Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

A convex parameterization for solving constrained min-max problems with a quadratic cost

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***Abstract*— This paper is concerned with the application of a recent result in the literature on robust optimization to the control of linear discrete-time systems, which are subject to unknown, persistent state disturbances and mixed constraints on the state and input. By parameterizing the control input sequence as an affine function of the disturbance sequence, it is shown that a certain class of finite horizon min-max control problems is convex and that the number of variables and constraints grows polynomially with the problem size. It is assumed that the constraint and the disturbance sets are polyhedral and that the cost is a suitably-chosen quadratic, where the disturbance is negatively weighted as in H_∞ control.**

I. INTRODUCTION

Consider the following discrete-time LTI system:

$$x^+ = Ax + Bu + w, \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $x^+ \in \mathbb{R}^n$ is the successor state, $u \in \mathbb{R}^m$ is the control input and $w \in \mathbb{R}^n$ is the disturbance. The actual values of the state, input and disturbance at a time instant k are denoted by $x(k)$, $u(k)$ and $w(k)$, respectively; where it is clear from the context, x, u and w will be used to denote the current or initial state, input and disturbance. It is assumed that (A,B) is stabilizable and that at each sample instant a measurement of the state is available. The current and future values of the disturbance are unknown and the disturbance is persistent, but contained in a polytope (bounded polyhedron) W. Without loss of generality and in order to simplify notation (see [1], [2] for ways of generalizing the results in this paper), we assume that W is a hypercube:

$$W := \{ w \in \mathbb{R}^n \mid \|w\|_\infty \le \eta \}. \tag{2}$$

The system is subject to polyhedral, mixed constraints on the state and input:

$$\mathscr{Y} := \{ (x,u) \in \mathbb{R}^n \times \mathbb{R}^m \mid Cx + Du \le b \}, \tag{3}$$

where the matrices $C \in \mathbb{R}^{s\times n}$, $D \in \mathbb{R}^{s\times m}$ and the vector $b \in \mathbb{R}^s$; s is the number of affine inequality constraints in (3).

For a given initial state, a time-varying control policy is to be designed, which guarantees that for all disturbance sequences of a length N, the state and input of the closed-loop system is in $\mathscr{Y}$ over the horizon $k = 0,\dots,N-1$. The state is required to be in a target/terminal constraint set X_f at the end of the horizon ($k = N$), where X_f is a polyhedron given by

$$X_f := \{ x \in \mathbb{R}^n \mid Yx \le z \}, \tag{4}$$

the matrix $Y \in \mathbb{R}^{r\times n}$, the vector $z \in \mathbb{R}^r$ and r is the number of affine inequality constraints that define X_f.

NOTATION: $\mathbf{1}$ is an appropriately-size column vector of ones. If A and B are matrices, then $\mathrm{abs}(A)$ is a matrix of the absolute values of the corresponding components of A, $B \succ 0$ denotes that B is positive definite and $A \le B$ is used to denote component-wise inequality.

II. AN AFFINE PARAMETERIZATION OF THE CONTROL INPUT SEQUENCE

Let N be a positive integer and the vectors $\mathbf{v} \in \mathbb{R}^{mN}$ and $\mathbf{w} \in \mathbb{R}^{nN}$ be defined as $[v_0^T \; v_1^T \; \cdots \; v_{N-1}^T]^T$ and $[w_0^T \; w_1^T \; \cdots \; w_{N-1}^T]^T$, where the vectors $v_i \in \mathbb{R}^m$ and $w_i \in \mathbb{R}^n$ for all $i \in \{0,\dots,N-1\}$. Let $\mathscr{W} := W^N := W \times \cdots \times W$.

We define the *strictly block lower triangular* matrix $\mathbf{M} := [M_{i,j}] \in \mathbb{R}^{mN\times nN}$, where the matrices $M_{i,j} \in \mathbb{R}^{m\times n}$ for all $i \in \{0,\dots,N-1\}$ and $j \in \{0,\dots,N-1\}$ and $M_{i,j} := 0$ for all $j \in \{i,\dots,N-1\}$. This constraint on $\mathbf{M}$ is assumed throughout the rest of this paper. The variable ψ is defined as the pair $\psi := (\mathbf{v},\mathbf{M})$.

