Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Control of Distributed Discrete-Time Systems on Graphs

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Abstract

This paper considers linear time-invariant dynamic systems with an interconnection structure specified by a directed graph. We formulate a semidefinite program for system analysis and controller synthesis. The structure of the resulting matrix inequalities is related to recent approaches for distributed control, and the results of this paper reduce to recent results in the case where the underlying graph structure is a rectangular array.

1 Introduction

This paper considers linear time-invariant dynamical systems with an interconnection structure specified by a directed graph. In particular, we develop computational tests for measuring the stability and performance of such systems, making use of semidefinite programming. We also develop a synthesis approach for designing distributed state feedback controllers.

This work builds on much recent work in distributed control. An important approach for analysis and synthesis of distributed controllers has made use of spatial symmetry [3, 4]. For distributed systems with certain spatial symmetries, a Fourier transform can be applied in the spatial directions. It was shown in [4] that the resulting controllers share the spatial symmetries of the underlying system. An approach using linear parameter-varying (LPV) control for spatially symmetric systems has been developed in [7, 8], using generalized Roesser state-space realizations [20] and semidefinite programming. Other approaches have also considered symmetry, including [1], where an approach for decentralized control using integral quadratic constraints is developed for spatially symmetric systems.

For systems which do not possess spatial symmetry, strongly related results have been developed [10, 11]. These generalize the semidefinite programs used in the symmetric case, making use of block-diagonal operators. The resulting framework is a multidimensional generalization of the operator framework of [12] which used shift operators to characterize robust controllers for time-varying systems. These results develop systems theory for dynamics on a *spatio-temporal grid*.

For systems associated with arbitrary graphs, an analysis and synthesis approach has been developed based on dissipation inequalities [6], applied to graphs in *extensive form*. Other recent work also includes the distributed control synthesis approaches presented in [14]. This framework generalizes the synthesis approach for designing decentralized controllers [16, 21].

In this paper, we generalize the multidimensional shift operator framework and the associated Roesser realizations to linear time-invariant systems evolving over a directed graph. In this paper, we do not work over the graph in extensive form, allowing analysis and decentralized control synthesis for systems evolving over infinite time. The results reduce to the results of [10, 11] when applied to dynamic systems evolving over a rectangular spatio-temporal grid.

Furthermore, we present a state feedback synthesis approach which uses semidefinite programming to construct controllers with a decentralized structure. These results have applications to distributed control of asymmetric formations of vehicles.

2 Preliminaries

The matrix Kronecker product is denoted $\otimes$ and the matrix I_n is the $n \times n$ identity. The notation $\mathbb{S}^n$ denotes the set of symmetric matrices in $\mathbb{R}^{n\times n}$. For $X \in \mathbb{S}^n$, the notation $X > 0$ means that X is positive definite.

Suppose that $(\mathcal{V}, \mathcal{E}^s)$ is a simple directed graph, with vertex set $\mathcal{V} = 1, \dots, N$, and let $\mathcal{E}^s \subset \mathcal{V} \times \mathcal{V}$. Here $\mathcal{V}$ is the set of ***nodes*** and $\mathcal{E}^s$ the set of ***edges***. Each edge $e \in \mathcal{E}^s$ is a pair of vertices $e = (i, j)$ where $i, j \in \mathcal{V}$, representing an edge from node i to node j; node i is called the ***tail*** of edge e, and node j is called the ***head*** of edge e. We number the edges $1, 2, \dots, M^s$, and will slightly abuse notion by using e to denote both the number of the edge $e \in \mathcal{E}$ and the pair $e = (i, j)$. We will also use the notation $e(1)$ and $e(2)$ to denote the tail and head of edge e respectively. The edge $e = (i, j)$ is called ***outgoing*** from node i and ***incoming*** on node

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Both authors were partially supported by the Stanford MICA *Coordination and Control for Networks of Interacting Automata*, DARPA/SPAWAR award number N66001-01-C-8080

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j. For $i \in \mathcal{V}$, define

$$\mathcal{O}^s(i) = \{\, e \in \mathcal{E}^s \mid e(1) = i \,\}$$
$$\mathcal{I}^s(i) = \{\, e \in \mathcal{E}^s \mid e(2) = i \,\}$$

as the set of edges outgoing from and incoming to node i in a simple graph, respectively. We define the ***spatial outgoing incidence matrix*** $F \in \mathbb{R}^{M^s \times N}$ and the ***spatial incoming incidence matrix*** $G \in \mathbb{R}^{M^s \times N}$ as

$$F_{ei} = \begin{cases} 1 & \text{if } e \in \mathcal{O}^s(i) \\ 0 & \text{otherwise} \end{cases} \qquad G_{ei} = \begin{cases} 1 & \text{if } e \in \mathcal{I}^s(i) \\ 0 & \text{otherwise} \end{cases}$$

for all $i \in \mathcal{V}$ and $e \in \mathcal{E}^s$.

