELIMINARIES

As usual, $\mathbb{C}^n$ stands for the n-dimensional Euclidean space over the field of complex numbers $\mathbb{C}$. We set $\mathcal{S} = \mathbb{N} = \{1, 2, ...\}$. In the case of control problems involving Markov Jump Linear Systems, $\mathcal{S}$ corresponds to the state space of the Markov chain, as we shall see in Section III. We use the superscript * for conjugate transpose of a matrix. We denote $\mathbb{M}(\mathbb{C}^m, \mathbb{C}^n)$ the normed linear space of all n by m complex matrices and, for simplicity, write $\mathbb{M}(\mathbb{C}^n)$ whenever $n = m$. The notation $L \geq 0$ and $L > 0$ is adopted if a self-adjoint matrix is positive or nonnegative definite, respectively, and we write $\mathbb{M}(\mathbb{C}^n)^+ = \{L \in \mathbb{M}(\mathbb{C}^n); L = L^* \geq 0\}$. Furthermore, I_n stands for the identity operator in $\mathbb{M}(\mathbb{C}^n)$.

We denote $\|\cdot\|$ the Euclidean norm in $\mathbb{C}^n$ or the spectral induced norm in $\mathbb{M}(\mathbb{C}^n)$. We set $\mathcal{H}_1^{m,n}$ (resp. $\mathcal{H}_\infty^{m,n}$) the linear space of all infinite sequences of complex matrices $H = (H_1, H_2, ...)$, $H_i \in \mathbb{M}(\mathbb{C}^m, \mathbb{C}^n)$ such that $\sum_{i=1}^{\infty} \|H_i\| < \infty$ (resp. $\sup\{\|H_i\|, \ i \in \mathcal{S}\} < \infty$) and write $\mathcal{H}_1^n$ and $\mathcal{H}_\infty^n$ whenever $n = m$. For $H \in \mathcal{H}_1^{m,n}$ (resp. $H \in \mathcal{H}_\infty^{m,n}$) we define $\|H\|_1 = \sum_{i=1}^{\infty} \|H_i\|$ (resp. $\|H\|_\infty = \sup\{\|H_i\|, \ i \in \mathcal{S}\}$) the norm in the Banach space $(\mathcal{H}_1^{m,n}, \|\cdot\|_1)$ (resp. $(\mathcal{H}_\infty^{m,n}, \|\cdot\|_\infty)$).

We define the nonnegative sets $\mathcal{H}_1^{n^+} = \{H \in \mathcal{H}_1^n, H_i \in \mathbb{M}(\mathbb{C}^n)^+, \ i \in \mathcal{S}\}$ and $\mathcal{H}_\infty^{n^+} = \{H \in \mathcal{H}_\infty^n, \ H_i \in \mathbb{M}(\mathbb{C}^n)^+, \ i \in \mathcal{S}\}$, the strictly positive set $\breve{\mathcal{H}}_\infty^{n^+} = \{H \in \mathcal{H}_\infty^{n^+}, \ H_i > \alpha_H I$ for some $\alpha_H > 0, \ i \in \mathcal{S}\}$ and the sets $\mathcal{H}_1^{n*} = \{H \in \mathcal{H}_1^n, \ H_i^* = H_i, \ i \in \mathcal{S}\}$ and $\mathcal{H}_\infty^{n*} = \{H \in \mathcal{H}_\infty^n, \ H_i^* = H_i, \ i \in \mathcal{S}\}$. For $H = (H_1, H_2, ...)$ and $L = (L_1, L_2, ...)$ in $\mathcal{H}_1^{n*}$ or $\mathcal{H}_\infty^{n*}$, we say that $H \leqslant L$ if $H_i \leqslant L_i$ for each i in $\mathcal{S}$. We have that $H \leqslant L \Rightarrow \|H\|_1 \leqslant \|L\|_1$ and $\|H\|_\infty \leqslant \|L\|_\infty$. For $C = (C_1, C_2 ...) \in \mathcal{H}_\infty^n$, we denote $C^* = (C_1^*, C_2^* ...) \in \mathcal{H}_\infty^n$.

