Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Approximate Solutions to Nonlinear Fluid Networks with Periodic Inputs

Olga I. Koroleva and Miroslav Krstić

***Abstract*— In this paper we use Kirchhoff's laws and pipe flow dynamics equations to describe a fluid flow network in the form of a nonlinear differential equation with a periodic right hand side. We apply the averaging method to find an approximate solution of this equation and analyze its stability properties. The approximate solution consists of three parts: a mean flow part due to the resistive effects of branches, a time-periodic part due to "inductive" effects, and a mean flow average correction due to the interaction of nonlinear and time varying effects. We present an example that may help explain the processes participating in the development of venous diseases. In particular, it is shown that the widening of a branch in a venous network leads to an increase in the AC flow and decrease in the DC flow through that branch, thus increasing the stress on venous valves, and consequently leading to further increase in the effective width of the vein.**

I. INTRODUCTION

We consider a fluid flow network driven by an ideal current (flow) source (generator). Combining Kirchhoff's laws for flows and pressure laws with the equations for pipe flow dynamics, we get a fluid flow network model in the form of a nonlinear differential equation with a periodic right hand side $\dot{Q} = \varepsilon f(t, Q, \varepsilon)$, where Q is the flow rate. The exact closed-form solution of this equation can not be found. Applying the averaging method, we find an approximate solution which is in closed form and can be found using algebraic calculations.

Each network can be divided into a set of *tree* branches, which connect all the nodes without creating loops, and the complement of the tree, called co-tree, whose branches are called *links*. From the conservation of mass in the nodes it follows that flow rates in the branches are not independent, so we choose the flows in the links as independent variables, and the flows in the tree branches are a function of those in the links.

The paper is organized as follows: in Section II we consider a simple introductory example, in Section III we derive a minimal model of the fluid flow network, using pipe flow dynamics equations and Kirchhoff's laws, in Section IV we do averaging analysis of the network. In Section V we consider three examples. First, a network with two branches and a generator branch, for which we can find closed-form solution; second, a network with four branches and a generator branch, for which we can find closed-form solution for a constant input and then apply the main result for a periodic input; third, a network with five branches and a generator branch, for which we calculate the approximate solution numerically.

Olga I Koroleva is with Department of MAE, University of California at San Diego, La Jolla, CA 92093-0411, USA. Fax: (858) 822–3107. Phone: (858) 822–1936. olga@ucsd.edu

Miroslav Krstić is with Department of MAE, University of California at San Diego, La Jolla, CA 92093-0411, USA. Fax: (858) 822–3107. Phone: (858) 822–1374. krstic@ucsd.edu

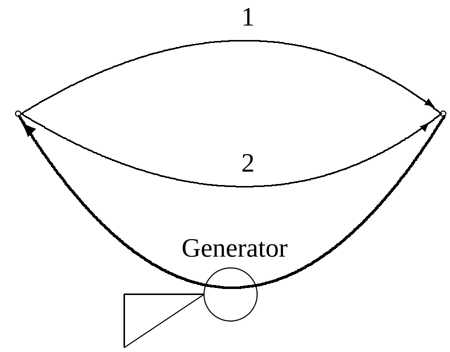

Fig. 1. Fluid flow network with a generator and two branches.

II. INTRODUCTORY EXAMPLE

Consider a network consisting of two parallel branches, as shown in Fig.1, with

$$Q_{in} = Q_1 + Q_2, \quad (1)$$
$$H_1 = H_2. \quad (2)$$

As will be explained later, the dynamic equations of the branches are

$$T_1\dot{Q}_1 + R_1Q_1^2 = H_1 \quad (3)$$
$$T_2\dot{Q}_2 + R_2Q_2^2 = H_2. \quad (4)$$

With (1) and (2) we obtain

$$T_1\dot{Q}_1 + R_1Q_1^2 = T_2\dot{Q}_{in} - T_2\dot{Q}_1 + R_2(Q_{in} - Q_1)^2. \quad (5)$$

In the case when there are no resistances, $R_1 = R_2 = 0$ we get a linear differential equation

$$(T_1 + T_2)\dot{Q}_1 = T_2 a\omega \cos(\omega t), \quad (6)$$

whose solution (for a zero initial condition) is

$$Q_1(t) = \frac{T_2}{T_1 + T_2} a \sin(\omega t). \quad (7)$$

In the static case, i.e., when $a = T_1 = T_2 = 0$, (5) becomes a quadratic equation

$$(R_1 - R_2)Q_1^2 + 2R_2Q_0Q_1 - R_2Q_0^2 = 0. \quad (8)$$

The solution of this equation is

$$Q_1 = \frac{\sqrt{R_2}}{\sqrt{R_1} + \sqrt{R_2}} Q_0. \quad (9)$$

0-7803-8335-4/04/$17.00 ©2004 AACC

Adding (7) and (9) we get an $O(a)$ approximation of the solution

$$\hat{Q}_1(t) = \frac{\sqrt{R_2}}{\sqrt{R_1}+\sqrt{R_2}}Q_0 + \frac{T_2}{T_1+T_2}a\sin(\omega t). \quad (10)$$

