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On the Duality of Constrained Estimation and Control

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***Abstract*— We establish that the Lagrangian dual of a constrained linear estimation problem is a particular nonlinear optimal control problem. The result has an elegant symmetry, which is revealed when the constrained estimation problem is expressed as an equivalent nonlinear optimisation problem.**

I. Introduction

The relationship between linear estimation and linear quadratic control is well known in the *unconstrained* case. Since the original work of Kalman and others [4], [5], many authors have contributed to further understand this relationship. For example, Kailath, Sayed and Hassibi [3] have explored duality in the unconstrained case using the geometrical concepts of dual bases and orthogonal complements. The connection between the two unconstrained optimisation problems using Lagrangian duality has also been established in, e.g., the recent work of Rao [6].

However, to the best of our knowledge, the duality between estimation and control remains an open question in the *constrained* case. Here we derive the Lagrangian dual of a constrained estimation problem and show that it leads to a particular nonlinear optimal control problem. We then show that the primal constrained estimation problem has an equivalent formulation as a nonlinear optimisation problem, exposing a clear symmetry with its dual.

II. Constrained Estimation

It is well known (e.g., [2]) that the unconstrained state estimation problem for linear systems can be set up as an optimisation problem. Specifically, consider

$$\begin{aligned} x_{k+1} &= Ax_k + Bw_k, \qquad && k = 0, \cdots, N-1\,, \\ y_k &= Cx_k + e_k, && k = 1, \cdots, N\,, \end{aligned} \tag{II.1}$$

where $x_k \in \boldsymbol{R}^n$, $w_k \in \boldsymbol{R}^m$, $y_k \in \boldsymbol{R}^p$, and where $\{w_k\}$, $\{e_k\}$ are i.i.d. sequences having Gaussian distributions $N(0, Q)$ and $N(0, R)$, respectively, and x_0 has a Gaussian distribution $N(\bar{x}_0, P_0)$. We will assume throughout the paper that Q, R and P_0 are symmetric, positive definite matrices. Given $\{y_k\} \triangleq \{y_0, \ldots, y_N\}$, then the minimum variance unbiased estimator of $\{x_k\} \triangleq \{x_0, \ldots, x_N\}$ satisfies

$$\hat{x}^* \triangleq \arg\min_{\hat{x}_k, \hat{e}_k, \hat{w}_k} J(\{\hat{x}_k\}, \{\hat{e}_k\}, \{\hat{w}_k\})\,, \tag{II.2}$$

where

$$J(\{\hat{x}_k\}, \{\hat{e}_k\}, \{\hat{w}_k\}) = \frac{1}{2}\|\hat{x}_0 - \bar{x}_0\|^2_{P_0^{-1}} + \frac{1}{2}\sum_{k=1}^{N}\|\hat{e}_k\|^2_{R^{-1}} + \frac{1}{2}\sum_{k=0}^{N-1}\|\hat{w}_k\|^2_{Q^{-1}}\,, \tag{II.3}$$

and where $\|x\|^2_M$ denotes $x^T Mx$ for a real vector x and real symmetric matrix M.

The optimisation in (II.2) is carried out subject to the linear constraints

$$\begin{aligned} \hat{x}_{k+1} &= A\hat{x}_k + B\hat{w}_k\,, \qquad && k = 0, \cdots, N-1\,, \\ \hat{e}_k &= y_k - C\hat{x}_k\,, && k = 1, \cdots, N\,. \end{aligned} \tag{II.4}$$

It is well known that the solution to (II.2)–(II.3) is the Kalman filter [2].