We will also define another directed graph on $\mathcal{V}$ by the edge set $\mathcal{E}^t \subset \mathcal{V} \times \mathcal{V}$. We view this graph as a bipartite graph, whose partite sets are each a copy of $\mathcal{V}$. We number the edges $1, 2, \dots, M^t$. For $i \in \mathcal{V}$, define

$$\mathcal{O}^t(i) = \{\, e \in \mathcal{E}^t \mid e(1) = i \,\}$$
$$\mathcal{I}^t(i) = \{\, e \in \mathcal{E}^t \mid e(2) = i \,\}$$

as the set of edges outgoing from and incoming to node i, respectively. Now we define the ***temporal outgoing incidence matrix*** $H \in \mathbb{R}^{M^t \times N}$ and the ***temporal incoming incidence matrix*** $J \in \mathbb{R}^{M^t \times N}$ as

$$H_{ei} = \begin{cases} 1 & \text{if } e \in \mathcal{O}^t(i) \\ 0 & \text{otherwise} \end{cases} \qquad J_{ei} = \begin{cases} 1 & \text{if } e \in \mathcal{I}^t(i) \\ 0 & \text{otherwise} \end{cases}$$

for all $i \in \mathcal{V}$ and $e \in \mathcal{E}^t$. In this paper, we assume that H is full column rank; that is, every node is the tail of at least one edge. This assumption does not restrict the class of physical systems which may be modeled, since states which do not have an edge outgoing may be removed from the realization.

Suppose $\mathcal{V} = \{1, 2, 3\}$, $\mathcal{E}^s = \{(1,2), (1,3), (2,1)\}$, and $\mathcal{E}^t = \{(1,1), (2,2), (3,3), (3,1)\}$. An example of how we interpret these two graphs is shown in Figure 1. We construct a graph consisting of many copies of $(\mathcal{V}, \mathcal{E}^s)$, with nodes in neighboring copies of $\mathcal{V}$ linked by edges of $\mathcal{E}^t$, as shown in Figure 1(c). In this paper, we use $(\mathcal{V}, \mathcal{E}^t, \mathcal{E}^s)$ to denote this kind of *spatio-temporal graph* and we will use this graph to represent the interconnected dynamic systems evolving over time.

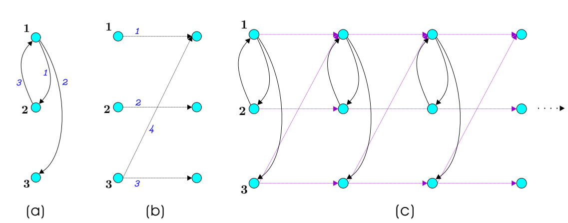

Figure 1: (a) $(\mathcal{V}, \mathcal{E}^s)$, (b) $(\mathcal{V}, \mathcal{E}^t)$ (c) Spatio-temporal graph

3 Dynamical Systems on Graphs

We will use the spatio-temporal graph to represent the *independent variables* associated with the system dynamics; for example, each node of $\mathcal{V}$ may represent a particular vehicle. A ***spatio-temporal system*** $\mathcal{P}$ is defined by $(\mathcal{V}, \mathcal{E}^s, \mathcal{E}^t)$, and matrices

$$\begin{aligned} &\bar{A}_1, \dots, \bar{A}_{M^t} \in \mathbb{R}^{n^t \times n}, && \bar{B}_1, \dots, \bar{B}_{M^t} \in \mathbb{R}^{n^t \times n^w}, \\ &\bar{S}_1, \dots, \bar{S}_{M^s} \in \mathbb{R}^{n^s \times n}, && \bar{T}_1, \dots, \bar{T}_{M^s} \in \mathbb{R}^{n^s \times n^w}, \\ &\bar{C}_1, \dots, \bar{C}_N \in \mathbb{R}^{n^z \times n}, && \bar{D}_1, \dots, \bar{D}_N \in \mathbb{R}^{n^z \times n}. \end{aligned}$$

At time k, the state, input, and output are given by $x(k) \in \mathbb{R}^{nN}$, $w(k) \in \mathbb{R}^{n^w N}$, and $z(k) \in \mathbb{R}^{n^w N}$, where

$$x(k) = \begin{bmatrix} x_1(k) \\ \vdots \\ x_N(k) \end{bmatrix}, \quad w(k) = \begin{bmatrix} w_1(k) \\ \vdots \\ w_N(k) \end{bmatrix}, \quad z(k) = \begin{bmatrix} z_1(k) \\ \vdots \\ z_N(k) \end{bmatrix}$$