We denote $(l_1, \|\cdot\|_1)$ and $(l_\infty, \|\cdot\|_\infty)$ the spaces made up of all infinite sequences of complex numbers $x = (x_1, x_2, ...)$ such that $\|x\|_1 = \sum_{i=1}^{\infty} |x_i| < \infty$ and $\|x\|_\infty = \sup\{|x_i|, \ i = 1, 2, ...\} < \infty$, respectively.

Remark 1: It is easy to verify that $(\mathcal{H}_\infty^{m,n}, \|\cdot\|_\infty)$ and $(l_\infty, \|\cdot\|_\infty)$ are uniformly homeomorphic. Since $(l_\infty, \|\cdot\|_\infty)$ is a Banach space, $(\mathcal{H}_\infty^{m,n}, \|\cdot\|_\infty)$ is also a Banach space. The same stands for $(\mathcal{H}_1^{m,n}, \|\cdot\|_1)$ and $(l_1, \|\cdot\|_1)$.

For any complex Banach space Y, we denote $Blt(Y)$ the Banach space of all bounded linear transformations of Y into Y equipped with the uniform induced norm, which we shall also denote by $\|\cdot\|$, and for $L \in Blt(Y)$ we denote $\sigma(L)$ the spectrum of L.

We define the product of an element $A \in \mathcal{H}_\eta^{m,n}$ by

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another element $B \in \mathcal{H}_{\nu}^{q,m}$ by

$$AB = (A_1B_1, A_2B_2, ...), \tag{1}$$

where η and ν stands either for ∞ or 1. AB then belongs either to $\mathcal{H}_{\infty}^{q,n}$ or $\mathcal{H}_1^{q,n}$, as we shall see below. $\mathcal{H}_{\infty}^{n}$ equipped with (1) is a Banach algebra with unitary element $(I_n, I_n, ...)$.

Lemma 1: For every $A\in \mathcal{H}_{\infty}^{m,n}$, $B\in \mathcal{H}_1^{m,n}$, $C \in \mathcal{H}_{\infty}^{q,m}$ and $D \in \mathcal{H}_1^{q,m}$
(i) AC belongs to $\mathcal{H}_{\infty}^{q,n}$ and $\|AC\|_{\infty} \leqslant \|A\|_{\infty} \|C\|_{\infty}$,
(ii) AD belongs to $\mathcal{H}_1^{q,n}$ and $\|AD\|_1 \leqslant \|A\|_{\infty} \|D\|_1$ and
(iii) BC belongs to $\mathcal{H}_1^{q,n}$ and $\|BC\|_1 \leqslant \|B\|_1 \|C\|_{\infty}$.

Proof: Each entry of A_iC_i, A_iD_i and B_iC_i is well defined and, reminding definition (1), we have that
(i) $\|AC\|_{\infty} = \sup_{i\in\mathcal{S}} \|A_iC_i\| \leqslant \sup_{i\in\mathcal{S}} \|A_i\| \sup_{i\in\mathcal{S}} \|C_i\| = \|A\|_{\infty} \|C\|_{\infty}$,
(ii) $\|AD\|_1 = \sum_{i=1}^{\infty} \|A_iD_i\| \leqslant \sum_{i=1}^{\infty} \|A_i\| \|D_i\| \leqslant \|A\|_{\infty} \|D\|_1$ and
(iii) $\|BD\|_1 = \sum_{i=1}^{\infty} \|B_iD_i\| \leqslant \sum_{i=1}^{\infty} \|B_i\| \|D_i\| \leqslant \|B\|_1 \|D\|_{\infty}$. ■