Here we consider a *constrained* version of the above problem, in which the sequence $\{w_k\}$ has a *truncated* Gaussian distribution, that is, the distribution is a scaled Gaussian distribution $N(0, Q)$ in a region $\Omega \subset \boldsymbol{R}^m$ and zero elsewhere. It can be shown that the appropriate optimisation problem becomes (II.2)–(II.3) subject to the additional constraint $\hat{w}_k \in \Omega$. This yields the estimate which maximises the joint probability density of the states $\{x_0, \ldots, x_N\}$ given the measurements $\{y_0, \ldots, y_N\}$, and leads to the following constrained estimation problem

$$\mathcal{P}_e: \quad \min_{\hat{x}_k, \hat{e}_k, \hat{w}_k} \left\{ \frac{1}{2}\|\hat{x}_0 - \bar{x}_0\|^2_{P_0^{-1}} + \frac{1}{2}\sum_{k=1}^{N}\|\hat{e}_k\|^2_{R^{-1}} + \frac{1}{2}\sum_{k=0}^{N-1}\|\hat{w}_k\|^2_{Q^{-1}} \right\}, \tag{II.5}$$

subject to:

$$\hat{x}_{k+1} = A\hat{x}_k + B\hat{w}_k\,, \quad k = 0, \cdots, N-1\,, \tag{II.6}$$

$$\hat{e}_k = y_k - C\hat{x}_k\,, \qquad k = 1, \cdots, N\,, \tag{II.7}$$

$$\{\hat{x}_0, \ldots, \hat{x}_N, \hat{e}_1, \ldots, \hat{e}_N, \hat{w}_0, \ldots, \hat{w}_{N-1}\} \in X\,, \tag{II.8}$$

where

$$X = \underbrace{\boldsymbol{R}^n \times \cdots \times \boldsymbol{R}^n}_{N+1} \times \underbrace{\boldsymbol{R}^p \times \cdots \times \boldsymbol{R}^p}_{N} \times \underbrace{\Omega \times \cdots \times \Omega}_{N}. \tag{II.9}$$

Provided Ω is a *polyhedral set*, that is, the intersection of a finite number of closed half-spaces, then the above problem is a well studied quadratic programme. Many authors have investigated this *primal* constrained estimation problem (e.g., [7]). In practice, the problem is usually formulated in a moving horizon sense, i.e., the fixed horizon problem (II.5)–(II.9) is solved over a window of size N that is "moved" at each time step. However, our interest here is in the underlying fixed horizon problem (II.5)–(II.9).

III. Background on Lagrangian Duality

Here we review well known facts about Lagrangian duality. Consider the *primal* problem

$$\mathcal{P}: \quad \min f(x)\,, \tag{III.1}$$

subject to:

$$g_i(x) \le 0\,,$$
$$h_i(x) = 0\,,$$
$$x \in X\,.$$

where $f : \boldsymbol{R}^n \to \boldsymbol{R}$, $g : \boldsymbol{R}^n \to \boldsymbol{R}^m$, $h : \boldsymbol{R}^n \to \boldsymbol{R}^l$ and X is a nonempty set in $\boldsymbol{R}^n$. Then the Lagrangian dual problem is

$$\mathcal{D}: \quad \max \theta(u, v)\,, \tag{III.2}$$

subject to:

$$u \ge 0\,.$$

where

$$\theta(u, v) \triangleq \inf\{f(x) + u^T g(x) + v^T h(x) \ : x \in X\} \tag{III.3}$$

is the Lagrangian dual function. The entries of the vectors $u \in \boldsymbol{R}^m$ and $v \in \boldsymbol{R}^l$ are known as the Lagrange multipliers. The following is a well known result (see, for example, Theorem 6.2.4 in [1]).

Theorem 3.1 (Strong Duality Theorem): Let X be a nonempty convex set in $\boldsymbol{R}^n$, and let $f : \boldsymbol{R}^n \to \boldsymbol{R}$, $g : \boldsymbol{R}^n \to \boldsymbol{R}^m$ be convex, and $h : \boldsymbol{R}^n \to \boldsymbol{R}^l$ be affine. Suppose that the following constraint qualification is satisfied. There exist an $\hat{x} \in X$ such that $g(\hat{x}) < 0$ and $h(\hat{x}) = 0$, and $0 \in \operatorname{int} h(X)$ (the interior of $h(X)$), where $h(X) = \{h(x) : x \in X\}$. Then,

$$\inf\{f(x) : x \in X, g(x) \le 0, h(x) = 0\} = \sup\{\theta(u, v) : u \ge 0\}\,. \tag{III.4}$$

Furthermore, if the inf is finite, then $\sup\{\theta(u, v) : u \ge 0\}$ is achieved at $(\bar{u}, \bar{v})$ with $\bar{u} \ge 0$. If the inf is achieved at $\bar{x}$, then $\bar{u}^T g(\bar{x}) = 0$.