As we shall show in Section V-B, the $O(a^2)$ approximation is

$$\begin{aligned}\overline{Q}_1(t) \quad = \quad & \frac{\sqrt{R_2}}{\sqrt{R_1}+\sqrt{R_2}}Q_0 + \frac{T_2}{T_1+T_2}a\sin(\omega t) \\ & + \frac{a^2}{4Q_0\sqrt{R_1R_2}}\frac{R_2T_1^2 - R_1T_2^2}{(T_1+T_2)^2}. \quad (11)\end{aligned}$$

The third term is the result of combined nonlinear and time varying effects. It is obtained by the method of averaging. The rest of this paper develops this idea for general networks and discusses a possible application.

III. PIPE FLOW DYNAMICS AND KIRCHHOFF'S LAWS FOR FLUID FLOW NETWORKS

In this section we present the basic components of a model of a fluid flow network. We consider a network driven by a single ideal current source with flow $Q_{in}(t)$.

We first introduce the dynamical equation of one branch. For simplicity, we make the following assumptions: A1. the fluid is incompressible; A2. the temperatures in all branches are identical. Under assumptions A1 and A2, one branch of the fluid flow network is described with the following equations [4], [5], [6], [7]

$$T_j\frac{dQ_j}{dt} + R_j\,|Q_j|\,Q_j = H_j, \quad (12)$$

where Q_j is flow through a branch j, R_j are aero/hydro dynamic resistances, H_j are pressure drops of the branches, T_j are inertia coefficients, $j = 1, \cdots, n$ and n is the number of network branches (excluding the generator branch).

We write this in vector form as

$$T\dot{Q} = -Q_D^2R + H, \quad (13)$$

where $T = \text{diag}\{T_j\}$, $R = \text{col}\{R_j\}$ and

$$Q_D^2 = \text{diag}\{Q_j\,|Q_j|\}. \quad (14)$$

Let n_c denote the number of nodes. Then $l = n - n_c + 1$ is the number of links (excluding the generator branch) and $n - l$ is the number of tree branches.

Like an electrical network, a fluid flow network must satisfy Kirchhoff's current law, i.e., the flow out of any node is equal to the flow into that node. Mathematically, Kirchhoff's current law for fluid flow networks can be expressed as:

$$E_{Q_{in}}\begin{bmatrix} Q_{in} \\ Q \end{bmatrix} = 0, \quad (15)$$

or

$$\sum_{j=1}^{n} E_{Qij}Q_j + e_{Q_{in_i}}Q_{in} = 0, \quad i = 1, \cdots, n-l, \quad (16)$$

where $n - l + 1$ is the number of nodes (of which one is a "reference" node and is not represented in (16)), Q is a vector of flow quantities, $E_{Q_{in}} = [e_{Q_{in}}\ E_Q]$, and $E_Q = [E_{Qij}]$ is a full rank matrix of order $(n-l)\times n$ where the values of E_{Qij} are defined as follows: $E_{Qij} = 1$ if branch j is connected to node i and the flow goes away from node i, $E_{Qij} = -1$ if it goes into node i, $E_{Qij} = 0$ if branch j is not connected to node i, and e_{Qin} is an $(n-l)\times 1$ vector where the values e_{Qin_i} are defined as follows: if generator branch is connected to node i and the flow goes away from node i then $e_{Qin_i} = 1$, if the flow goes into node i then $e_{Qin_i} = -1$, and $e_{Qin_i} = 0$ if generator branch is not connected to node i.

Similarly, the fluid flow network also satisfies Kirchhoff's voltage law, i.e., the sum of the pressure drops around any loop in the network must be equal to zero, or mathematically,

$$E_HH = 0, \quad (17)$$

or

$$\sum_{j=1}^{n} E_{Hij}H_j = 0, \quad i = 1, \cdots, l, \quad (18)$$

where H_j is the pressure drop of the branch j, H is a vector of pressure drops, $E_H = [E_{Hij}]$ is an $l\times n$ mesh matrix, in which each mesh (loop) is formed by a link and a unique chain in the tree connecting the two nodes of the link. The elements of E_{Hij} are defined as follows: $E_{Hij} = 1$ if branch j is contained in mesh i and has the same direction, $E_{Hij} = -1$ if branch j is contained in mesh i and has the opposite direction, $E_{Hij} = 0$ if branch j is not contained in mesh i.