Let us now consider $\hat{F} \in \mathcal{H}_{\infty}^{n}$ and $\Gamma = (\Gamma_1, \Gamma_2, ...) \in Blt(\mathcal{H}_1^n)$ such that $\Gamma_i(H) = \sum_{j=1, j\neq i}^{\infty} \lambda_{ji} H_j$ for $H = (H_1, H_2, ...)$, where $[\lambda_{ij}]_{i,j\in\mathcal{S}}$ is the infinitesimal matrix of a standard conservative Markov chain $\{\theta\}$ with values in $\mathcal{S}$, $\lambda_{ij} \geq 0$, $i \neq j$, $0 < -\lambda_{ii} = \sum_{j=1, j\neq i}^{\infty} \lambda_{ij} \leqslant cte < \infty$. For $Q = (Q_1, Q_2, ...) \in \mathcal{H}_1^n$, we define the operator $\mathcal{D} \in Blt\,(\mathcal{H}_1^n)$ such that $\mathcal{D}(Q) = (\mathcal{D}_1(Q), \mathcal{D}_2(Q), ...)$,

$$\mathcal{D}_i(Q) = F_iQ_i + Q_iF_i^* + \Gamma_i(Q), \quad i \in \mathcal{S},$$

and $F \in \mathcal{H}_{\infty}^{n}$ where $F_i = \hat{F}_i + \frac{1}{2}\lambda_{ii} I_n$ (to see that $\mathcal{D} \in Blt\,(\mathcal{H}_1^n)$ refer to [10]). Alternatively (see (1)), we may write

$$\mathcal{D}(Q) = FQ + QF^* + \Gamma(Q). \tag{2}$$

Now consider the differential equation

$$\dot{Q}(t) = \mathcal{D}(Q(t)), \quad t \geq 0. \tag{3}$$

It is closely related to Markov jump linear systems and describes the behavior of a version of the state correlation matrix running in these systems. In order to preserve the nomenclature in the MJLS scenario, we define the following L_2-type of stability:

Definition 1 (Stochastic Stabilizability (SS)): We say that system $(\hat{A}, B, \Gamma)$ is stochastically stabilizable (SS) if there exists a stabilizing $K \in \mathcal{H}_{\infty}^{n,m}$ in that, for any $Q(0) \in \mathcal{H}_1^n$,

$$\int_0^{\infty} \|Q(t)\|_1 \, dt < \infty, \tag{4}$$

where $Q(t) \in \mathcal{H}_1^n$ uniquely satisfies (3) with $\hat{F} = \hat{A} - BK$ (clearly $\hat{F} \in \mathcal{H}_{\infty}^{n}$). Also, we say that $(\hat{F},\Gamma)$ is stochastically stable (SS) if (4), with Q(t) given by (3), holds.

Definition 2: We say that S is a stabilizing solution of the Banach space perturbed algebraic Riccati equation (BPARE) (9) if S belongs to $\mathcal{H}_{\infty}^{n+}$, is a solution to the BPARE and $K = R^{-1}B^*S$ stabilizes $(\hat{A}, B, \Gamma)$ (clearly $K \in \mathcal{H}_{\infty}^{n,m}$).

Remark 2: It is worth noticing that Γ, whose matrix representation is $((\Lambda - \, diag(\lambda_{ii})) \otimes I_n)^*$, is the adjoint operator of χ in the finite dimensional case, more explicitly, the Hilbert space of finite sequences of matrices in $\mathbb{M}(\mathbb{C}^n)$.