Using the same affine parameterization of the control input sequence originally proposed in [1], let the current value of the state x define the set of admissible ψ, which will be used to define a feedback policy, as:

$$\Psi_N(x) := \left\{ \psi \;\middle|\; \begin{aligned} & x_{i+1} = Ax_i + Bu_i + w_i,\; x_0 = x, \\ & u_i = v_i + \sum_{j=0}^{i-1} M_{i,j} w_j, \\ & (x_i,u_i) \in \mathscr{Y},\; x_N \in X_f, \\ & \forall i \in \{0,\dots,N-1\},\; \forall \mathbf{w} \in \mathscr{W} \end{aligned} \right\}. \tag{5}$$

The above parameterization has a number of advantages and system-theoretic properties, compared to the case if one were to set $\mathbf{M} = 0$ as in so-called "open-loop" finite horizon control. The reader is referred to [1]–[3] for further details.

By eliminating x_i and u_i from (5), it is easy to find matrices $F \in \mathbb{R}^{q\times mN}$, $G \in \mathbb{R}^{q\times nN}$, $L \in \mathbb{R}^{q\times n}$ and a vector $c \in \mathbb{R}^q$, where $q := sN + r$, such that one can rewrite $\Psi_N(x)$ in (5) as

$$\Psi_N(x) = \{ \psi \mid F\mathbf{v} + (F\mathbf{M} + G)\mathbf{w} \le c + Lx,\; \forall \mathbf{w} \in \mathscr{W} \} \tag{6}$$

$$= \{ \psi \mid F\mathbf{v} + \eta\,\mathrm{abs}(F\mathbf{M} + G)\mathbf{1} \le c + Lx \}. \tag{7}$$

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Remark 1: Note that $\text{abs}(F\mathbf{M}+G)\mathbf{1}$ is a vector formed from the 1-norms of the rows of $F\mathbf{M}+G$. In going from (6) to (7) we have used the well-known fact (see, for example, [1]–[3]) that $a^T\mathbf{w} \leq d$ for all $\mathbf{w} \in \mathscr{W}$ if and only if $\max_{\mathbf{w}} \left\{ a^T\mathbf{w} \mid \|\mathbf{w}\|_\infty \leq \eta \right\} = \eta\|a\|_1 \leq d$, where a is any vector in $\mathbb{R}^{nN}$ and d is any real scalar.

It follows immediately from (7) that $\psi \in \Psi_N(x)$ if and only if there exists a matrix $\Lambda \in \mathbb{R}^{q\times nN}$ such that

$$-\Lambda \leq F\mathbf{M}+G \leq \Lambda \text{ and } F\mathbf{v}+\eta\Lambda\mathbf{1} \leq c+Lx. \quad (8)$$

III. CONTRIBUTION OF THIS PAPER

Consider now the following finite horizon quadratic cost, as encountered in the literature on H_∞ control:

$$J_N(x,\gamma,\psi,\mathbf{w}) := \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i - \gamma^2 w_i^T w_i + x_N P x_N \quad (9)$$

where $x_0 = x$, $x_{i+1} = Ax_i + Bu_i + w_i$ and $u_i = v_i + \sum_{j=0}^{i-1} M_{i,j} w_j$ for all $i \in \{0,\ldots,N-1\}$. The matrices P, Q and R are positive definite and γ is a scalar.

One can eliminate x_i and u_i in (9) to get matrices H_{xx}, $H_{x\mathbf{u}}$, $H_{x\mathbf{w}}$, $H_{\mathbf{uu}}$, $H_{\mathbf{uw}}$, $H_{\mathbf{ww}}$ of suitable dimensions such that

$$\begin{aligned} J_N(x,\gamma,\psi,\mathbf{w}) &= x^T H_{xx} x + 2x^T H_{x\mathbf{u}}\mathbf{v} + \mathbf{v}^T H_{\mathbf{uu}}\mathbf{v} \\ &+ 2x^T(H_{x\mathbf{u}}\mathbf{M} + H_{x\mathbf{w}})\mathbf{w} + 2\mathbf{v}^T(H_{\mathbf{uu}}\mathbf{M}+H_{\mathbf{uw}})\mathbf{w} \\ &- \mathbf{w}^T(\gamma^2 I - H_{\mathbf{ww}} - 2\mathbf{M}^T H_{\mathbf{uw}} - \mathbf{M}^T H_{\mathbf{uu}}\mathbf{M})\mathbf{w}, \end{aligned} \quad (10)$$

where H_{xx} and $H_{\mathbf{uu}}$ are positive definite and $H_{\mathbf{ww}}$ is positive semi-definite.