and each $x_i(k)$, $w_i(k)$, $z_i(k)$ represents the state, disturbance and output of node i respectively. We partition $x_i(k)$ as $x_i(k) = \begin{bmatrix} x_i^t(k)^T & x_i^s(k)^T \end{bmatrix}^T$ where $x_i^t(k) \in \mathbb{R}^{n^t}$, $x_i^s(k) \in \mathbb{R}^{n^s}$ and $n = n^s + n^t$. Then the dynamic equations are given by

$$\begin{aligned} x_i^t(k+1) &= \sum_{\alpha \in \mathcal{I}_T(i)} \left(\bar{A}_\alpha x_{\alpha(1)}(k) + \bar{B}_\alpha w_{\alpha(1)}(k) \right) \\ x_i^s(k) &= \sum_{\alpha \in \mathcal{I}_S(i)} \left(\bar{S}_\alpha x_{\alpha(1)}(k) + \bar{T}_\alpha w_{\alpha(1)}(k) \right) \\ z_i(k) &= C_i x_i(k) + D_i w_i(k) \end{aligned} \tag{1}$$

with the initial condition $x_i^t(0) = q_i$ for $i = 1, \dots, N$. Here $q_i \in \mathbb{R}^{n^t}$ for each i. When $q = 0$, this defines a linear system $\mathcal{P}$ mapping from w to z.

We can simplify this representation as follows. Given the above matrices, let

$$A_i = \begin{bmatrix} \bar{A}_i \\ 0 \end{bmatrix}, \quad A_i \in \mathbb{R}^{n \times n} \qquad S_i = \begin{bmatrix} 0 \\ \bar{S}_i \end{bmatrix}, \quad S_i \in \mathbb{R}^{n \times n}$$
$$B_i = \begin{bmatrix} \bar{B}_i \\ 0 \end{bmatrix}, \quad B_i \in \mathbb{R}^{n \times n^w} \qquad T_i = \begin{bmatrix} 0 \\ \bar{T}_i \end{bmatrix}, \quad T_i \in \mathbb{R}^{n \times n^w}.$$

Then define the block diagonal matrices

$$\begin{aligned} A &= \mathrm{diag}(A_1, \dots, A_{M^t}), & B &= \mathrm{diag}(B_1, \dots, B_{M^t}), \\ S &= \mathrm{diag}(S_1, \dots, S_{M^s}), & T &= \mathrm{diag}(T_1, \dots, T_{M^s}), \\ C &= \mathrm{diag}(C_1, \dots, C_N) & D &= \mathrm{diag}(D_1, \dots, D_N), \end{aligned}$$

and the above system (1) is equivalent to

$$\begin{aligned} x(k+1) &= J^T A H x(k) + J^T B H w(k) \\ &\quad + G^T S F x(k+1) + G^T T F w(k+1) \\ z(k) &= C x(k) + D w(k). \end{aligned} \tag{2}$$

with initial conditions

$$x(0) = (I - G^T SF)^{-1} G^T TFw(0) + \begin{bmatrix} q_1^T & 0 & \dots & q_N^T & 0 \end{bmatrix}^T$$

The system is ***well-posed*** if $I - G^T SF$ is invertible.

4 Stability and performance analysis

We first state a standard preliminary lemma.

Lemma 1. *Given $A \in \mathbb{R}^{n\times n}$, if there exists $P \in \mathbb{S}^n$ such that $A^T PA - P < 0$, then $I - A$ is invertible.*

Proof. Suppose there exists $P \in \mathbb{S}^n$ such that $A^T PA - P < 0$. Assume $I - A$ is not invertible which implies there exists $x \neq 0$ such that $(I - A)x = 0$. Left and right multiply x^T and x to $A^T PA - P$ gives $x^T(A^T PA - P)x = x^T Px - x^T Px = 0$ which is a contradiction. ∎

We now state the main analysis result in this paper.

Theorem 2. *Suppose $\mathcal{P}$ is a spatio-temporal system with the initial condition $q = 0$. Then $\mathcal{P}$ is well-posed, stable and $\|\mathcal{P}\| < 1$ if there exist symmetric matrices $P_1, \dots, P_N \in \mathbb{S}^n$, $Q^{xx} \in \mathbb{S}^{nM^t}$, $Q^{ww} \in \mathbb{S}^{n^w M^t}$, $W^{xx} \in \mathbb{S}^{nM^s}$, $W^{ww} \in \mathbb{S}^{n^w M^s}$, $Q^{xw} \in \mathbb{R}^{nM^t \times n^w M^t}$, $W^{xw} \in \mathbb{R}^{nM^s \times n^w M^s}$ such that*

$$\mathcal{M}^T P \mathcal{M} - \begin{bmatrix} Q & \\ & W \end{bmatrix} \leq 0 \quad (3)$$

$$\begin{bmatrix} H & 0 \\ 0 & H \\ F & 0 \\ 0 & F \end{bmatrix}^T \begin{bmatrix} Q & \\ & W \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & H \\ F & 0 \\ 0 & F \end{bmatrix} + \mathcal{N} - \begin{bmatrix} P & \\ & I \end{bmatrix} < 0 \quad (4)$$

