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On avoiding saturation in the control of vehicular platoons

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***Abstract*— We investigate some fundamental limitations and tradeoffs in the control of vehicular platoons. These systems are of considerable practical importance as they represent an example of systems on lattices in which different units are dynamically coupled only through feedback controls. We demonstrate that in very long platoons, to avoid large position, velocity, and control amplitudes, one needs to explicitly account for the initial deviations of vehicles from their desired trajectories. We further derive explicit constraints on feedback gains–for any given set of initial conditions–to achieve desired position transients without magnitude and rate saturation. These constraints are used to generate the trajectories around which the states of the platoon system are driven towards their desired values without the excessive use of control effort. All results are illustrated using computer simulations of platoons containing a large number of vehicles.**

***Index Terms*— Vehicular Platoons; Spatially Interconnected Systems; Distributed Control.**

I. Introduction

Control of vehicular platoons has been an intensive area of research for almost four decades [1]–[6]. These systems belong to the class of systems on lattices in which the interactions between different subsystems originate because of a specific control objective that designer wants to accomplish. Additional examples of systems on lattices with this property include unmanned aerial vehicles in formation [7]–[9] and satellites in synchronous orbit [10]–[12]. These interactions often generate surprisingly complex responses that cannot be inferred by analyzing the individual plant units. Rather, intricate behavioral patterns, an instance of which is the so-called *string instability* [13] (or, more generally, the *spatio-temporal instability* [14]), arise because of the aggregate effects. Another particularity of this class of systems is that every subsystem is equipped with sensing and actuating capabilities. The controller design problem is thus dominated by architectural questions such as the choice of localized vs centralized control.

Recent article [15] addressed some fundamental design limitations in vehicular platoons. In particular, it was shown that string stability of a finite platoon with linear dynamics cannot be achieved with any linear controller that uses only information about relative distance between the vehicle on which it acts and its immediate predecessor. A similar result was previously established for a spatially invariant infinite string of vehicles with static feedback controllers having the same information passing properties [3]. This necessitates use of distributed strategies for control of platoons and underscores the importance of developing distributed schemes with favorable architectures. We refer the reader to the references above for a fuller discussion of various algorithms that can be used for platoon control. Additional information about recent work on distributed control of systems on lattices can be found in [14], [16]–[20] and references therein.

In this paper we study some fundamental limitations and tradeoffs in the control of vehicular platoons. We illustrate that in very long platoons one needs to account explicitly for the initial distances of vehicles from their desired trajectories in order to avoid large position and velocity deviations and the excessive use of control effort. We further derive an initial condition dependent set of requirements that the control gains need to satisfy to guarantee the desired quality of position transient response, and rule out saturation in both velocity and control. These requirements are used to generate the trajectories around which the states of the string of vehicles are driven to their desired values without the excessive use of control effort.

Our presentation is organized as follows: in § II, we formulate a control problem and propose a distributed control strategy that solves it. In § III, we illustrate that commanding a uniform rate of convergence for all vehicles towards their desired trajectories may require large control efforts. In § IV, we remark on some basic design limitations and tradeoffs in vehicular platoons and determine the conditions that control gains need to satisfy to provide operation within the imposed saturation limits. In § V, we redesign the controller of § II to provide the desired quality of transient response and avoid large control excursions. We summarize the major contributions and the ongoing research directions in § VI.

II. Control of vehicular platoons

A system of identical unit mass vehicles in an infinite string is shown in Figure 1. The dynamics of this system can be captured by representing each vehicle as a moving mass with the second order dynamics

$$\ddot{x}_n \;=\; u_n, \quad n \,\in\, \mathbb{N}_0, \tag{1}$$

where x_n represents the position of the n-th vehicle, u_n is the control applied on the n-th vehicle, and $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, \ldots\}$.

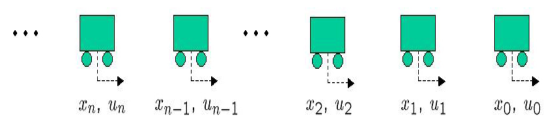

Fig. 1. Platoon of vehicles.

A control objective is to provide a desired constant cruising velocity v_d and to keep the inter-vehicular distance at a constant pre-specified level L. A coordinate transformation of the form

$$e_n(t) \;:=\; \alpha_n \xi_n(t) \;+\; \beta_n \eta_n(t), \quad n \in \mathbb{N}_0, \tag{2}$$

renders (1) into

$$\left.\begin{array}{rcl} \dot{\phi}_n & = & \psi_n \\ \dot{\psi}_n & = & -\,\beta_n u_{n-1} \;+\; (\alpha_n + \beta_n) u_n \end{array}\right\} \quad n \in \mathbb{N}_0, \tag{3}$$

where α_n and β_n represent nonnegative parameters that are not simultaneously equal to zero, with $\beta_0 \equiv 0$, and $\{\phi_n := e_n, \;\; \psi_n := \dot{e}_n\}$. On the other hand, ξ_n and η_n respectively denote the position error variables of the n-th vehicle with respect to the absolute and relative reference frames: $\xi_n(t) := x_n(t) - v_d t + nL$, $\eta_n(t) := x_n(t) - x_{n-1}(t) + L$. By assigning different values to α_n and β_n we can adjust the relative importance of these two quantities.

The appropriate state-space for system (3) is a Banach space $l_\infty \times l_\infty$. This choice of the state-space implies that initially, at $t = 0$, no vehicle is infinitely far from its desired absolute position. Control design should provide boundedness of position and velocity error variables for every $t > 0$ and their asymptotic convergence to zero. In § III, we illustrate, by means of an example, that choosing a Hilbert space $l_2 \times l_2$ rather than a Banach space $l_\infty \times l_\infty$ for the underlying state-space can be rather restrictive. Namely, this choice of the state-space can exclude an entire set of relevant initial conditions from the analysis.