IV. MAIN RESULT

In order to establish a dynamic model of minimal order, one has to find independent variables as states of the system. We take the flows of link (co-tree) branches as state variables. We also include the generator branch into the set of links since its flow is given:

$$Q_{in}(t) = Q_0 + a\sin\omega t, \quad (19)$$

For convenience of analysis, we label the link branches (except the generator branch) from 1 to l. Define

$$Q = \begin{bmatrix} Q_c \\ Q_a \end{bmatrix}, \quad H = \begin{bmatrix} H_c \\ H_a \end{bmatrix}, \quad (20)$$

so that Q_c and H_c vectors describe flow and pressure drop, respectively, in the links, excluding the generator branch, and Q_a and H_a vectors describe them in the tree branches.

The matrices E_H and $E_{Q_{in}}$ can be split into blocks,

$$E_H = [E_{Hc}\quad E_{Ha}], \quad (21)$$

$$E_{Q_{in}} = [e_{Q_{in}}\quad E_{Qc}\quad E_{Qa}], \quad (22)$$

where [2], [5], [6]

$$E_{Qa} = I_{(n-l)\times(n-l)},\ \ E_{Hc} = I_{l\times l},\ \ E_{Ha} = -E_{Qc}^T. \quad (23)$$

Hence, the structure of the network can be expressed in the matrix form as

$$E=\begin{bmatrix} 0 & I & -E_{Qc}^T \\ e_{Q_{in}} & E_{Qc} & I \end{bmatrix}. \tag{24}$$

Furthermore,

$$T=\begin{bmatrix} T_c & 0 \\ 0 & T_a \end{bmatrix}, \quad R=\begin{bmatrix} R_c^T & R_a^T \end{bmatrix}^T. \tag{25}$$

We are ready now to state the main result.

Theorem 4.1: Let

$$\begin{aligned} T_0(T) &= T_c+E_{Qc}^T T_a E_{Qc}, & (26)\\ B_c(T,E) &= -T_0^{-1}E_{Qc}^T T_a e_{Qin}, & (27)\\ B_a(T,E) &= -(I-E_{Qc}T_0^{-1}E_{Qc}^T T_a)e_{Qin}, & (28)\\ U(R,T,E) &= \mathrm{col}\left\{B_{c_i}^2 R_{c_i}\right\}-E_{Qc}^T\mathrm{col}\left\{B_{a_i}^2 R_{a_i}\right\}, & (29)\\ V(R,E,Q_0) &= \mathrm{diag}\left\{Q_{c0_i}R_{c_i}\right\}+E_{Qc}^T W, & (30) \end{aligned}$$

where

$$W=\left\{E_{Qc_{ij}}(-E_{Qc_i}Q_{c0}-e_{Qin_i}Q_0)R_{a_i}\right\}_{(n-l)\times l}, \tag{31}$$

and $Q_{c0}(R,E,Q_0)$ denotes a solution to the l-dimensional quadratic equation

$$Q_{c0D}^2 R_c-E_{Qc}^T\mathrm{diag}\{(E_{Qc_i}Q_{c0}+e_{Qin_i}Q_0)^2\}R_a=0 \tag{32}$$

such that V is nonsingular and $-T_0^{-1}V$ is Hurwitz. Then for a given $Q_0>0$, for sufficiently small a and sufficiently large ω the solutions of the system (12)-(19) locally exponentially converge to an $O\left(\frac{1}{\omega}+a^4\right)$ neighborhood of

$$\begin{aligned} \overline{Q}_c(t) &= Q_{c0}-\frac{a^2}{4}V^{-1}U+B_c a\sin\omega t, & (33)\\ \overline{Q}_a(t) &= (-E_{Qc}Q_{c0}-e_{Qin}Q_0)+\frac{a^2}{4}E_{Qc}V^{-1}U \\ &\quad +B_a a\sin\omega t. & (34) \end{aligned}$$

Proof: With (20), (22) the flow rates through tree branches can be expressed by flows through links:

$$\begin{aligned} Q_a &= -e_{Qin}Q_{in}-E_{Qc}Q_c, & (35)\\ \dot{Q}_a &= -e_{Qin}\dot{Q}_{in}-E_{Qc}\dot{Q}_c. & (36) \end{aligned}$$

From branch dynamics equation (13), with (20) and (36) we get

$$\begin{aligned} \begin{bmatrix} H_c \\ H_a \end{bmatrix} &= \begin{bmatrix} T_c\dot{Q}_c \\ -T_a e_{Q_{in}}\dot{Q}_{in}-T_a E_{Qc}\dot{Q}_c \end{bmatrix} \\ &\quad +\begin{bmatrix} Q_{cD}^2 R_c \\ Q_{aD}^2(Q_c,Q_{in})R_a \end{bmatrix}. & (37) \end{aligned}$$