$$Q^{xx} > 0 \quad (5)$$

where

$$\mathcal{N} = \begin{bmatrix} C & D \end{bmatrix}^T \begin{bmatrix} C & D \end{bmatrix},$$

$$\mathcal{M} = \begin{bmatrix} J \\ G \end{bmatrix}^T \begin{bmatrix} A & B & & \\ & & S & T \end{bmatrix} \quad P = \operatorname{diag}(P_1, \dots, P_N),$$

$$W = \begin{bmatrix} W^{xx} & W^{xw} \\ (W^{xw})^T & W^{ww} \end{bmatrix} \qquad Q = \begin{bmatrix} Q^{xx} & Q^{xw} \\ (Q^{xw})^T & Q^{ww} \end{bmatrix}.$$

Proof. We first consider well-posedness. From the upper-left block of (4), we have

$$H^T Q^{xx} H + F^T W^{xx} F < P - C^T C.$$

We know $F^T W^{xx} F < P$ since $Q^{xx} > 0$. Let $\mathcal{V} = \begin{bmatrix} 0 & 0 & F^T & 0 \end{bmatrix}^T$. Pre- and post-multiplying (3) by $\mathcal{V}^T$, and $\mathcal{V}$ gives

$$F^T S^T GPG^T SF \leq F^T W^{xx} F < P.$$

Using Lemma 1, this shows $I - G^T SF$ is invertible as required. We now consider stability. We define for convenience the notation

$$\mathcal{U}_k = \begin{bmatrix} Hx(k) & Hw(k) & Fx(k+1) & Fw(k+1) \end{bmatrix}^T$$
$$\mathcal{V}_k = \begin{bmatrix} x(k) & w(k) \end{bmatrix}^T$$

and also define the functions

$$R(x) = x^T Px, \qquad V(x) = x^T H^T Q^{xx} Hx,$$

$$L^t(x, w) = \begin{bmatrix} Hx \\ Hw \end{bmatrix}^T Q \begin{bmatrix} Hx \\ Hw \end{bmatrix},$$

$$L^s(x, w) = \begin{bmatrix} Fx \\ Fw \end{bmatrix}^T W \begin{bmatrix} Fx \\ Fw \end{bmatrix}.$$

Pre- and post-multiply (3) by $\mathcal{U}_k^T$ and $\mathcal{U}_k$ to give

$$R(x(k+1)) - L^t(x(k), w(k)) + L^s(x(k+1), w(k+1)) \leq 0.$$

Suppose $x(k + 1) \neq 0$, then multiply (4) by $\mathcal{V}_{k+1}^T$ and $\mathcal{V}_{k+1}$ to give

$$L^t(x(k+1), w(k+1)) + L^s(x(k+1), w(k+1)) - R(x(k+1)) - w(k+1)^T w(k+1) + z(k+1)^T z(k+1) < 0$$

These two inequalities hold for any solution trajectories x, w of the system equation (2) such that $x(k + 1)$ is nonzero. Since we are analyzing stability, let $w(k) = 0$ for all k. Then summing the above two inequalities gives

$$V(x(k+1)) - V(x(k)) < 0$$

Since Q^{xx} is positive definite, a standard Lyapunov argument implies that the system is exponentially stable.

Finally, we consider contractiveness. Assume the initial condition $q = 0$. For $k > 0$, pre- and post-multiply (3) by $\mathcal{U}_{k-1}^T$ and $\mathcal{U}_{k-1}$ to give

$$R(x(k)) - L^t(x(k-1), w(k-1)) - L^s(x(k), w(k)) \leq 0 \quad (6)$$

for $k > 0$. Define $\mathcal{U}_{-1} = \begin{bmatrix} 0 & 0 & Fx(0) & Fw(0) \end{bmatrix}^T$ and pre- and post-multiplying (3) by this gives

$$R(x(0)) - L^s(x(0), w(0)) \leq 0 \quad (7)$$

Also pre- and post-multiply (4) by $\mathcal{V}_k^T$ and $\mathcal{V}_k$ give

$$L^t(x(k), w(k)) + L^s(x(k), w(k)) - R(x(k)) + z(k)^T z(k) - (1 - \gamma) w(k)^T w(k) \leq 0 \quad (8)$$

for some positive γ. Summing (6) and (8) gives

$$L^t(x(k), w(k)) - L^t(x(k-1), w(k-1)) + z(k)^T z(k) - (1 - \gamma) w(k)^T w(k) \leq 0 \quad (9)$$

for $k > 0$, and summing (7) and (8) gives

$$L^t(x(0), w(0)) + z(0)^T z(0) - (1-\gamma) w(0)^T w(0) \leq 0 \quad (10)$$

Sum (9) and (10) for all $k \geq 0$, we have

$$\sum_{k=0}^{\infty} z(k)^T z(k) - (1-\gamma) \sum_{k=0}^{\infty} w(k)^T w(k) \leq \lim_{k \to \infty} L^t(x(k), w(k)) \quad (11)$$