Notice that, under the conditions of the theorem, both the primal problem (III.1) and the dual problem (III.2) achieve identical optima. This is often referred to as absence of a *duality gap*.

IV. Duality of Constrained Estimation and Control

The following result establishes strong duality between the constrained estimation problem (II.5)–(II.9) and a particular nonlinear optimal control problem.

Theorem 4.1: Assume that Ω in (II.9) is a nonempty convex set. Given the primal *constrained* fixed horizon estimation problem $\mathcal{P}_e$, defined by Equations (II.5)–(II.9), the Lagrangian dual problem is:

$$\mathcal{D}_e: \quad \min_{\lambda_k, u_k} \left\{ \frac{1}{2}\|A^T\lambda_0 + P_0^{-1}\bar{x}_0\|^2_{P_0} + \frac{1}{2}\sum_{k=1}^{N} \|u_k - R^{-1}y_k\|^2_R + \sum_{k=0}^{N-1}\left[\frac{1}{2}\|\bar{\zeta}_k\|^2_Q + (\zeta_k - \bar{\zeta}_k)^T Q\bar{\zeta}_k\right]\right\} + \gamma \tag{IV.1}$$

subject to:

$$\lambda_{k-1} = A^T\lambda_k + C^T u_k, \quad k = 1, \cdots, N, \tag{IV.2}$$
$$\lambda_N = 0, \tag{IV.3}$$
$$\zeta_k = B^T\lambda_k\,, \qquad k = 0, \cdots, N-1\,, \tag{IV.4}$$
$$\bar{\zeta}_k = Q^{-1/2}\,\Pi_{\tilde{\Omega}}\,Q^{1/2}\,\zeta_k\,, \quad k = 0, \cdots, N-1\,, \tag{IV.5}$$

where γ is the constant term

$$\gamma \triangleq -\frac{1}{2}\|\bar{x}_0\|^2_{P_0^{-1}} - \frac{1}{2}\sum_{k=1}^{N}\|y_k\|^2_{R^{-1}}\,,$$

and where $\Pi_{\tilde{\Omega}}$ denotes the minimum Euclidean distance projection onto $\tilde{\Omega} \triangleq \{z = Q^{-1/2}w : w \in \Omega\}$, that is,

$$\begin{aligned}\Pi_{\tilde{\Omega}} : \boldsymbol{R}^m &\longrightarrow \tilde{\Omega}\\ v &\longmapsto \bar{v} = \Pi_{\tilde{\Omega}}\, v \triangleq \arg\min_{z\in\tilde{\Omega}} \|z - v\|\,.\end{aligned} \tag{IV.6}$$

Moreover, there is no duality gap, that is, the minimum achieved in (II.5) is equal to minus the minimum achieved in (IV.1).

Proof: Consider the primal *constrained* fixed horizon estimation problem $\mathcal{P}_e$, defined by Equations (II.5)–(II.9). From (III.3) and (II.9), the Lagrangian dual function θ is given by

$$\theta(\{\lambda_k\}, \{u_k\}) = \inf_{\hat{w}_k\in\Omega, \hat{x}_k, \hat{e}_k} L(\{\hat{w}_k\}, \{\hat{x}_k\}, \{\hat{e}_k\}, \{\lambda_k\}, \{u_k\})\,, \tag{IV.7}$$