In particular, a choice of control law of the form

$$u_n \;=\; \frac{\beta_n}{\alpha_n + \beta_n} u_{n-1} \;-\; \frac{1}{\alpha_n + \beta_n}(a_n \phi_n \;+\; b_n \psi_n), \quad (4)$$

with $\{a_n > 0,\, b_n > 0,\, \forall\, n \in \mathbb{N}_0\}$, yields an infinite number of stable, fully decoupled second order subsystems

$$\left.\begin{array}{rcl} \dot{\phi}_n & = & \psi_n \\ \dot{\psi}_n & = & -\,a_n \phi_n \;-\; b_n \psi_n \end{array}\right\} \quad n \in \mathbb{N}_0. \quad (5)$$

Therefore, we conclude boundedness of both $e_n(t)$ and $\dot{e}_n(t)$, $\forall\, t \geq 0$, and their asymptotic convergence to zero, for every $n \in \mathbb{N}_0$.

In the remainder of this section, we assume that $\{\alpha_n := \alpha = \text{const.},\, \beta_n := \beta = \text{const.},\, \forall\, n \in \mathbb{N};\, \alpha_0 \neq 0,\, \beta_0 \equiv 0\}$. In this case, controller (4) simplifies to

$$u_0 \;=\; -\,\frac{1}{\alpha_0}(a_0 \phi_0 + b_0 \psi_0), \quad (6a)$$

$$u_n \;=\; \frac{\beta}{\alpha + \beta} u_{n-1} \;-\; \frac{1}{\alpha + \beta}(a_n \phi_n + b_n \psi_n), \quad n \in \mathbb{N},$$

which in turn implies

$$u_n \;=\; \frac{\beta^n}{(\alpha + \beta)^n} u_0 \;-\; \sum_{k=1}^{n} \frac{\beta^{n-k}}{(\alpha + \beta)^{n-k+1}} (a_k \phi_k + b_k \psi_k), \quad (7)$$

for every $n \in \mathbb{N}$. If $\{\alpha \neq 0,\, \beta = 0\}$, controller (6a,7) does not take information about the preceding vehicle into consideration since it only accounts for the error variable with respect to the absolute desired trajectory. Therefore, this control strategy is unsafe and because of that we choose $\beta \neq 0$. On the other hand, if $\{\alpha = 0,\, \beta \neq 0\}$, the information about all vehicles enters into (7) with the same importance which is not desirable from communication point of view. For these reasons, we consider the situation in which both α and β have positive values and rewrite (7) as

$$u_n \;=\; q^n u_0 - \frac{1}{\beta} \sum_{k=1}^{n} q^{n-k+1} (a_k \phi_k + b_k \psi_k), \;\; n \in \mathbb{N}, \quad (8)$$

where $q := \beta/(\alpha + \beta) < 1$. With this choice of design parameters α and β the gains in (8) decay geometrically as a function of spatial location. Thus, in the very long string of vehicles, the positions and velocities of vehicles in the beginning of the platoon do not have a significant influence on controls that act on vehicles in the end of the platoon. This feature is of paramount importance for practical implementations.

Using (8), we can establish the following bound on the infinity-norm of $u_n(t)$:

$$\|u_n\|_\infty \;\leq\; q^n \|u_0\|_\infty + \frac{1}{\beta}\big(\sup_k \|\phi_k\|_\infty \sup_k a_k + \sup_k \|\psi_k\|_\infty \sup_k b_k\big) \frac{q(1-q^n)}{1-q}, \;\; n \in \mathbb{N}, \quad (9)$$

which, together with the properties of system (5), implies boundedness of $u_n(t)$ for all $t \geq 0$, $n \in \mathbb{N}_0$. Based on (2), we also conclude boundedness of both $\xi_n(t)$ and $\eta_n(t)$ for all $t \geq 0$, $n \in \mathbb{N}_0$. Asymptotic convergence of these two quantities and their temporal derivatives to zero follows from the following expressions: $x_0(t) = v_d t + \frac{1}{\alpha_0} e_0(t)$, $\dot{x}_0(t) = v_d + \frac{1}{\alpha_0} \dot{e}_0(t)$, $x_n(t) \;=\; v_d t \;-\; nL \;+\; \frac{1}{\beta} \sum_{k=1}^{n} q^{n-k+1} e_k(t) \;+\; \frac{q^n}{\alpha_0} e_0(t)$, $\dot{x}_n(t) = v_d \;+\; \frac{1}{\beta} \sum_{k=1}^{n} q^{n-k+1} \dot{e}_k(t) \;+\; \frac{q^n}{\alpha_0} \dot{e}_0(t)$, $n \in \mathbb{N}$, and the fact that $\lim_{t\to\infty} e_n(t) = 0$, $\lim_{t\to\infty} \dot{e}_n(t) = 0$, for every $n \in \mathbb{N}_0$. Therefore, controller (6a,8) provides boundedness of all signals in the closed-loop and asymptotic convergence of the platoon of vehicles to its desired cruising formation.

In § III, we illustrate that, even though our design provides boundedness of controls for all times and all vehicles, the lack of uniform bound on $\|u_n\|_\infty$ may result in an excessive use of control effort.

III. Issues arising in control strategies with uniform convergence rates

In this section, we show that imposing a uniform rate of convergence for all vehicles towards their desired trajectories may generate large control magnitudes. To illustrate this, we consider a platoon of vehicles with controller (6a,8) and $\{a_n := a = \text{const.},\, b_n := b = \text{const.},\, \forall\, n \in \mathbb{N}_0\}$. Clearly, in this case both $e_n(t)$ and $\dot{e}_n(t)$ converge towards zero with the rates that are independent of the spatial location. Furthermore, we assume that each vehicle has a limited amount of control effort at its disposal, that is $u_n(t) \in [-\,u_{\max},\, u_{\max}]$, for all $t \geq 0$, $n \in \mathbb{N}_0$, with $u_{\max} > 0$.