Since $w \in l_2$, $w(k) \to 0$ as $k \to \infty$. From the above, we know the system is stable and so $x_i(k) \to 0$ as $k \to \infty$. Therefore, $L^t(x(k), w(k)) \to 0$ as $k \to \infty$. Thus, we have $\|z\|_2^2 \leq (1-\gamma)\|u\|_2^2$ as desired. ■

4.1 Example: Two-Dimensional Models.

We now consider a two-dimensional state-space model, introduced by Roesser [20]. It is given by

$$\begin{bmatrix} \xi_0(k+1,i) \\ \xi_1(k,i+1) \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} \xi_0(k,i) \\ \xi_1(k,i) \end{bmatrix} + \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} w(k,i)$$

$$z(k,i) = \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix} \begin{bmatrix} \xi_0(k,i) \\ \xi_1(k,i) \end{bmatrix} + \hat{D} w(k,i)$$

Here $k \in \mathbb{N}$ and $i = 0, \ldots, V-1$ are temporal and spatial independent variables. We can write this as

$$\begin{aligned} x_i^t(k+1) &= \bar{A} x_i(k) + \bar{B} w_i(k) \\ x_i^s(k) &= \bar{S} x_{i-1}(k) + \bar{T} w_{i-1}(k) \\ z_i(k) &= \bar{C} x_i(k) + \bar{D} w_i(k), \end{aligned}$$

where $\bar{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \end{bmatrix}$, $\bar{B} = \hat{B}_1$, $\bar{S} = \begin{bmatrix} \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}$, $\bar{T} = \hat{B}_2$, $\bar{C} = \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix}$, $\bar{D} = \hat{D}$. The nodes are arranged in a rectangular array. Define $x_i(k) = \begin{bmatrix} \xi_0(k,i) & \xi_1(k,i) \end{bmatrix}^T$ and

$$A_\alpha = \begin{bmatrix} \bar{A} \\ 0 \end{bmatrix}, \quad S_\beta = \begin{bmatrix} 0 \\ \bar{S} \end{bmatrix}, \quad B_\alpha = \begin{bmatrix} \bar{B} \\ 0 \end{bmatrix}, \quad T_\beta = \begin{bmatrix} 0 \\ \bar{T} \end{bmatrix}.$$

for $\alpha = 1, \ldots, V$ and $\beta = 1, \ldots, V-1$. Then the Roesser dynamical system is equivalent to

$$\begin{aligned} x(k+1) &= J^T A H x(k) + J^T B H w(k) \\ &\quad + G^T B F x(k+1) + G^T B F w(k+1), \\ z(k) &= C x(k) + D u(k) \end{aligned}$$

where $J = I, H = I$,

$$G = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & & 0 \\ & \ddots & \vdots \\ & 1 & 0 \end{bmatrix},$$

and A, B, S, T, C, D are block diagonal matrices defined by $A_\alpha, B_\alpha, S_\beta, T_\beta$. Suppose there exist symmetric matrices P and Q, W satisfying

$$\mathcal{L}^T P \mathcal{L} - \begin{bmatrix} Q & 0 \\ 0 & W \end{bmatrix} \leq 0 \quad (13)$$

$$Q + W - \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} C & D \end{bmatrix}^T \begin{bmatrix} C & D \end{bmatrix} < 0 \quad (14)$$

$$Q^{xx} > 0 \quad (15)$$

where $\mathcal{L} = \begin{bmatrix} A_\alpha & B_\alpha & S_\alpha & T_\alpha \end{bmatrix}$, which is independent of α. Then the system is well-posed, stable and contractive from Theorem 2.

We can write this LMI in a more standard form, as previously used for nodes arranged in a rectangular array in [10]. It is straightforward to show that there exists a P of the form $P = \operatorname{diag}(X_0, X_1)$ with $X_0 > 0$, and matrices Q and W satisfying the matrix inequalities (13) to (15) if and only if there exist symmetric matrices X_0 and X_1 with $X_0 > 0$ such that

$$\begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \hat{B}_1 \\ \hat{A}_{21} & \hat{A}_{22} & \hat{B}_2 \\ \hat{C}_1 & \hat{C}_2 & \hat{D} \end{bmatrix}^T \begin{bmatrix} X_0 & 0 & 0 \\ 0 & X_1 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \hat{B}_1 \\ \hat{A}_{21} & \hat{A}_{22} & \hat{B}_2 \\ \hat{C}_1 & \hat{C}_2 & \hat{D} \end{bmatrix} - \begin{bmatrix} X_0 & 0 & 0 \\ 0 & X_1 & 0 \\ 0 & 0 & I \end{bmatrix} < 0. \quad (16)$$

The proof is obtained by noticing that given X satisfying (16), the matrix P is then defined and one may construct Q and W immediately from (13). Conversely, given P, Q, W satisfying (13) to (15), one has the matrix inequality (16) by pre- and post-multiplying (13) by $\begin{bmatrix} I & I \end{bmatrix}$ and $\begin{bmatrix} I & I \end{bmatrix}^T$ respectively.