III. PROBLEM STATEMENT AND MAIN THEOREM

We consider the perturbed Riccati operator $\mathcal{T}$, defined on the infinite dimensional Banach space $\mathcal{H}_{\infty}^{n}$ such that, for every $H = (H_1, H_2, ...) \in \mathcal{H}_{\infty}^{n}$, $\mathcal{T}(H) = (\mathcal{T}_1(H), \mathcal{T}_2(H), ...)$ where, for each $i \in \mathcal{S}$,

$$\mathcal{T}_i(H) = A_i^*H_i + H_iA_i + \chi_i(H) + Q_i - H_iB_iR_i^{-1}B_i^*H_i \tag{5}$$

Parameters are uniformly norm-bounded, more precisely, $B = (B_1, B_2, ...) \in \mathcal{H}_{\infty}^{m,n}$, $Q = (Q_1, Q_2, ...) \in \mathcal{H}_{\infty}^{n+}$ and $R = (R_1, R_2, ...) \in \mathcal{H}_{\infty}^{m+}$. We specify $\chi = (\chi_1, \chi_2, ...) \in Blt(\mathcal{H}_{\infty}^{n})$, a positive operator in that it maps $\mathcal{H}_{\infty}^{n*}$ into $\mathcal{H}_{\infty}^{n*}$ and $\mathcal{H}_{\infty}^{n+}$ into $\mathcal{H}_{\infty}^{n+}$, as it appears in control problems involving Markov jump linear systems with infinite countable state space $\mathcal{S}$ (see, e.g., [10] and references therein): $\chi_i(H) = \sum_{j=1, j\neq i}^{\infty} \lambda_{ij} H_j$, or else, viewing H as an infinite column of matrices, $\chi H = ((\Lambda - \, diag(\lambda_{ii})) \otimes I_n)H$, where $\Lambda = [\lambda_{ij}]_{i,j\in\mathcal{S}}$ and I_n stands for the identity operator in $\mathbb{M}(\mathbb{C}^n)$. Note that χ_i responds for the interconnection among the individual components $\mathcal{T}_i$. Furthermore, as part of the control-like specification, we define $A \in \mathcal{H}_{\infty}^{n}$ such that

$$A_i = \hat{A}_i + \frac{1}{2}\lambda_{ii} I_n \tag{6}$$

for arbitrary $\hat{A} = (\hat{A}_1, \hat{A}_2, ...) \in \mathcal{H}_{\infty}^{n}$. Since all elements in (5) are uniformly norm bounded on $i \in \mathcal{S}$ and $\mathcal{T}_i(H)^* = \mathcal{T}_i(H)$ if $H^* = H$, it follows easily that $\mathcal{T}$ maps $\mathcal{H}_{\infty}^{n}$ into $\mathcal{H}_{\infty}^{n}$ and $\mathcal{H}_{\infty}^{n^*}$ into $\mathcal{H}_{\infty}^{n^*}$.

In view of (1) and (5), $\mathcal{T}$ also writes

$$\mathcal{T}(H) = A^*H + HA + \chi(H) + Q - HBR^{-1}B^*H \tag{7}$$

with $R^{-1} := (R_1^{-1}, R_2^{-1}, ...)$.

Remark 3: In the above control problem scenario (i) $\hat{A}_i$ and B_i, $i \in \mathcal{S}$, stand for the parameters that drive the jump-linear dynamic $\dot{x}(t) = \hat{A}_{\theta(t)}x(t) + B_{\theta(t)}u(t)$ where $x(t) \in \mathbb{C}^n$ denotes the state vector, $u(t) \in \mathbb{C}^m$ the control input and, for $\theta(t) = i$, we have that $\hat{A}_{\theta(t)} = \hat{A}_i$ and $B_{\theta(t)} = B_i$ and (ii) Q and R are the associated state and control penalties.

Before we get into our theorem, let us state an equivalence result, which is proved in [10] and [12].

Lemma 2: The following assertions are equivalent.

(a) $(\hat{A}, B, \Gamma)$ is stochastically stabilizable (SS) with stabilizing K.

(b) Given any $V \in \breve{\mathcal{H}}_\infty^{n^+}$, there is $S \in \breve{\mathcal{H}}_\infty^{n^+}$, unique in $\mathcal{H}_\infty^{n*}$, satisfying the countably infinite set of perturbed coupled Lyapunov equations given by

$$(A_i - B_iK_i)^*S_i + S_i(A_i - B_iK_i)_i + \chi_i(S) + V_i = 0,\quad i \in \mathcal{S}, \tag{8}$$

(c) Given some $V \in \breve{\mathcal{H}}_\infty^{n^+}$, there is $S \in \breve{\mathcal{H}}_\infty^{n^+}$ satisfying (8).