In particular, we study the situation in which at $t = 0$ the string of vehicles cruises at the desired velocity v_d with the lead vehicle being at its desired spatial location. We also assume that the distance between the vehicles indexed by n and $n-1$, for every $n \in \{1, \ldots, N\}$, $N \in \mathbb{N}$, is equal to $L + S_n$. The position initial conditions of the remaining vehicles can be chosen to ensure boundedness of $e_n(0)$ for $n > N$. For simplicity, we assume that for $n > N$ the spacing between the neighboring vehicles is at the desired level L. In other words, we consider the initial conditions of the form

$$\dot{x}_n(0) \;=\; v_d, \quad \forall\, n \in \mathbb{N}_0,$$

$$x_n(0) = \left\{ \begin{array}{cl} 0 & n = 0, \\ -\,(nL + \sum_{k=1}^{n} S_k) & n \in \{1, \ldots, N\}, \\ -\,(nL + \sum_{k=1}^{N} S_k) & n > N, \end{array} \right. \quad (10)$$

which translates into: $\{\dot{e}_n(0) = 0,\;\; \forall\, n \in \mathbb{N}_0\}$, and $e_n(0) = \{0, n = 0;\; -(\alpha \sum_{k=1}^{n} S_k + \beta S_n),\;\; n \in \{1, \ldots, N\};\; -\alpha \sum_{k=1}^{N} S_k,\; n > N\}$ Clearly, for this choice of initial condition $\{e_n(0)\}_{n \in \mathbb{N}_0} \notin l_2$, unless $\sum_{k=1}^{N} S_k \equiv 0$. Hence, despite the fact that the entire platoon cruises at the desired velocity v_d and the inter-vehicular spacing for most of the vehicles is kept at the desired level L, a relevant initial condition that does not belong to $l_2 \times l_2$ can be easily constructed. This implies that a Hilbert space $l_2 \times l_2$ represents a rather restrictive choice for the underlying state-space.

The absence of uniform bound in (9) indicates that the large states will lead to the large control signals if the feedback gains are not appropriately selected. In particular, we observe that large initial states are readily obtained if there exists $m \in \{1, \ldots, N\}$ such that $\{S_1, \ldots, S_m\}$ sum to a big number. Moreover, it is not difficult to see that if $\{\text{sign}\, S_n = \text{const.} \neq 0,\, \forall\, n \in \{1, \ldots, N\}\}$, then $|\sum_{k=1}^{N} S_k|$ represents a monotonically increasing function of N. Thus, for N large enough, very large initial conditions are possible if, for example, all S_n's are either positive or negative. Because of that, we study a spatially constant non-zero sequence of S_n's, that is we assume $\{S_n := S = \text{const.} \neq 0,\; \forall\, n \in \{1, \ldots, N\}\}$. In this case, (10) simplifies to

$$\dot{x}_n(0) \;=\; v_d, \quad \forall\, n \in \mathbb{N}_0,$$

$$x_n(0) \;=\; \left\{ \begin{array}{rl} 0 & n = 0, \\ -\,n(L + S) & n \in \{1, \ldots, N\}, \\ -\,(nL + NS) & n > N, \end{array} \right. \quad (11)$$

or equivalently $\{\dot{e}_n(0) = 0, \ \forall n \in \mathbb{N}_0\}$, $e_n(0) = \{0, \ n = 0; \ -(n\alpha + \beta)S, \ n \in \{1, \ldots, N\}; \ -N\alpha S, \ n > N\}$. For this choice of initial conditions $u_0 \equiv 0$, whereas the initial amount of control effort for the remaining vehicles is given by

$$u_n(0) \ = \ \frac{aS}{\beta}(n\frac{\alpha q}{1-q} \ + \ \frac{q(1-q^n)(\beta \ - \ (\alpha+\beta)q)}{(1-q)^2}),$$

for every $n \in \{1, \ldots, N\}$, and

$$u_n(0) \ = \ \frac{aS}{\beta}(N\frac{\alpha q}{1-q} + \frac{q^{n-N+1}(1-q^N)(\beta-(\alpha+\beta)q)}{(1-q)^2}),$$

for every $n > N$, which implies that for any choice of design parameter a there exist $m \in \mathbb{N}$ such that $|u_n(0)| > u_{\max}$, for every $n > m$, provided that N is large enough. For example, if $\alpha = \beta = a = 1$, $b = 2$, $S = 0.5$, $u_{\max} = 5$, for $N = 50$, $u_{10}(0) = u_{\max} = 5$, and $|u_n(0)| > u_{\max}$, $\forall n > 10$, with $\lim_{n\to\infty} |u_n(0)| = 25$. Simulation results for this choice of design parameters and initial conditions given by (11), using controller (6a,8), are shown in Figure 2.

A. LQR design for a platoon on a circle

To illustrate that the above raised issues are not caused by the specific control strategy, we also consider a Linear Quadratic Regulator (LQR) design for a platoon on a circle that consists of M vehicles. By exploiting the *spatial invariance*, we *analytically* establish that *any* LQR design leads to large control signals for the appropriately selected set of initial conditions.

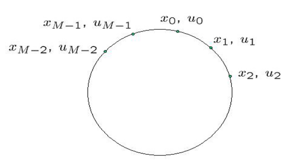

Fig. 3. Circular platoon of M vehicles.