It is easy to show that $J_N(x,\gamma,\psi,\mathbf{w})$ is a convex function in $\psi := (\mathbf{v},\mathbf{M})$. To see why this is the case, note that it is sufficient to show that $f(\psi,\mathbf{w}) := \mathbf{v}^T H_{\mathbf{uu}}\mathbf{v} + 2\mathbf{v}^T H_{\mathbf{uu}}\mathbf{M}\mathbf{w} + \mathbf{w}^T\mathbf{M}^T H_{\mathbf{uu}}\mathbf{M}\mathbf{w}$ is convex in ψ. Consider now the function $g(\mathbf{u}) := \mathbf{u}H_{\mathbf{uu}}\mathbf{u}$, which is convex in $\mathbf{u}$. Since $f(\psi,\mathbf{w}) = g(\mathbf{v}+\mathbf{M}\mathbf{w})$ and the fact that convexity of a function is preserved under an affine map, it follows that $f(\psi,\mathbf{w})$ is convex in ψ.

Since the pointwise supremum of an arbitrary, infinite set of convex functions is convex, it follows that

$$V_N(x,\gamma,\psi) := \max_{\mathbf{w}\in\mathscr{W}} J_N(x,\gamma,\psi,\mathbf{w}) \quad (11)$$

is a *convex* function in ψ.

Note also that γ can be chosen sufficiently large such that

$$\gamma^2 I - H_{\mathbf{ww}} - \mathbf{M}^T H_{\mathbf{uw}} - H_{\mathbf{uw}}^T\mathbf{M} - \mathbf{M}^T H_{\mathbf{uu}}\mathbf{M} \succ 0. \quad (12)$$

Clearly, if (12) is satisfied, then $J_N(x,\gamma,\psi,\mathbf{w})$ is a *strictly concave* function in $\mathbf{w}$. This implies that $V_N(x,\gamma,\psi)$ can be computed by defining and solving a *tractable, strictly convex* quadratic programming (QP) problem.

Note that the number of variables and constraints in (8) is polynomial in N, n, m, r and s. Observe also that (12) is a quadratic matrix inequality (QMI) that, by Schur complement, can be converted to a linear matrix inequality (LMI) in $\mathbf{M}$ and γ^2. This implies that, for a given initial state $x = x(0)$, a sufficiently large γ and an admissible ψ can be found by solving an LMI defined from (8) and (12).

We can now state the min-max problem that is of interest to us. For a given initial state $x = x(0)$ and γ, let

$$V_N^*(x,\gamma) := \min_{(\psi,\Lambda)} \{V_N(x,\gamma,\psi) \mid (\psi,\Lambda) \text{ satisfy (8) and (12)}\}$$

Recalling from the above that $V_N(x,\gamma,\psi)$ can be calculated efficiently by solving a tractable QP, it follows that one can compute $V_N^*(x,\gamma)$ efficiently using standard tools from convex optimization, such as cutting plane, interior-point and bundle methods.

IV. FINITE ℓ_2 GAIN

As a final, motivating point for this paper, let $\psi^*(x,\gamma)$ and $\Lambda^*(x,\gamma)$ be minimizers of the above min-max problem. Let the initial state $x = x(0)$ and a *time-varying* control policy be given by

$$u(k) = v_k^*(x,\gamma) + \sum_{j=0}^{k-1} M_{k,j}^*(x,\gamma)w(j) \quad (13)$$

for all $k \in \{0,\ldots,N-1\}$. Note that (13) is a causal feedback policy that is dependent on the current state as well as past values of the state and input; since measurements of the state are available and past inputs are known, $w(j)$ in (13) is given by $w(j) = x(j+1) - Ax(j) - Bu(j)$ for all $j \in \{0,\ldots,N-1\}$.

It follows from the optimality of $\psi^*(x,\gamma)$ that if the disturbance sequence $\{w(0),\ldots,w(N-1)\}$ takes on values in W and the input sequence $\{u(0),\ldots,u(N-1)\}$ is defined as in (13), then one has the following finite ℓ_2 gain property: $\sum_{k=0}^{N-1} x(k)^T Q(k) + u(k)^T Ru(k) + x(N)^T P(N) \leq \gamma^2 \sum_{k=0}^{N-1} w(k)^T w(k) + V_N^*(x,\gamma)$. Furthermore, $(x(k),u(k)) \in \mathscr{Y}$ for all $k \in \{0,\ldots,N-1\}$ and $x(N) \in X_f$.

Further research may involve extending the results in this paper to H_∞ receding horizon control [4, Sect. 4.7]. The reader is referred to [2] for some initial results on the robust invariance of receding horizon controllers that are based on the parameterization in (5).

V. ACKNOWLEDGEMENTS

The authors would like to thank David Mayne and Sasa Rakovic for bringing the finite horizon H_∞ problem to their attention and would like to thank them for sharing some of their preliminary results.

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