(d) $\sup\{ Re\lambda : \lambda \in \sigma(\mathcal{D})\} < 0$, with $\mathcal{D}$ equipped with $\hat{F} = \hat{A} - BK$.

We say that $\bar{S}$ is maximal in some subset of $\mathcal{H}_\infty^{n*}$ if $\bar{S} \geq \hat{S}$ for every $\hat{S}$ in that subset. Clearly $\bar{S}$ is unique. Our main result reads as follows:

Theorem 1: Let $(\hat{A}, B, \Gamma)$ be SS. Then the BPARE

$$\mathcal{T}(S) = 0, \tag{9}$$

with $\mathcal{T}$ defined as above, has a (unique) maximal solution in the set of all solutions in $\mathcal{H}_\infty^{n*}$. Moreover this solution belongs to $\mathcal{H}_\infty^{n+}$ and is such that $\sup\{ Re\lambda : \lambda \in \sigma(\mathcal{D})\} \leqslant 0$, with $\mathcal{D}$ equipped with $\hat{F} = \hat{A} - BR^{-1}B^*S$.

Proof: Let us first write (9) as the following system.

$$(A - BK)^*S + S(A - BK) + K^*RK = -\chi(S) - Q \tag{10}$$

$$K = R^{-1}B^*S \tag{11}$$

Now, set some stabilizing K^1 for $(\hat{A}, B, \Gamma)$. It exists since $(\hat{A}, B, \Gamma)$ is SS. Starting from K^1 we construct the sequences $\{K^i\}_{i\in\mathbb{N}}$ and $\{S^i\}_{i\in\mathbb{N}}$ according to the following equations.

$$(A - BK^i)^*S^i + S^i(A - BK^i) + K^{i*}RK^i = -\chi(S^i) - Q \tag{12}$$

and

$$K^{i+1} = R^{-1}B^*S^i, \tag{13}$$

$i \in \mathcal{S}$. We use induction to prove that $\{S^i\}_{i\in\mathbb{N}}$ exists, is uniquely defined and nonincreasing and $\{K^i\}_{i\in\mathbb{N}}$ is stabilizing for $(\hat{A}, B, \Gamma)$. An essential property in this part of the proof is that, for arbitrarily fixed $S \in \mathcal{H}_\infty^{n*}$, $K_0 = R^{-1}B^*S$ is a point of minimum, in that, for every $K \in \mathcal{H}_\infty^{n,m}$

$$\begin{aligned}&(A - BK)^*S + S(A - BK) + K^*RK - U \\ &= (A - BK_0)^*S + S(A - BK_0) + K_0^*RK_0,\end{aligned} \tag{14}$$

where $U := (K - K_0)^*R(K - K_0) \in \breve{\mathcal{H}}_\infty^{n^+}$, $K_0 \neq K$.

So, for arbitrary $i \in \mathbb{N}$, assume that an element $K^i \in \mathcal{H}_\infty^{n,m}$ stabilizes $(\hat{A}, B, \Gamma)$. Note that $V^i := K^{i*}RK^i + Q$ belongs to $\breve{\mathcal{H}}_\infty^{n+}$. Then, from Lemma 2 ($(a) \Rightarrow (b)$), there exists a unique $S^i \in \breve{\mathcal{H}}_\infty^{n+}$ that satisfies

$$(A - BK^i)^*S^i + S^i(A - BK^i) + \chi(S^i) + V^i = 0, \tag{15}$$

or else, (12). Let us now show that $S^i \geq \hat{S}$, $\hat{S} \in \mathcal{H}_\infty^{n*}$ being an arbitrary solution of (9) or, equivalently, system (10)/(11). From the minimum property given by (14),

$$\begin{aligned}&(A - BK^i)^*\hat{S} + \hat{S}(A - BK^i) + K^{i*}RK^i \\ &- (K^i - \hat{K})^*R(K^i - \hat{K}) = -\chi(\hat{S}) - Q\end{aligned} \tag{16}$$