The control objective is the same as in § II: to drive the entire platoon at the constant cruising velocity v_d, and keep the distance between the neighboring vehicles at a pre-specified constant level L. Clearly, this is possible only if the radius of a circle is given by $r_M = ML/2\pi$. We rewrite system (1) for $n \in \{0, \ldots, M-1\}$ in terms of a state-space realization of the form

$$\begin{aligned} \begin{bmatrix} \dot{\xi}_n \\ \dot{\varsigma}_n \end{bmatrix} \ &= \ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_n \\ \varsigma_n \end{bmatrix} \ + \ \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_n \\ &=: \ A_n \varphi_n \ + \ B_n u_n, \end{aligned} \tag{12}$$

where $\xi_n(t) := x_n(t) - v_d t - nL$ and $\varsigma_n(t) := \dot{x}_n(t) - v_d$ denote the absolute position and velocity errors of the n-th vehicle, respectively. We propose the following cost functional

$$\begin{aligned} J \ := \ & \frac{1}{2}\int_0^\infty \sum_{n=0}^{M-1}\sum_{m=0}^{M-1} \varphi_n^*(t) Q_{n-m} \varphi_m(t)\, \mathrm{d}t \ + \\ & \frac{1}{2}\int_0^\infty \sum_{n=0}^{M-1}\sum_{m=0}^{M-1} u_n^*(t) R_{n-m} u_m(t)\, \mathrm{d}t, \end{aligned} \tag{13}$$

where $\{\sum_{n=0}^{M-1}\sum_{m=0}^{M-1} \varphi_n^* Q_{n-m} \varphi_m \ \geq \ 0, \ Q_{-n}^* = Q_n\}$, for all sequences φ_n, and $\{\sum_{n=0}^{M-1}\sum_{m=0}^{M-1} u_n^* R_{n-m} u_m \ > \ 0, \ R_{-n}^* = R_n\}$, for all non-zero sequences u_n.

We utilize the fact that system (12) has spatially invariant dynamics over a circle. This implies that the Discrete Fourier Transform (DFT) can be used to convert analysis and quadratic design problems into those for a parameterized family of second order systems [14]. DFT is defined by: $\hat{x}_k \ := \ \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} x_n \mathrm{e}^{-j\frac{2\pi n k}{M}}$, $k \in \{0, \ldots, M-1\}$, and the inverse DFT is defined by: $x_n \ := \ \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} \hat{x}_k \mathrm{e}^{j\frac{2\pi n k}{M}}$, $n \in \{0, \ldots, M-1\}$. Using this, system (12) and quadratic performance index (13) transform to

$$\begin{aligned} \begin{bmatrix} \dot{\hat{\xi}}_k \\ \dot{\hat{\varsigma}}_k \end{bmatrix} \ &= \ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\xi}_k \\ \hat{\varsigma}_k \end{bmatrix} \ + \ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u}_k \\ &=: \ \hat{A}_k \hat{\varphi}_k \ + \ \hat{B}_k \hat{u}_k, \quad k \in \{0, \ldots, M-1\}, \end{aligned} \tag{14}$$

and

$$J \ = \ \frac{\sqrt{M}}{2} \sum_{k=0}^{M-1} \int_0^\infty \left(\hat{\varphi}_k^*(t) \hat{Q}_k \hat{\varphi}_k(t) + \hat{u}_k^*(t) \hat{R}_k \hat{u}_k(t)\right) \mathrm{d}t,$$

where $\hat{R}_k > 0$ and

$$\hat{Q}_k \ := \ \begin{bmatrix} \hat{q}_{11k} & \hat{q}_{21k}^* \\ \hat{q}_{21k} & \hat{q}_{22k} \end{bmatrix} \ \geq \ 0,$$

for every $k \in \{0, \ldots, M-1\}$. Clearly, the pair $(\hat{A}_k, \hat{B}_k)$ is stabilizable for every $k \in \{0, \ldots, M-1\}$. On the other hand, the pair $(\hat{Q}_k, \hat{A}_k)$ is detectable if and only if $\hat{q}_{11k} > 0$ for every $k \in \{0, \ldots, M-1\}$. These conditions are necessary and sufficient for the existence of a stabilizing optimal solution to the LQR problem (12,13).

It is readily shown that for $\dot{x}_n(0) \equiv v_d$, i.e. for $\varsigma_n(0) \equiv 0$, we have: $\sum_{n=0}^{M-1} u_n^2(0) = \sum_{k=0}^{M-1} \frac{\hat{q}_{11k}}{\hat{R}_k} \hat{\xi}_k^*(0) \hat{\xi}_k(0)$, which in turn implies

$$\inf_k \frac{\hat{q}_{11k}}{\hat{R}_k} \sum_{n=0}^{M-1} \xi_n^2(0) \ \leq \ \sum_{n=0}^{M-1} u_n^2(0) \ \leq \ \sup_k \frac{\hat{q}_{11k}}{\hat{R}_k} \sum_{n=0}^{M-1} \xi_n^2(0).$$

Thus, we have established the lower and upper bounds on the initial amount of control effort for a formation that cruises at the desired velocity v_d. These bounds are determined by the deviations of vehicles from their absolute desired trajectories at $t = 0$, and by the LQR design parameters $\hat{q}_{11k}$ and $\hat{R}_k$. Clearly, since $\hat{q}_{11k} > 0$ (for detectability) $\inf_k \hat{q}_{11k}/\hat{R}_k$ is always greater than zero. We note that this quantity can be made smaller by increasing the control penalty. In particular, for $x_n(0) = n(L - S)$, $0 < S < L$, we have

$$\sum_{n=0}^{M-1} u_n^2(0) \ \geq \ \frac{S^2}{6} M(M-1)(2M-1) \inf_k \frac{\hat{q}_{11k}}{\hat{R}_k},$$

which illustrates an unfavorable scaling of the initial amount of control effort with the number of vehicles in formation. Hence, unless $u_{\max} \geq S^2(M-1)(2M-1)\inf_k \hat{q}_{11k}/6\hat{R}_k$, there exist at least one vehicle for which $|u_n(0)| > u_{\max}$.

The results of this section illustrate that in very large platoons one needs to take into account the initial distance of vehicles from their desired trajectories and to adjust the control gains accordingly in order to avoid large velocity deviations and the excessive use of control effort. In the next section, we give conditions that the feedback gains need to satisfy to prevent saturation in both velocity and control and discuss some design limitations and tradeoffs in vehicular platoons.