where the function L is defined as,

$$\begin{aligned}L(\{\hat{w}_k\}, \{\hat{x}_k\}, \{\hat{e}_k\}, \{\lambda_k\}, \{u_k\}) &= \frac{1}{2}\|\hat{x}_0 - \bar{x}_0\|^2_{P_0^{-1}} + \frac{1}{2}\sum_{k=1}^{N}\|\hat{e}_k\|^2_{R^{-1}} + \frac{1}{2}\sum_{k=0}^{N-1}\|\hat{w}_k\|^2_{Q^{-1}}\\ &\quad + \sum_{k=0}^{N-1}\lambda_k^T\Big[\hat{x}_{k+1} - A\hat{x}_k - B\hat{w}_k\Big] + \sum_{k=1}^{N}u_k^T\Big[y_k - C\hat{x}_k - \hat{e}_k\Big],\end{aligned} \tag{IV.8}$$

where $\{\lambda_k\}$ and $\{u_k\}$ are the Lagrangian multipliers corresponding, respectively, to the linear equalities (II.6) and

(II.7). The function L can be rewritten as,

$$\begin{aligned}L(\{\hat{w}_k\},\{\hat{x}_k\},\{\hat{e}_k\},\{\lambda_k\},\{u_k\}) &= \frac{1}{2}\|\hat{x}_0-\bar{x}_0\|^2_{P_0^{-1}} \\ &+ \sum_{k=1}^{N}\{\lambda_{k-1}^T\hat{x}_k-\lambda_{k-1}^TA\hat{x}_{k-1}-u_k^TC\hat{x}_k\} \\ &+ \sum_{k=1}^{N}\left\{\frac{1}{2}\|\hat{e}_k\|^2_{R^{-1}}-u_k^T\hat{e}_k+u_k^Ty_k\right\} \\ &+ \sum_{k=0}^{N-1}\left\{\frac{1}{2}\|\hat{w}_k\|^2_{Q^{-1}}-\lambda_k^TB\hat{w}_k\right\}. \end{aligned} \quad \text{(IV.9)}$$

Notice that the terms that depend on the constrained variables $\hat{w}_k$ are independent of the other variables, $\hat{x}_k$ and $\hat{e}_k$, with respect to which the minimisation (IV.7) is carried out. Hence, from the convexity of the function L, the values that achieve the infimum in (IV.7), denoted $\hat{w}_k^*$, $\hat{x}_k^*$ and $\hat{e}_k^*$, can be computed from:

$$\hat{w}_k^* = \arg\min_{\hat{w}_k\in\Omega}\left\{\frac{1}{2}\|\hat{w}_k\|^2_{Q^{-1}}-\lambda_k^TB\hat{w}_k\right\}, \quad k=0,\cdots,N-1, \quad \text{(IV.10)}$$

$$\frac{\partial L(\cdot)}{\partial\hat{x}_0} = P_0^{-1}\left(\hat{x}_0^*-\bar{x}_0\right)-A^T\lambda_0=0, \quad \text{(IV.11)}$$

$$\frac{\partial L(\cdot)}{\partial\hat{x}_k} = \lambda_{k-1}-A^T\lambda_k-C^Tu_k=0, \quad k=1,\cdots,N-1, \quad \text{(IV.12)}$$

$$\frac{\partial L(\cdot)}{\partial\hat{x}_N} = \lambda_{N-1}-C^Tu_N=0, \quad \text{(IV.13)}$$

$$\frac{\partial L(\cdot)}{\partial\hat{e}_k} = R^{-1}\hat{e}_k^*-u_k=0, \quad k=1,\cdots,N. \quad \text{(IV.14)}$$