Rewrite (18) as

$$\begin{aligned} 0 &= E_{Hc}H_c+E_{Ha}H_a=H_c-E_{Qc}^T H_a \\ &= T_c\dot{Q}_c+Q_{cD}^2R_c+E_{Qc}^T T_a E_{Qc}\dot{Q}_c \\ &\quad -E_{Qc}^T Q_{aD}^2 R_a+E_{Qc}^T T_a e_{Q_{in}}\dot{Q}_{in}, & (38) \end{aligned}$$

or, rearranging this

$$-T_0\dot{Q}_c-E_{Qc}^T T_a e_{Q_{in}}\dot{Q}_{in}=Q_{cD}^2R_c-E_{Qc}^T Q_{aD}^2 R_a. \tag{39}$$

where T_0 is invertible [5]. Denote

$$\begin{aligned} \widetilde{Q}_c &= Q_c+T_0^{-1}E_{Qc}^T T_a e_{Q_{in}}a\sin(\omega t)-Q_{c0} \\ &= Q_c-Q_{c0}-B_c a\sin(\omega t). & (40) \end{aligned}$$

Then (39) can be rewritten as

$$-T_0\dot{\widetilde{Q}}_c=Q_{cD}^2R_c-E_{Qc}^T Q_{aD}^2 R_a. \tag{41}$$

Denote the RHS of (41) by

$$\begin{aligned} f\left(\omega t,\widetilde{Q}_c,a^2,\frac{1}{\omega}\right) &= Q_{cD}^2R_c-E_{Qc}^T Q_{aD}^2R_a \\ &= \mathrm{diag}\{2Q_{c0_i}(\widetilde{Q}_{c_i}+B_{c_i}a\sin(\omega t)) \\ &\quad +(\widetilde{Q}_{c_i}+B_{c_i}a\sin(\omega t))^2\}R_c \\ &\quad -E_{Qc}^T\mathrm{diag}\{2(e_{Qin_i}Q_0 \\ &\quad +E_{Qc_i}(\widetilde{Q}_c+Q_{c0})) \\ &\quad \times(e_{Qin_i}+E_{Qc_i}B_c)a\sin\omega t\}R_a \\ &\quad -E_{Qc}^T\mathrm{diag}\{(e_{Qin_i}+E_{Qc_i}B_c)^2 \\ &\quad \times a^2\sin^2\omega t\}R_a \\ &\quad -E_{Qc}^T\mathrm{diag}\{(E_{Qc_i}\widetilde{Q}_c)^2 \\ &\quad +2E_{Qc_i}\widetilde{Q}_c(e_{Qin_i}Q_0 \\ &\quad +E_{Qc_i}Q_{c0})\}R_a \\ &\quad +f_1(Q_{c0}), & (42) \end{aligned}$$

where

$$\begin{aligned} f_1(Q_{c0}) &= Q_{c0D}^2R_c-E_{Qc}^T\mathrm{diag}\{(E_{Qc_i}Q_{c0} \\ &\quad +e_{Qin_i}Q_0)^2\}R_a=0 & (43) \end{aligned}$$

according to (32). Equation (41) becomes

$$\dot{\widetilde{Q}}_c=-T_0^{-1}f\left(\omega t,\widetilde{Q}_c,a^2,\frac{1}{\omega}\right), \tag{44}$$

which rewrite (44) as

$$\frac{d\widetilde{Q}_c}{d(\omega t)}=-\frac{1}{\omega}T_0^{-1}f\left(\omega t,\widetilde{Q}_c,a^2,\frac{1}{\omega}\right). \tag{45}$$

Next we calculate

$$\begin{aligned} f_{av}(\widetilde{Q}_c,a^2) &= \frac{1}{2\pi}\int_0^{2\pi}f(\omega t,\widetilde{Q}_c,a^2,0)d(\omega t) \\ &= -E_{Qc}^T\mathrm{diag}\{(E_{Qc_i}\widetilde{Q}_c)^2 \\ &\quad +2E_{Qc_i}\widetilde{Q}_c(e_{Qin_i}Q_0+E_{Qc_i}Q_{c0})\}R_a \\ &\quad +\mathrm{diag}\{2Q_{co_i}\widetilde{Q}_{c_i}+\widetilde{Q}_{c_i}^2\}R_c \\ &\quad +\frac{a^2}{2}\mathrm{diag}\{B_{c_i}^2\}R_c \\ &\quad -\frac{a^2}{2}E_{Qc}^T\mathrm{diag}\left\{B_{a_i}^2\right\}R_a. & (46) \end{aligned}$$