IV. Design limitations and tradeoffs in vehicular platoons

In this section, we determine the conditions that control gains need to satisfy to provide operation within the imposed saturation limits. Our analysis yields the explicit constraints on these gains−for any given set of initial conditions−to achieve desired position transients without magnitude and rate saturation. We also remark on some of the basic limitations and tradeoffs that need to be addressed in the control of vehicular platoons.

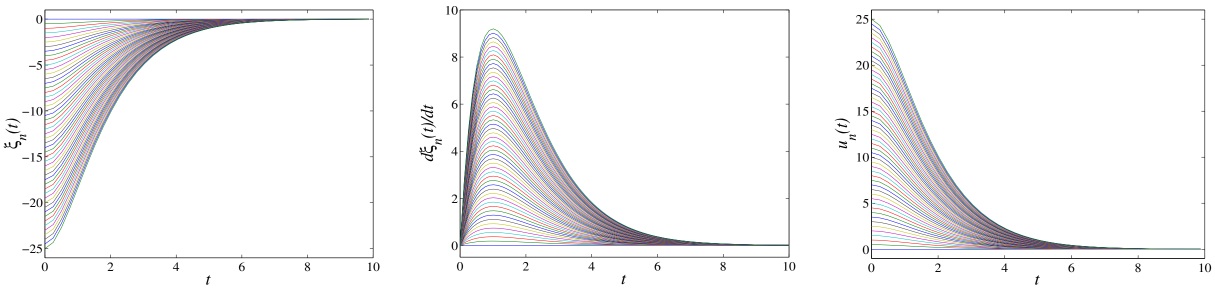

Fig. 2. Simulation results of vehicular platoon using control law (6a,8) with $\alpha = \beta = a = 1$, and $b = 2$. The initial state of the platoon system is given by (11) with $N = 50$ and $S = 0.5$.

We rewrite system (1) as

$$\ddot{\bar{x}}_n \;=\; \bar{u}_n, \quad n \;\in\; \mathbb{N}_0. \tag{15}$$

The justification for the notation used in (15) is given in § V. We want to drive each vehicle towards its desired absolute position $v_d t - nL$, and its desired velocity v_d. For the time being we are not concerned with the relative spacing between the vehicles. If we introduce the error variable $\{r_n(t) := \bar{x}_n(t) - v_d t + nL,\, n \in \mathbb{N}_0\}$, we can rewrite (15) as

$$\ddot{r}_n \;=\; \bar{u}_n, \quad n \;\in\; \mathbb{N}_0, \tag{16}$$

and choose $\bar{u}_n$ to meet the control objective. In particular, we take $\bar{u}_n$ of the form

$$\bar{u}_n \;=\; -\,p_n^2 r_n \;-\; 2 p_n \dot{r}_n, \quad n \;\in\; \mathbb{N}_0, \tag{17}$$

where, for every $n \in \mathbb{N}_0$, p_n represents a positive design parameter. With this choice of control, the solution of system (16,17) is for any $n \in \mathbb{N}_0$ given by

$$r_n(t) \;=\; (c_n \;+\; d_n t)\mathrm{e}^{-p_n t}, \tag{18a}$$
$$\dot{r}_n(t) \;=\; (d_n \;-\; c_n p_n \;-\; d_n p_n t)\mathrm{e}^{-p_n t}, \tag{18b}$$
$$\bar{u}_n(t) \;=\; (c_n p_n^2 \;-\; 2 d_n p_n \;+\; d_n p_n^2 t)\mathrm{e}^{-p_n t}, \tag{18c}$$

where for every $n \in \mathbb{N}_0$

$$\begin{aligned} c_n &:= r_n(0) \;=\; \bar{x}_n(0) \;+\; nL, \\ d_n &:= r_n(0) p_n \;+\; \dot{r}_n(0) \\ &\;=\; (\bar{x}_n(0) \;+\; nL) p_n \;+\; (\dot{\bar{x}}_n(0) \;-\; v_d). \end{aligned} \tag{19}$$

We want to determine conditions that the sequence of positive numbers $\{p_n\}_{n \in \mathbb{N}_0}$ has to satisfy to guarantee

$$|r_n(t)| \;\leq\; r_{n,\max}, \qquad \forall\, t \;\geq\; 0, \tag{20a}$$
$$|\dot{r}_n(t)| \;\leq\; v_{\max}, \qquad \forall\, t \;\geq\; 0, \tag{20b}$$
$$|\bar{u}_n(t)| \;\leq\; u_{\max}, \qquad \forall\, t \;\geq\; 0, \tag{20c}$$

with $\{r_{n,\max} > 0,\ \forall\, n \in \mathbb{N}_0\}$, $v_{\max} > 0$, and $u_{\max} > 0$ being the pre-specified numbers. For notational convenience, we have assumed that all vehicles have the same velocity and control saturation limits, given by $v_{\max}$ and $u_{\max}$, respectively. Typically, the sequence $\{r_{n,\max}\}_{n \in \mathbb{N}_0}$ is given in terms of position initial conditions $\{r_n(0)\}_{n \in \mathbb{N}_0}$ as $\{r_{n,\max} := \gamma_n |r_n(0)|\}_{n \in \mathbb{N}_0}$, where sequence of numbers $\{\gamma_n > 1,\, \forall\, n \in \mathbb{N}_0\}$ determines the allowed overshoot with respect to the desired position trajectory of the n-th vehicle. Clearly, for this choice of $\{r_{n,\max}\}_{n \in \mathbb{N}_0}$, $\{r_n(0)\}_{n \in \mathbb{N}_0}$ satisfies (20a). Based on (18a), $r_n(t)$ asymptotically goes to zero, so we only need to determine conditions under which (20a) is violated for finite non-zero times. If (18a) achieves an extremum for some $\bar{t}_n \in (0, \infty)$, the absolute value of r_n at that point is given by:

$$|r_n(\bar{t}_n)| \;=\; \frac{|d_n|}{p_n}\mathrm{e}^{-p_n \bar{t}_n} \;\leq\; \frac{|d_n|}{p_n} \;\leq\; \frac{|\dot{r}_n(0)|}{p_n} \;+\; |r_n(0)|.$$

Therefore, if sequence of positive numbers $\{p_n\}_{n \in \mathbb{N}_0}$ is chosen such that

$$\frac{|\dot{r}_n(0)|}{p_n} \;+\; |r_n(0)| \;\leq\; r_{n,\max}, \quad \forall\, n \;\in\; \mathbb{N}_0, \tag{21}$$

condition (20a) will be satisfied for every $t \geq 0$. This implies that, for good position transient response (that is, for small position overshoots), design parameters p_n have to assume large enough values determined by (21).