Subtracting (16) from (12), we have that

$$(A - BK^i)^*\phi^i + \phi^i(A - BK^i) + \chi(\phi^i) + W^i = 0, \tag{17}$$

$\phi^i := S^i - \hat{S}$ and $W^i := (K^i - \hat{K})^*R(K^i - \hat{K})$. Now, W^i belongs to $\breve{\mathcal{H}}_\infty^{n+}$ if $K^i \neq \hat{K}$ and by assumption K^i stabilizes $(\hat{A}, B, \Gamma)$. Then the solution of (17) is unique and belongs to $\breve{\mathcal{H}}_\infty^{n+}$, as Lemma 2 ($(a) \Rightarrow (b)$) shows. Hence this is the case of $\phi^i := S^i - \hat{S}$. Since both S^i and $\hat{S}$ belong to $\mathcal{H}_\infty^{n*}$, it follows that $S^i \geq \hat{S}$. If $K^i = \hat{K}$, then $S^i = \hat{S}$.

Again from the minimum property, $K^{i+1} = R^{-1}B^*S^i$ minimizes the left hand side of (12), so we have that

$$\begin{aligned}&(A - BK^{i+1})^*S^i + S^i(A - BK^{i+1}) + K^{i+1*}RK^{i+1} \\ &+ \chi(S^i) + Z^i = 0,\end{aligned} \tag{18}$$

$Z^i := (K^{i+1} - K^i)^*R(K^{i+1} - K^i) + Q \in \breve{\mathcal{H}}_\infty^{n+}$ if $K^{i+1} \neq K^i$. Hence, for some element in $\breve{\mathcal{H}}_\infty^{n+}$ (Z^i), there is an element in $\breve{\mathcal{H}}_\infty^{n+}$ (S^i) that satisfies (18). From Lemma 2 ($(c) \Rightarrow (a)$), K^{i+1} stabilizes $(\hat{A}, B, \Gamma)$. If $K^{i+1} = K^i$, the same conclusion is obvious.

For the step $i + 1$, we note as before that $V^{i+1} := K^{i+1*}RK^{i+1} + Q$ belongs to $\breve{\mathcal{H}}_\infty^{n+}$ and K^{i+1} stabilizes $(\hat{A}, B, \Gamma)$ so that, from Lemma 2 ($(a) \Rightarrow (b)$), there exists a unique $S^{i+1} \in \breve{\mathcal{H}}_\infty^{n+}$ that satisfies

$$\begin{aligned}&(A - BK^{i+1})^*S^{i+1} + S^{i+1}(A - BK^{i+1}) + \chi(S^{i+1}) \\ &+ V^{i+1} = 0,\end{aligned}$$

or else,

$$\begin{aligned}&(A - BK^{i+1})^*S^{i+1} + S^{i+1}(A - BK^{i+1}) \\ &+ K^{i+1*}RK^{i+1} = -\chi(S^{i+1}) - Q.\end{aligned} \tag{19}$$

Subtracting (19) from (18) we obtain

$$(A - BK^{i+1})^*\Delta^i + \Delta^i(A - BK^{i+1}) + \chi(\Delta^i) + U^i = 0, \tag{20}$$

where Δ^i exists and is defined as $\Delta^i := S^i - S^{i+1}$ and $U^i := (K^{i+1} - K^i)^* R (K^{i+1} - K^i)$. Now, U^i belongs to $\breve{\mathcal{H}}_\infty^{n+}$ if $K^{i+1} \neq K^i$ and K^{i+1} stabilizes $(\hat{A}, B, \Gamma)$, so that, using Lema 2 ($(a) \Rightarrow (b)$), the solution of (20) is unique and belongs to $\breve{\mathcal{H}}_\infty^{n+}$. Hence this is the case of Δ^i. Since both S^i and S^{i+1} belong to $\mathcal{H}_\infty^{n*}$, it follows that $S^i \geq S^{i+1}$. Clearly, if $K^{i+1} = K^i$,then $S^i = S^{i+1}$. This completes the induction.