With the definitions of U and V given by (29), (30) we can rewrite (46) as

$$\begin{aligned} f_{av}(\widetilde{Q}_c,a^2) &= 2V\widetilde{Q}_c+\frac{a^2}{2}U-E_{Qc}^T\mathrm{diag}\{(E_{Qc_i}\widetilde{Q}_c)^2\}R_a \\ &\quad +\mathrm{diag}\{\widetilde{Q}_{c_i}^2\}R_c \\ &= 2V\widetilde{Q}_c+\frac{a^2}{2}U+\mathrm{col}\{\widetilde{Q}_c^T Z_i\widetilde{Q}_c\}, & (47) \end{aligned}$$

where

$$Z_i = -\sum_{j=1}^{n-l} E_{Qc_{ji}} E_{Qc_j}^T E_{Qc_j} R_{a_j} + \text{diag}\{0,\ldots,0,R_{c_i},0,\ldots,0\}. \quad (48)$$

So, the average system is

$$\frac{d\widetilde{Q}_c}{d(\omega t)} = -\frac{1}{\omega} T_0^{-1} f_{av}(\widetilde{Q}_c, a^2). \quad (49)$$

To find the equilibrium points of (49) one needs to find the solution $\widetilde{Q}_c(a^2)$ of $f_{av}(\widetilde{Q}_c, a^2) = 0$. By the implicit function theorem, since V is assumed to be invertible, there exists such a solution. It can be written as

$$\widetilde{Q}_c^{av} = -\frac{a^2}{4} V^{-1} U + O(a^4). \quad (50)$$

The Jacobian of the average system (49) at $\widetilde{Q}_c^{av}$ is

$$\begin{aligned} J &= -\frac{2}{\omega} T_0^{-1} \left(V + \begin{bmatrix} Z_1 \widetilde{Q}_c^{av} \\ \vdots \\ Z_l \widetilde{Q}_c^{av} \end{bmatrix} \right) \\ &= -\frac{2}{\omega} T_0^{-1} \left(V - \frac{a^2}{4} \begin{bmatrix} (V^{-1}U + O(a^2))^T Z_1 \\ \vdots \\ (V^{-1}U + O(a^2))^T Z_l \end{bmatrix} \right) \\ &= -\frac{2}{\omega} T_0^{-1} V + O(a^2). \end{aligned} \quad (51)$$

Since $-T_0^{-1}V$ is Hurwitz, for sufficiently small a the Jacobian J will also be Hurwitz. By the averaging theorem [3] there exists an exponentially stable periodic solution $\widetilde{Q}_c^{2\pi/\omega}(t)$ of period $2\pi/\omega$ in the $1/\omega$-neighborhood of the average equilibrium $\widetilde{Q}_c^{av}$, that is,

$$\begin{aligned} \widetilde{Q}_c(t) &= \widetilde{Q}_c^{2\pi/\omega}(t) + \epsilon^{-t} \\ &= \widetilde{Q}_c^{av} + O\left(\frac{1}{\omega}\right) + \epsilon^{-t} \\ &= -\frac{a^2}{4} V^{-1} U + O\left(\frac{1}{\omega} + a^4\right) + \epsilon^{-t}, \end{aligned} \quad (52)$$

where ϵ^{-t} denotes exponentially decaying terms. Rewrite (40) as

$$Q_c(t) = \widetilde{Q}_c(t) + Q_{c0} + B_c a \sin(\omega t). \quad (53)$$

With (50) this becomes

$$Q_c(t) = Q_{c0} - \frac{a^2}{4} V^{-1} U + B_c a \sin(\omega t) + O\left(\frac{1}{\omega} + a^4\right) + \epsilon^{-t}. \quad (54)$$

Substitution of (54) into (35) gives

$$Q_a(t) = -e_{Qin} Q_0 - E_{Qc} Q_{c0} + \frac{a^2}{4} E_{Qc} V^{-1} U + B_a a \sin \omega t + O\left(\frac{1}{\omega} + a^4\right) + \epsilon^{-t}. \quad (55)$$

■

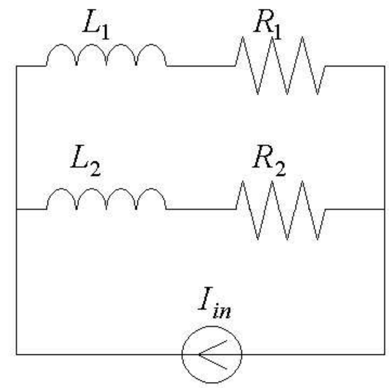

Fig. 2. Electrical circuit with two branches and current generator.