Clearly, (20b) is going to be violated unless $|\dot{r}_n(0)| \leq v_{\max}$, for every $n \in \mathbb{N}_0$. If $\dot{r}_n$ has a maximum or a minimum at some non-zero finite time $\bar{t}_n$, the absolute value of (18b) at that point can be upper bounded by

$$|\dot{r}_n(\bar{t}_n)| \;=\; |d_n|\mathrm{e}^{-p_n \bar{t}_n} \;\leq\; |d_n| \;\leq\; |r_n(0)| p_n \;+\; |\dot{r}_n(0)|.$$

Thus, to avoid velocity saturation, sequence of positive design parameters $\{p_n\}_{n \in \mathbb{N}_0}$ has to be small enough to satisfy

$$|r_n(0)| p_n \;+\; |\dot{r}_n(0)| \;\leq\; v_{\max}, \quad \forall\, n \;\in\; \mathbb{N}_0. \tag{22}$$

Finally, to rule out saturation in control we need to make sure that condition (20c) is satisfied for both $t = 0$ and $\bar{t}_n > 0$, where the potential extremum of $\bar{u}_n$ takes place. The absolute values of (18c) at these two time instants are respectively given by $|\bar{u}_n(0)| = |-\,r_n(0) p_n^2 \;-\; 2\dot{r}_n(0) p_n| \leq |r_n(0)| p_n^2 \;+\; 2|\dot{r}_n(0)| p_n$, and $|\bar{u}_n(\bar{t}_n)| = |d_n| p_n \mathrm{e}^{-p_n \bar{t}_n} \leq |d_n| p_n \leq |r_n(0)| p_n^2 \;+\; |\dot{r}_n(0)| p_n$. Since $p_n > 0$, for every $n \in \mathbb{N}_0$, condition (20c) is met if

$$|r_n(0)| p_n^2 \;+\; 2|\dot{r}_n(0)| p_n \;\leq\; u_{\max}, \quad \forall\, n \;\in\; \mathbb{N}_0. \tag{23}$$

Inequalities (21), (22), and (23) establish conditions for positive design parameters p_n to prevent saturation in velocity and control, and guarantee a good position transient response. We remark that these conditions can be somewhat conservative, but they are good enough to illustrate the major point. Clearly, for small excursions from the desired position trajectories control gains have to assume large values, determined by (21). On the other hand, for small velocity deviations and small control efforts these gains have to be small enough to satisfy (22) and (23). These facts illustrate some basic tradeoffs that designer faces in the control of vehicular platoons. In particular, the set of control gains that satisfies (22) and (23) determines the maximal position deviations and the rates of convergence towards the desired trajectories. In other words, the position overshoots and settling times can be significantly increased in the presence of stringent requirements on velocity and control saturation limits.

For the example considered in § III with the initial conditions of the form

$$\begin{aligned} \dot{\bar{x}}_n(0) \;&=\; v_d, \quad \forall\, n \in \mathbb{N}_0, \\ \bar{x}_n(0) &= \begin{cases} 0 & n = 0, \\ -\,(nL + \sum_{k=1}^{n} S_k) & n \in \{1, \ldots, N\}, \\ -\,(nL + \sum_{k=1}^{N} S_k) & n > N, \end{cases} \end{aligned} \tag{24}$$

condition (21) is always satisfied, which implies that the largest deviation for all vehicles from their desired absolute positions takes place at $t = 0$. Therefore, the chosen initial conditions do not impose any lower bounds on the control gains. On the other hand, whereas conditions (22) and (23) do not put any constraints on p_0, they respectively dictate the following upper bounds on $\{p_n\}_{n\in\mathbb{N}}$: $p_n \leq \{v_{\max}/|\sum_{k=1}^{n} S_k|,\ n \in \{1,\ldots,N\};\ v_{\max}/|\sum_{k=1}^{N} S_k|,\ n > N\}$, and $p_n \leq \{\sqrt{u_{\max}/|\sum_{k=1}^{n} S_k|},\ n \in \{1,\ldots,N\};\ \sqrt{u_{\max}/|\sum_{k=1}^{N} S_k|},\ n > N\}$. In particular, the following choice of $\{p_n\}_{n\in\mathbb{N}}$

$$p_n = \left\{ \begin{array}{ll} \min\left\{ \dfrac{\varrho_n v_{\max}}{|\sum_{k=1}^{n} S_k|}, \sqrt{\dfrac{\sigma_n u_{\max}}{|\sum_{k=1}^{n} S_k|}} \right\} & n < N, \\ \min\left\{ \dfrac{\varrho_n v_{\max}}{|\sum_{k=1}^{N} S_k|}, \sqrt{\dfrac{\sigma_n u_{\max}}{|\sum_{k=1}^{N} S_k|}} \right\} & n \geq N, \end{array} \right. \tag{25}$$

with $\{0 < \varrho_n \leq 1,\ 0 < \sigma_n \leq 1,\ \forall\, n \in \mathbb{N}\}$, clearly satisfies the above requirements. Figure 4 illustrates the solution of system (16,17) for initial conditions determined by (24) with $N = 50$ and $S_n = 0.5$, for every $n \in \{1,\ldots,N\}$. The control gains are chosen using (25) with $v_{\max} = u_{\max} = 5$, $\{\varrho_n = 1,\ \sigma_n = 0.8,\ \forall\, n \in \mathbb{N}\}$, to prevent reaching imposed velocity and control saturation limits. The dependence of these gains on discrete spatial variable n is also illustrated in Figure 4.