We shall now show that $\{S^i\}_{i \in \mathbb{N}}$ converges to the maximal solution of (9) or, equivalently, system (10)/(11), and that the other assertions of the theorem follows. Since $\{S^i\}_{i \in \mathbb{N}}$ is nonincreasing and bounded from below by zero, we have, from a finite dimensional result on nonnegative matrices, that there exists the limit in $\mathbb{M}(\mathbb{C}^n)$

$$S_j := \lim_{i \to \infty} S_j^i, \quad \forall j \in \mathcal{S} \tag{21}$$

From this and the results obtained above, $S := (S_1, S_2, ...)$ is such that $S \geq \hat{S}$, $\hat{S} \in \mathcal{H}_\infty^{n*}$ being an arbitrary solution of (9). Since S^i belongs to $\breve{\mathcal{H}}_\infty^{n+}$ then $S_j \geq 0$ $\forall j \in \mathcal{S}$. Moreover, we justify that S belongs to $\mathcal{H}_\infty^{n}$ noting that $\{\|S^i\|_\infty\}_{i \in \mathbb{N}}$ is nonincreasing so that $\|S^i\|_\infty$ is uniformly bounded on i, say by c_1. From (21) and taking any $\varepsilon > 0$, there are n_j such that $\left| \|S_j\| - \|S_j^i\| \right| \leqslant \|S_j^i - S_j\| \leqslant \varepsilon, \quad \forall i \geq n_j,\ j \in \mathbb{N}$, and so

$$\|S_j\| \leqslant \|S_j^i\| + \varepsilon \leqslant \sup_{j \in \mathbb{N}} \|S_j^i\| + \varepsilon = \|S^i\|_\infty + \varepsilon \leqslant c_1 + \varepsilon$$

$\forall j \in \mathbb{N}$.

Hence $S \in \mathcal{H}_\infty^{n+}$. Now, from (13), there exists the limit $K_j := \lim_{i \to \infty} K_j^{i+1} = \lim_{i \to \infty} R_j^{-1} B_j^* S_j^i = R_j^{-1} B_j^* S_j$, and $K := (K_1, K_2, ...) \in \mathcal{H}_\infty^{n,m}$. Also, using a monotonicity property in finite dimensional spaces,

$$\chi_j(S^i) = \lim_{M \to \infty} \sum_{r=1, r \neq i}^{M} \lambda_{ir} S_r^i \to \lim_{M \to \infty} \sum_{r=1, r \neq i}^{M} \lambda_{ir} S_r$$
$$=: \chi_j(S) \quad \text{as } i \to \infty.$$

So, passing (12) and (13) to the limit for an arbitrarily fixed entry $j \in \mathcal{S}$, we have that

$$\mathrm{K}_j = R_j^{-1} B_j^* S_j,$$

or else, S satisfies (10)/(11). Now, K^i, $i \in \mathcal{S}$, stabilize $(\hat{A}, B, \Gamma)$ and so Lemma (2) ($(a) \Rightarrow (d)$) gives us that $\sup\{\ Re\lambda : \lambda \in \sigma(\mathcal{D}^i)\} < 0$, where $\mathcal{D}^i$ is defined as (2) with $\hat{F}^i = \hat{A} - BK^i$. From the continuity property of the spectrum, $\sup\{\ Re\lambda : \lambda \in \sigma(\mathcal{D}^i)\} \leq 0$. ■

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