Remark 4.1: The vector field

$$f_{av}(\widetilde{Q}_c, a^2) = \text{col}\{\widetilde{Q}_c^T Z_i \widetilde{Q}_c\} + 2V\widetilde{Q}_c + \frac{a^2}{2} U, \quad (56)$$

is a vector-valued quadratic form in $\widetilde{Q}_c$. In the scalar case the solution to the quadratic equation $f_{av}(\widetilde{Q}_c, a^2) = 0$ would be explicit. While in some special multivariable cases an explicit solution might be possible, providing an exact average equilibrium, in general only an approximate solution (for small a) can be obtained. □

Remark 4.2: It is worth noting that in (54), (55) the first (respective) terms are due to the resistive part of the network (R), the third (sinusoidal) terms are due to the "inductive" part of the network (T), and the second terms are due to both resistance and inductivity. □

V. EXAMPLES

A. Two Branch Electric Circuit

Consider the circuit in Figure 2, which is driven by

$$I_{in}(t) = I_0 + a \sin(\omega t). \quad (57)$$

The Laplace transform of the current in the first branch is

$$I_1(s) = \frac{R_2 + L_2 s}{R_1 + R_2 + (L_1 + L_2)s} I_{in}(s), \quad (58)$$

The time response of the current in the first branch can be calculated as

$$\begin{aligned} I_1(t) = & \frac{R_2}{R_1 + R_2} I_0 \\ & + \frac{L_2 R_1 - L_1 R_2}{(R_1 + R_2)(L_1 + L_2)} I_0 e^{-\frac{R_1 + R_2}{L_1 + L_2} t} \\ & - \frac{L_2 R_1 - L_1 R_2}{(R_1 + R_2)^2 + \omega^2 (L_1 + L_2)^2} a\omega e^{-\frac{R_1 + R_2}{L_1 + L_2} t} \\ & + \frac{(R_1 + R_2) R_2 + \omega^2 (L_1 + L_2) L_2}{(R_1 + R_2)^2 + \omega^2 (L_1 + L_2)^2} a \sin(\omega t) \\ & + \frac{L_2 R_1 - L_1 R_2}{(R_1 + R_2)^2 + \omega^2 (L_1 + L_2)^2} a\omega \cos(\omega t). \end{aligned} \quad (59)$$

For large ω and t (and for any a), (59) becomes

$$I_1(t) \approx \frac{R_2}{R_1+R_2}I_0 + \frac{L_2}{L_1+L_2}a\sin(\omega t). \quad (60)$$

Thus, the DC response depends only on the resistive effects, and, for fast forcing, the AC response depends only on the inductive effects. As we shall see in the next section, the former is not the case for the fluid networks, where the resistive effect is nonlinear.

B. Two Branch Fluid Network

Let us now consider a fluid network example where we can calculate the flows in closed form. The network consists of two parallel branches, as shown on Fig.1. Branch 1 is the link and branch 2 is the tree of the network. Thus

$$T_c = T_1, \quad T_a = T_2, \quad (61)$$

$$R_c = R_1, \quad R_a = R_2, \quad (62)$$

and, from (1), (2) we get

$$\begin{array}{ll} E_{Hc} = 1, & E_{Ha} = -1, \\ E_{Qc} = 1, & E_{Qa} = 1, \\ e_{Qin} = -1. & \end{array} \quad (63)$$

Following the procedure given in the statement of Theorem 4.1, we get

$$\begin{aligned} T_0 &= T_1 + T_2, && (64)\\ B_c &= \frac{T_2}{T_1+T_2}, && (65)\\ B_a &= \frac{T_1}{T_1+T_2}, && (66)\\ U &= \frac{T_2^2R_1 - T_1^2R_2}{T_1+T_2}, && (67)\\ W &= (Q_{c0}-Q_0)R_2, && (68)\\ V &= Q_{c0}R_1 - W = Q_{c0}(R_1-R_2) - Q_0R_2, && (69) \end{aligned}$$

where Q_{c0} is the solution of quadratic equation

$$Q_1^2R_1 - (Q_1-Q_0)^2R_2 = 0, \quad (70)$$

such that $V \neq 0$ and $-T_0^{-1}V < 0$. We obtain

$$Q_{c0} = \frac{\sqrt{R_2}}{\sqrt{R_1}+\sqrt{R_2}}Q_0, \quad (71)$$

with

$$V = \sqrt{R_1R_2}Q_0 \neq 0, \quad (72)$$

and

$$-T_0^{-1}V = -\frac{\sqrt{R_1R_2}}{T_1+T_2}Q_0 < 0. \quad (73)$$

With (64)–(67) and (72) we get

$$\begin{aligned} \overline{Q}_1(t) = &\frac{\sqrt{R_2}}{\sqrt{R_1}+\sqrt{R_2}}Q_0 \\ &+\frac{a^2}{4Q_0\sqrt{R_1R_2}}\frac{R_2T_1^2-R_1T_2^2}{(T_1+T_2)^2} \\ &+\frac{T_2}{T_1+T_2}a\sin(\omega t). \end{aligned} \quad (74)$$