Thus we have shown that controller (17) with the gains satisfying (21), (22), and (23) precludes saturation in both velocity and control and takes into account the desired quality of position transient response. However, this control strategy is unsafe, since it does not account for the inter-vehicular spacings. Because of that, in § V we redesign controller (6a,7) by incorporating the constraints imposed by (20) in the synthesis.

V. Control without reaching saturation

We again consider system (1), and introduce an error variable of the form

$$\varepsilon_n(t) \;:=\; \alpha_n \zeta_n(t) \;+\; \beta_n \chi_n(t), \quad n \in \mathbb{N}_0, \tag{26}$$

where: $\{\zeta_n(t) := x_n(t) - v_d t + nL - r_n(t) = \xi_n(t) - r_n(t),\ n \in \mathbb{N}_0\}$, $\{\chi_n(t) := \zeta_n(t) - \zeta_{n-1}(t) = x_n(t) - x_{n-1}(t) + L - r_n(t) + r_{n-1}(t),\ n \in \mathbb{N}\}$, with $r_n(t)$ being defined by (18a,19), $\{p_n\}_{n\in\mathbb{N}_0}$ satisfying (21), (22), and (23), and parameters $\{\alpha_n\}_{n\in\mathbb{N}_0}$ and $\{\beta_n\}_{n\in\mathbb{N}_0}$ having the properties discussed in § II. The initial conditions on these two variables and their first derivatives are given by: $\{\zeta_n(0) = x_n(0) - \bar{x}_n(0) =: \mu_n,\ n \in \mathbb{N}_0\}$, $\{\dot{\zeta}_n(0) = \dot{x}_n(0) - \dot{\bar{x}}_n(0) =: \nu_n,\ n \in \mathbb{N}_0\}$, $\{\chi_n(0) = \zeta_n(0) - \zeta_{n-1}(0) = \mu_n - \mu_{n-1},\ n \in \mathbb{N}\}$, $\{\dot{\chi}_n(0) = \dot{\zeta}_n(0) - \dot{\zeta}_{n-1}(0) = \nu_n - \nu_{n-1},\ n \in \mathbb{N}\}$, where $\{x_n(0), \dot{x}_n(0)\}_{n\in\mathbb{N}_0}$ and $\{\bar{x}_n(0), \dot{\bar{x}}_n(0)\}_{n\in\mathbb{N}_0}$ represent the actual and the measured initial conditions, respectively. If perfect information about the initial positions and velocities is available, then clearly $\{\mu_n = \nu_n = 0,\ \forall\, n \in \mathbb{N}_0\}$. However, since initial condition uncertainties are always present we want to design a controller to guard against them.

Double differentiation of (26) with respect to time yields

$$\begin{aligned} \ddot{\varepsilon}_n \;&=\; (\alpha_n \;+\; \beta_n)(u_n \;-\; \bar{u}_n) \;-\; \beta_n(u_{n-1} \;-\; \bar{u}_{n-1}) \\ &=: (\alpha_n \;+\; \beta_n)\tilde{u}_n \;-\; \beta_n \tilde{u}_{n-1}, \quad n \in \mathbb{N}_0, \end{aligned} \tag{27}$$

where $\bar{u}_n$ is given by (17). System (27) can be represented in terms of its state-space realization of the form

$$\left. \begin{aligned} \dot{\vartheta}_n \;&=\; \upsilon_n \\ \dot{\upsilon}_n \;&=\; -\beta_n \tilde{u}_{n-1} \;+\; (\alpha_n + \beta_n)\tilde{u}_n \end{aligned} \right\} \; n \in \mathbb{N}_0, \tag{28}$$

where $\{\vartheta_n := \varepsilon_n,\ \upsilon_n := \dot{\varepsilon}_n\}$. In particular, this system can be stabilized by the following feedback

$$\tilde{u}_0 \;=\; -\,\frac{1}{\alpha_0}(a_0 \vartheta_0 + b_0 \upsilon_0), \tag{29a}$$

$$\tilde{u}_n \;=\; \tilde{u}_0 \prod_{k=1}^{n} q_k - \sum_{k=1}^{n} \frac{1}{\beta_k}(a_k \vartheta_k + b_k \upsilon_k) \prod_{i=k}^{n} q_i, \tag{29b}$$

with $q_k := \beta_k/(\alpha_k + \beta_k)$, provided that $\{\beta_n \neq 0,\ \forall\, n \in \mathbb{N}\}$. It is noteworthy that, if parameters α_n and β_n are such that $\{\alpha_n := \alpha = \text{const.},\ \beta_n := \beta = \text{const.},\ \forall\, n \in \mathbb{N}\}$, then controller (29) has the same properties as controller (6a,7). For the same choices of design parameters $\{a_n\}_{n\in\mathbb{N}_0}$ and $\{b_n\}_{n\in\mathbb{N}_0}$, these two control strategies are only distinguished by the regions from where the states of systems (3) and (28) have to be brought to the origin. Namely, due to different formulations of control objectives, the initial states of system (3) may occupy a portion of the state-space that is significantly larger than a region to which the initial conditions of system (28) belong. In the former case, this region is determined by the maximal deviations from the desired absolute trajectories at $t = 0$, whereas, in the latter case, it is determined by the precision of measurement devices, that is their ability to yield an accurate information about the initial positions and velocities. As illustrated in § III, the initial conditions may have an unfavorable scaling with discrete spatial variable n, which may result in the very large initial position deviations (and consequently, a large amount of the initial control effort) for large n's, unless the size of the initial conditions is explicitly accounted for. We have shown in § IV how to generate the initial condition dependent trajectories around which the states of vehicular platoon can be driven to zero without extensive use of control effort and large position and velocity overshoots.