5 State Feedback Synthesis

In this section, we will use the analytical result from the previous section to construct a state feedback synthesis procedure. The system considered in this section is

$$\begin{aligned} x_i^t(k+1) &= \sum_{\alpha \in \mathcal{I}^t(i)} \Big(\bar{A}_\alpha x_{\alpha(1)}(k) \\ &\qquad + \bar{B}_\alpha^u u_{\alpha(1)}(k) + \bar{B}_\alpha^w w_{\alpha(1)}(k) \Big) \\ x_i^s(k) &= \sum_{\alpha \in \mathcal{I}^s(i)} \Big(\bar{S}_\alpha x_{\alpha(1)}(k) \\ &\qquad + \bar{T}_\alpha^u u_{\alpha(1)}(k) + \bar{T}_\alpha^w w_{\alpha(1)}(k) \Big) \\ z_i(k) &= C_i x_i(k) + D_i^u u_i(k) + D_i^w w_i(k) \end{aligned} \quad (17)$$

with the initial condition $x_i^t(0) = 0$ for $i = 1, \ldots, N$. Here $u_i(k)$ and $w_i(k)$ are the control input and disturbance at node i respectively. The goal is to find a state feedback controller of the form $u_i(k) = K_i x_i(k)$ such that the closed-loop system is well-posed, stable and contractive. Let

$$A_\alpha = \begin{bmatrix} \bar{A}_\alpha \\ 0 \end{bmatrix}, \quad B_\alpha^u = \begin{bmatrix} \bar{B}_\alpha^u \\ 0 \end{bmatrix}, \quad B_\alpha^w = \begin{bmatrix} \bar{B}_\alpha^w \\ 0 \end{bmatrix},$$

$$S_\alpha = \begin{bmatrix} 0 \\ \bar{S}_\alpha \end{bmatrix}, \quad T_\alpha^u = \begin{bmatrix} 0 \\ \bar{T}_\alpha^u \end{bmatrix}, \quad T_\alpha^w = \begin{bmatrix} 0 \\ \bar{T}_\alpha^w \end{bmatrix},$$

and define

$$\begin{aligned}
A &= \operatorname{diag}(A_1, \ldots, A_{M^t}), & B^u &= \operatorname{diag}(B_1^u, \ldots, B_{M^t}^u),\\
B^w &= \operatorname{diag}(B_1^w, \ldots, B_{M^t}^w), & S &= \operatorname{diag}(S_1, \ldots, S_{M^s}),\\
T^u &= \operatorname{diag}(T_1^u, \ldots, T_{M^s}^u), & T^w &= \operatorname{diag}(T_1^w, \ldots, T_{M^s}^w),\\
C &= \operatorname{diag}(C_1, \ldots, C_N), & D^u &= \operatorname{diag}(D_1^u, \ldots, D_N^u),\\
D^w &= \operatorname{diag}(D_1^w, \ldots, D_N^w).
\end{aligned}$$

Now we state a state feedback synthesis result.

Theorem 3. *Given the system (17), there exist state feedback controllers $K_1, \ldots, K_N$ such that the closed-loop system is well-posed, stable and contractive if there exist symmetric matrices $P_1, \ldots, P_N \in \mathbb{S}^n$, $Y \in \mathbb{S}^{(n+n^w)M^t}$, $Z \in \mathbb{S}^{(n+n^w)M^s}$, $R_1, \ldots, R_N \in \mathbb{R}^{n^u \times n}$ such that*

$$\begin{bmatrix} -Y & 0 & \mathcal{U}^T J \\ 0 & -Z & \mathcal{V}^T G \\ J^T \mathcal{U} & G^T \mathcal{V} & -P \end{bmatrix} < 0 \qquad (18)$$

$$\begin{bmatrix} -P & 0 & (CP + D^u R)^T \\ 0 & -I & (D^w)^T \\ CP + D^u R & D^w & -I \end{bmatrix} + \begin{bmatrix} H & 0 & 0 \\ 0 & H & 0 \\ F & 0 & 0 \\ 0 & F & 0 \end{bmatrix}^T \begin{bmatrix} Y & \\ & Z \end{bmatrix} \begin{bmatrix} H & 0 & 0 \\ 0 & H & 0 \\ F & 0 & 0 \\ 0 & F & 0 \end{bmatrix} < 0 \qquad (19)$$