Let us now write the average system of (5)

$$\begin{aligned} (T_1+T_2)\dot{\widetilde{Q}}_1 &= f_{av}(\widetilde{Q}_1, a^2) \\ &= (R_2-R_1)\widetilde{Q}_1^2 - 2\sqrt{R_1R_2}Q_0\widetilde{Q}_1 \\ &\quad -\frac{a^2}{2}\frac{R_1T_2^2-R_2T_1^2}{(T_1+T_2)^2}, \end{aligned} \quad (75)$$

where

$$\begin{aligned} \widetilde{Q}_1 &= Q_1 - \frac{\sqrt{R_2}}{\sqrt{R_1}+\sqrt{R_2}}Q_0 \\ &\quad -\frac{T_2}{T_1+T_2}a\sin(\omega t), \end{aligned} \quad (76)$$

While the second term in (75) represents an approximate average equilibrium, the average equilibrium in this scalar situation can be found exactly, providing a more accurate, $O(1/\omega)$–approximation

$$\begin{aligned} \check{Q}_1(t) = &\frac{\sqrt{R_2}}{\sqrt{R_1}+\sqrt{R_2}}Q_0 \\ &+\frac{a^2}{2Q_0\sqrt{R_1R_2}}\frac{R_2T_1^2-R_1T_2^2}{(T_1+T_2)^2} \\ &\times\frac{1}{\left(1+\sqrt{1+\frac{a^2}{2Q_0^2}\frac{R_1-R_2}{R_1R_2}\frac{R_2T_1^2-R_1T_2^2}{(T_1+T_2)^2}}\right)} \\ &+\frac{T_2}{T_1+T_2}a\sin(\omega t), \end{aligned} \quad (77)$$

whereas (75) is an $O(1/\omega + a^4)$ approximation.

The expression (74) is very similar to (60) for $L_i = T_i$ and $I_0 = Q_0$. One minor difference is between R_i and $\sqrt{R_i}$, where the latter appears due to the quadratic nature of resistive losses in the fluid network. The other difference is the absence of the a^2-order DC term in (60). This term appears in (74) due to the nonlinearities in the fluid network. We note that this term depends on R_i, T_i, a, Q_0—i.e., all the problem data except the frequency ω. This term becomes significant when Q_0 becomes relatively small in comparison to a. It should be also noted that this term has no effect when $R_2T_1^2 \approx R_1T_2^2$. Physically, this is the case where two branches are of equal length (for example) and one is wider and smoother, while the other is narrower and rougher. To see this we recall that

$$T_i = \frac{\rho l_i}{S_i}, \quad R_i = r_il_i, \quad (78)$$

where S_i is the cross-section of the branch i, l_i is the length, ρ is the fluid density and r_i is the specific resistance of the branch i.

To examine (75) further and analyze its meaning for blood flow networks, let us introduce a quantity we call *peakiness*

$$P = \frac{\frac{T_2}{T_1+T_2}a}{\frac{\sqrt{R_2}}{\sqrt{R_1}+\sqrt{R_2}}Q_0 + \frac{a^2}{4Q_0\sqrt{R_1R_2}}\frac{R_2T_1^2-R_1T_2^2}{(T_1+T_2)^2}} \quad (79)$$

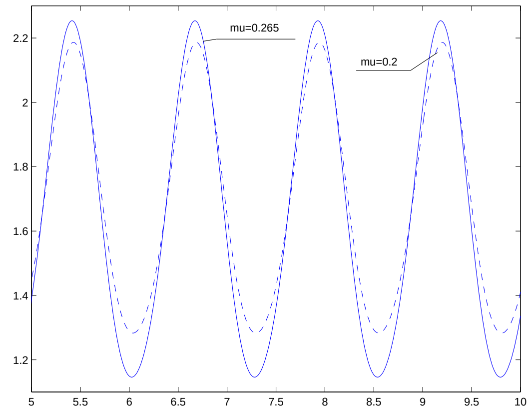

Fig. 3. Flow in the first branch with $R_1 = R_2 = 2$, $T_1 = T_2/\mu$, $T_2 = 1$ and $\mu = 0.2$ and $\mu = 0.265$.

as the ratio of AC and DC components of the flow. Clearly, the peakiness of the input $Q_{in}(t)$ is $P_0 = a/Q_0$. Normalized peakiness is defined as the ratio between P and P_0:

$$\frac{P}{P_0} = \frac{\frac{T_2}{T_1+T_2}}{\frac{\sqrt{R_2}}{\sqrt{R_1}+\sqrt{R_2}} + \frac{P_0^2}{4\sqrt{R_1R_2}}\frac{R_2T_1^2 - R_1T_2^2}{(T_1+T_2)^2}}. \tag{80}$$