Using the definition of $\tilde{u}_n$, we finally give the expressions for $\{u_n\}_{n\in\mathbb{N}_0}$

$$u_n \;=\; \bar{u}_n \;+\; \tilde{u}_n, \tag{30}$$

where $\bar{u}_n$ and $\tilde{u}_n$ are respectively given by (17) and (29). We remark that $\{u_n \equiv \bar{u}_n,\ \forall\, n \in \mathbb{N}_0\}$ if perfect information about the initial conditions is available. The only role of $\{\tilde{u}_n\}_{n\in\mathbb{N}_0}$ is to account for the discrepancies in the initial conditions due to measurement imperfections.

Asymptotic convergence of ζ_n, χ_n, $\dot{\zeta}_n$, and $\dot{\chi}_n$ to the origin for every $n \in \mathbb{N}_0$ can be easily established. Therefore, controller (30) provides operation within the imposed saturation bounds and asymptotic convergence of the platoon of vehicles to its desired cruising formation.

Simulation results of the platoon system with 101 vehicles ($M = 100$) using controller (30) with $\{\alpha_0 = 1,\ \alpha_n = \beta_n = 1,\ \forall\, n \in \{1,\ldots,M\}\}$, $\{a_n = 1,\ b_n = 2,\ \forall\, n \in \{0,\ldots,M\}\}$ are shown in Figure 5. The measured initial condition is given by (24) with $N = 50$ and $S_n = 0.5$, for every $n \in \{1,\ldots,N\}$, whereas the numbers μ_n and ν_n that determine the actual positions and velocities at $t = 0$ are randomly selected. The rates of convergence towards the origin are chosen using (25) with $v_{\max} = u_{\max} = 5$, $\{\varrho_n = 1,\ \sigma_n = 0.8,\ \forall\, n \in \{1,\ldots,M\}\}$, to prevent reaching imposed velocity and control saturation limits, and p_0 is set to 1. These convergence rates are shown in the far right plot in Figure 4. Clearly, the desired control objective is successfully accomplished with the quality of the transient response determined by the prescribed saturation bounds.

VI. Concluding remarks

The main purpose of this paper is to illustrate some fundamental design limitations and tradeoffs in automated highway systems. We show that in very large platoons the designer needs to pay attention to the initial deviations of vehicles from their desired trajectories when selecting control gains. We also establish explicit constraints on these gains−for any given set of initial conditions−to assure the desired quality of position transients without magnitude and rate saturation.

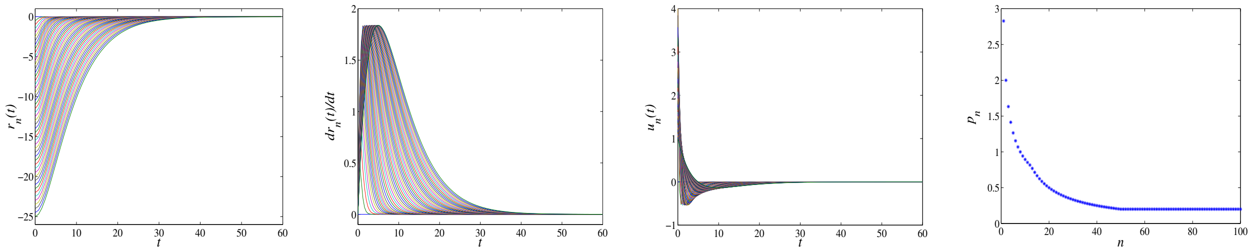

Fig. 4. Solution of system (16,17) for initial conditions determined by (24) with $N = 50$ and $S_n = 0.5$, for every $n \in \{1, \dots, N\}$. The control gains (far right plot) are determined using (25) with $v_{\max} = u_{\max} = 5$, and $\{\varrho_n = 1,\ \sigma_n = 0.8,\ \forall\, n \in \mathbb{N}\}$.

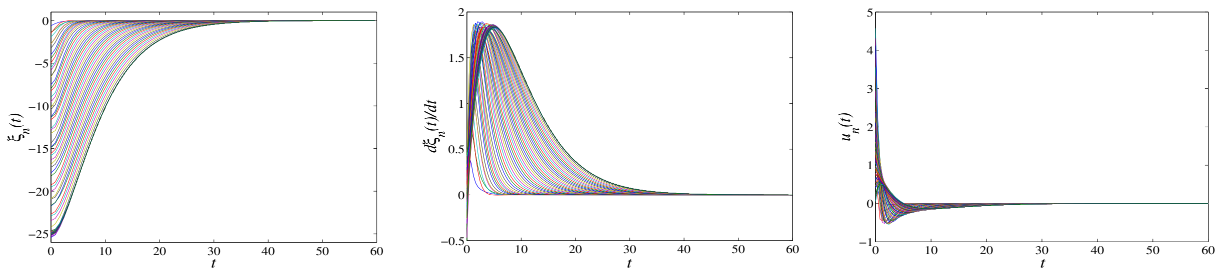

Fig. 5. Simulation results of platoon with 101 vehicles ($M = 100$) using controller (30) with $\alpha_0 = 1$, $\{\alpha_n = \beta_n = 1, \forall\, n \in \{1, \dots, M\}\}$, $\{a_n = 1, b_n = 2, \forall\, n \in \{0, \dots, M\}\}$. The measured initial condition is given by (24) with $N = 50$ and $S_n = 0.5$, for every $n \in \{1, \dots, N\}$, whereas the numbers μ_n and ν_n that determine the actual initial condition are randomly selected.

These requirements are used to generate the trajectories around which the states of the platoon system are driven towards their desired values without the excessive use of control effort.

Ongoing research effort is directed towards the design of robust controllers for vehicular platoons with favorable architecture. The main drawback of the control strategy employed in this paper is that it only guards against the initial condition uncertainties. The robust design will also provide satisfactory performance in the presence of external disturbances and unmodelled dynamics. Sensitivity of distributed control strategies to communication noise and delays is another topic worth considering.

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