where

$$\begin{aligned}
P &= \operatorname{diag}(P_1, \ldots, P_N), & R &= \operatorname{diag}(R_1, \ldots, R_N),\\
\mathcal{U} &= \begin{bmatrix} A\mathcal{P}^t + B^u \mathcal{R}^t & B^w \end{bmatrix}, & &\\
\mathcal{V} &= \begin{bmatrix} S\mathcal{P}^s + T^u \mathcal{R}^s & T^w \end{bmatrix}, & &\\
\mathcal{P}^t &= \operatorname*{diag}_{\alpha \in \mathcal{E}^t} P_{\alpha(1)}, & \mathcal{R}^t &= \operatorname*{diag}_{\alpha \in \mathcal{E}^t} R_{\alpha(1)},\\
\mathcal{P}^s &= \operatorname*{diag}_{\alpha \in \mathcal{E}^s} P_{\alpha(1)}, & \mathcal{R}^s &= \operatorname*{diag}_{\alpha \in \mathcal{E}^s} R_{\alpha(1)}.
\end{aligned}$$

These inequalities define a semidefinite program in variables Y, Z, $P_1, \ldots, P_N$, and $R_1, \ldots, R_N$. After solving this SDP, the controller is given by $K_i = R_i P_i^{-1}$.

The theorem follows directly from Theorem 2. The proof is very similar to the synthesis proof in [6] and so is omitted.

5.1 Mechanical Example.

Consider the mass-spring system shown in Figure 2. The mass of each node is 1 and the spring and damping constants of each spring are $k = 0.4$, $b = 0.5$. We will design a decentralized controller to perform disturbance rejection, such that K_1 uses measurements from nodes 2, and 4, K_2 uses measurements from nodes 1 and 3, K_3 uses measurements from nodes 4 and 2, and

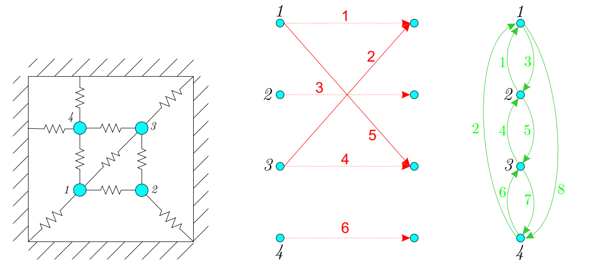

Figure 2: Mass-spring system, and associated temporal and spatial graphs

K_4 uses measurements from nodes 3 and 1. In particular, notice that even though nodes 1 and 3 are directly connected, neither uses information from the other.

The linearized model for perturbations of the system from the equilibrium point, discretized with sampling rate 0.5, can be expressed as

$$\begin{aligned}
x_1(k+1) &= A_1 x_1(k) + A_2 x_2(k) + A_3 x_3(k) + A_4 x_4(k)\\
&\quad + \bar{B}_1 u_1(k) + \bar{B}_1^w w_1(k)\\
x_2(k+1) &= A_6 x_1(k) + A_5 x_2(k) + A_7 x_3(k)\\
&\quad + \bar{B}_2 u_2(k) + \bar{B}_2^w w_2(k)\\
x_3(k+1) &= A_9 x_1(k) + A_{10} x_2(k) + A_8 x_3(k) + A_{11} x_4(k)\\
&\quad + \bar{B}_3 u_3(k) + \bar{B}_3^w w_3(k)\\
x_4(k+1) &= A_{13} x_1(k) + A_{14} x_3(k) + A_{12} x_4(k)\\
&\quad + \bar{B}_4 u_4(k) + \bar{B}_4^w w_4(k)
\end{aligned}$$

and $z_i(k) = C x_i(k)$ for $i = 1, \ldots, 4$. Group the states at each node by letting

$$\bar{x}_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix}, \quad \bar{x}_2 = \begin{bmatrix} x_2 \\ x_1 \\ x_3 \end{bmatrix}, \quad \bar{x}_3 = \begin{bmatrix} x_3 \\ x_2 \\ x_4 \end{bmatrix}, \quad \bar{x}_4 = \begin{bmatrix} x_4 \\ x_1 \\ x_3 \end{bmatrix}.$$

This gives the dynamic equations below, where the spatial and temporal graphs are illustrated in Figure 2.

$$\begin{aligned}
\bar{x}(k+1) &= J^T A_T H \bar{x}(k) + G^T A_S F \bar{x}(k+1)\\
&\quad + J^T B H u(k)\\
\bar{z}(k) &= C \bar{x}(k)
\end{aligned}$$

Let $A_T = \operatorname{diag}(\bar{A}_1, \ldots, \bar{A}_6)$, $A_S = \operatorname{diag}(S_1, \ldots, S_8)$,

$S_7 = S_5 = S_3 = S_1$, and $S_8 = S_6 = S_4 = S_2$, where

$$S_1 = \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad S_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix},$$

$$\bar{A}_1 = \begin{bmatrix} A_1 & A_2 & A_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} A_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\bar{A}_3 = \begin{bmatrix} A_5 & A_6 & A_7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{A}_4 = \begin{bmatrix} A_8 & A_{10} & A_{11} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\bar{A}_5 = \begin{bmatrix} A_9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \bar{A}_6 = \begin{bmatrix} A_{12} & A_{13} & A_{14} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

The H_∞ norm of open loop system is 299.38. Theorem 3 gives a decentralized controller achieving a closed-loop norm of 0.7, and a centralized state-feedback controller achieves a closed-loop H_∞ norm of 0.5.