Let the first branch be σ^2 times longer and μ times wider than second branch, i.e.,

$$R_1 = rl\sigma^2, \qquad R_2 = rl, \tag{81}$$

$$T_1 = \rho\frac{\sigma^2 l}{\mu S}, \qquad T_2 = \rho\frac{l}{S}. \tag{82}$$

(Taking $\sigma < 1$ or $\mu < 1$ means that the first branch is shorter, or, respectively, narrower.) Then (80) can be rewritten as

$$\frac{P}{P_0} = \mu\frac{(\sigma+1)(\sigma^2+\mu)}{(\sigma^2+\mu)^2 + \frac{P_0^2}{4}\sigma(\sigma+1)(\sigma^2-\mu^2)}. \tag{83}$$

The partial derivative of (83) with respect to μ is

$$\begin{aligned}\frac{\partial(P/P_0)}{\partial\mu} &= \frac{1}{[(\sigma^2+\mu)^2 + \frac{P_0^2}{4}\sigma(\sigma+1)(\sigma^2-\mu^2)]^2} \\ &\quad \times\sigma^2(\sigma+1)[(\sigma^2+\mu)^2 \\ &\quad +\frac{P_0^2}{4}\sigma(\sigma+1)(\sigma^2+2\mu+\mu^2)]. \end{aligned} \tag{84}$$

One can see that expression (84) is always positive. This shows that when a vein gets wider, the flow through it gets more ”peaky”. The extra stress promotes valve failure, which further increases the effective width of the vein. This ”positive feedback” scenario may explain the processes participating in the development of venous diseases.

Since our $\overline{Q}_1(t)$ is only an estimate, we present next an exact numerical simulation illustrating the above phenomenon. Consider the case where the branch lengths are equal, $l_1 = l_2$, and the first branch is narrower than second one: $S_1 = \mu S_2$, $\mu < 1$. Let the flow in generator branch be $Q_{in} = 3 + 2\sin 5t$, which gives $P_0 = 2/3$. Also, consider two cases: $\mu = 0.2$, and $\mu = 0.265$. This increase in cross-section corresponds to a 15% increase in diameter. Then the analytically estimated peakiness will be

$$\begin{aligned}\frac{P}{P_0}(\mu=0.2) &= 0.2903, \\ \frac{P}{P_0}(\mu=0.265) &= 0.3711.\end{aligned} \tag{85}$$

This is a 27% increase in peakiness due to a 15% increase in branch diameter. The exact responses of the system are shown in Fig.3. From it we can find peakiness:

$$\begin{aligned}\frac{P}{P_0}(\mu=0.2) &= \frac{0.455}{1.735} = 0.2622, \\ \frac{P}{P_0}(\mu=0.265) &= \frac{0.56}{1.7} = 0.3294.\end{aligned} \tag{86}$$

So, the increase in diameter by 15% results in an increase in peakiness of 25%. In this example we chose μ small because venous disease tends to develop in narrower ”superficial” veins, rather than in wider ”deep” veins [1].

VI. CONCLUSION

In this paper we developed an analytical form of approximate solutions for a model of fluid flow networks with periodic forcing. The approximation can be used as a quantitative predictive tool (when the network parameters are known, which is unfortunately seldom the case in blood flow networks) or as a qualitative tool that may assist in explaining phenomena participating in the progression of some vascular diseases. The main future challenge would be in extending these results to more realistic models of venous flow.

REFERENCES

[1] N.L. Browse, K.G. Burnand, A.T. Irvine, N.M. Wilson, *Diseases of the Veins*. Oxford University Press, New York, USA, 1999.

[2] C.A. Desoer, E.S. Kuh, *Basic Circuit Theory*, McGraw-hill, 1969.

[3] H.K. Khalil, *Nonlinear Systems*, 3rd ed., Prentice Hall, 2002.

[4] D.D. Kocić, On the autonomy of local systems in mine ventilation control, *2nd Mine Ventilation Congress*, Reno, Nevada, 1979.

[5] Yu. Hu, O.I. Koroleva, M. Krstić, ”Nonlinear Control of Mine Ventilation Networks”, *Systems and Control Letters*, vol.49, no.4, 2003, pp. 239-254.

[6] O.I. Koroleva, M. Krstić, G.W. Schmid-Schonbein, ”Decentralized and Adaptive Control of Nonlinear Fluid Flow Networks”, submitted to *Journal of Dynamic Systems, Measurement and Control*, April 2003.

[7] A.J. Ward-Smith, *Internal Fluid Flow-The Fluid Dynamics of Flow in Pipes and Ducts*, Clarendon press, Oxford, 1980.