6 Conclusions

In this paper, we have constructed a representation for discrete-time linear dynamic systems which evolve over graphs. We have also presented a semidefinite program which may be used to test whether the system is well-posed, stable and contractive. We have further shown the result can be reduced to recent LMIs used for performance analysis of multidimensional models. In addition, we have demonstrated an approach for state feedback synthesis, which can also be cast as a semidefinite program. The resulting controllers are decentralized.

References

[1] G. Ayres and F. Paganini. Convex synthesis of localized controllers for spatially invariant systems. *Automatica*, 38(3):445–56, March 2002.

[2] G. Ayres and F. Paganini. An IQC characterization of decentralized Lyapunov functions in spatially invariant systems. In *Proc. IEEE Conference on Decision and Control*, 2002.

[3] B. Bamieh. The structure of optimal controllers of spatially invariant distributed parameter systems. In *Proc. IEEE Conference on Decision and Control*, pages 1056–1061, 1997.

[4] B. Bamieh, F. Paganini, and M.A. Dahleh. Distributed control of spatially invariant systems. *IEEE Trans. Autom. Control*, 47(7):1091–1107, 2002.

[5] C. L. Beck and R. D'Andrea. Simplification of spatially distributed systems. In *Proc. IEEE Conference on Decision and Control*, pages 620–625, 1999.

[6] B. D. Chen and S. Lall. Dissipation inequalities for distributed systems on graphs. In *Proc. IEEE Conference on Decision and Control*, volume 3, pages 3084–3090, 2003.

[7] R. D'Andrea. A linear matrix inequality approach to decentralized control of distributed parameter systems. In *American Control Conference*, pages 1350–1354, 1998.

[8] R. D'Andrea and G.E. Dullerud. Distributed control of spatially interconnected systems. *IEEE Transactions on Automatic Control*, 48(9):1478–1495, 2003.

[9] G. Dullerud and R. D'Andrea. Distributed control of inhomogeneous systems, with boundary conditions. In *Proc. IEEE Conference on Decision and Control*, pages 186–190, 1999.

[10] G. E. Dullerud, R. D'Andrea, and S.G. Lall. Control of spatially varying distributed systems. In *Proc. IEEE Conference on Decision and Control*, volume 2, pages 1889–1893, 1998.

[11] G.E. Dullerud and R. D'Andrea. Distributed control of spatially heterogeneous systems. Preprint; submitted to *IEEE Transactions on Automatic Control*, 2002.

[12] G.E. Dullerud and S. Lall. A new approach for analysis and synthesis of time-varying systems. *IEEE Transactions on Automatic Control*, 44(8):1486–1497, 1999.

[13] D. Gorinevsky and G. Stein. Structured uncertainty analysis of robust stability for spatially distributed systems. In *Proc. IEEE Conference on Decision and Control*, volume 4, pages 3757–3762, 2000.

[14] C. Langbort and R. D'Andrea. Distributed control of heterogeneous systems interconnected over an arbitrary graph. In *Proc. IEEE Conference on Decision and Control*, volume 3, pages 2835–2840, 2003.

[15] A. Megretski and A. Rantzer. System analysis via integral quadratic constraints. *IEEE Trans. Aut. Control*, 42(6):819–830, June 1997.

[16] P. J. Moylan and D. J. Hill. Stability criteria for large scale systems. *IEEE Transactions on Automatic Control*, 23(2):143–149, 1978.

[17] F. Paganini. A recursive information flow system for distributed control arrays. In *American Control Conference*, volume 6, pages 3821–3825, 1999.

[18] J. H. Park. Design of robust decentralized dynamic controller for uncertain large-scale interconnected systems with time-delays. *IEICE Trans. Fundamentals*, E84-A(7), 2001.

[19] G. Dullerud R. D'Andrea and S. Lall. Convex l_2 synthesis for multidimensional systems. In *Proc. IEEE Conference on Decision and Control*, pages 1883–1888, 1998.

[20] R. P. Roesser. A discrete state-space model for linear image processing. *IEEE Transactions on Automatic Control*, 20:1–10, 1975.

[21] G. Scorletti and G. Duc. An LMI approach to decentralized control. *International Journal of Control*, 74(3):211–224, 2001.