We will express the optimisation problem (IV.10) in a more convenient way. To this end, define the variables

$$\zeta_k \triangleq B^T\lambda_k, \quad \text{(IV.15)}$$

$$v \triangleq Q^{-1/2}\hat{w}_k, \quad \text{(IV.16)}$$

$$v^* \triangleq Q^{-1/2}\hat{w}_k^*, \quad \text{(IV.17)}$$

to transform (IV.10) into the minimum Euclidean distance problem

$$v^* = \arg\min_{v\in\tilde{\Omega},}\left\{\frac{1}{2}v^Tv-(\zeta_k^TQ^{1/2})v\right\}, \quad \text{(IV.18)}$$

where $\tilde{\Omega} \triangleq \{v : Q^{1/2}v\in\Omega\}$. The solution to (IV.18) can be expressed as

$$v^* = \bar{v} \triangleq \Pi_{\tilde{\Omega}}Q^{1/2}\zeta_k, \quad \text{(IV.19)}$$

where $\Pi_{\tilde{\Omega}}$ is the Euclidean projection (IV.6). Using (IV.17) and (IV.19), the solution to (IV.10) is then

$$\hat{w}_k^* = Q^{1/2}\Pi_{\tilde{\Omega}}Q^{1/2}\zeta_k. \quad \text{(IV.20)}$$

Finally, we define

$$\bar{\zeta}_k \triangleq Q^{-1}\hat{w}_k^* = Q^{-1/2}\Pi_{\tilde{\Omega}}Q^{1/2}\zeta_k. \quad \text{(IV.21)}$$

and introduce an extra variable, $\lambda_N \triangleq 0$, for ease of notation. Thus, from (IV.10)–(IV.15), and (IV.21), we obtain:

$$\hat{w}_k^* = Q\bar{\zeta}_k, \quad k=0,\cdots,N-1, \quad \text{(IV.22)}$$

$$\bar{\zeta}_k \triangleq Q^{-1/2}\Pi_{\tilde{\Omega}}Q^{1/2}\zeta_k, \quad k=0,\cdots,N-1, \quad \text{(IV.23)}$$

$$\zeta_k \triangleq B^T\lambda_k, \quad k=0,\cdots,N-1, \quad \text{(IV.24)}$$

$$\lambda_N \triangleq 0, \quad \text{(IV.25)}$$

$$\lambda_{k-1} = A^T\lambda_k+C^Tu_k, \quad k=1,\cdots,N, \quad \text{(IV.26)}$$

$$\hat{x}_0^* = P_0A^T\lambda_0+\bar{x}_0, \quad \text{(IV.27)}$$

$$\hat{e}_k^* = Ru_k, \quad k=1,\cdots,N. \quad \text{(IV.28)}$$

Substituting (IV.22)–(IV.28) into (IV.9) we obtain, after some manipulations, the Lagrangian dual function:

$$\begin{aligned}\theta(\{\lambda_k\},\{u_k\}) &= L(\{\hat{w}_k^*\},\{\hat{x}_k^*\},\{\hat{e}_k^*\},\{\lambda_k\},\{u_k\}) \\ &= -\frac{1}{2}\{\|A^T\lambda_0\|^2_{P_0}+2\lambda_0^TA\bar{x}_0\} \\ &\quad -\frac{1}{2}\sum_{k=1}^{N}\{\|u_k\|^2_R-2u_k^Ty_k\} \\ &\quad +\sum_{k=0}^{N-1}\{\frac{1}{2}\bar{\zeta}_k^TQ\bar{\zeta}_k-\zeta_k^TQ\bar{\zeta}_k\}.\end{aligned} \quad \text{(IV.29)}$$

Finally, completing the squares in (IV.29), and after further algebraic manipulations, we obtain:

$$\begin{aligned}\theta(\{\lambda_k\},\{u_k\}) &= -\frac{1}{2}\|A^T\lambda_0+P_0^{-1}\bar{x}_0\|^2_{P_0} \\ &\quad -\frac{1}{2}\sum_{k=1}^{N}\|u_k-R^{-1}y_k\|^2_R \\ &\quad -\sum_{k=0}^{N-1}\left[\frac{1}{2}\|\bar{\zeta}_k\|^2_Q+(\zeta_k-\bar{\zeta}_k)^TQ\bar{\zeta}_k\right] \\ &\quad +\frac{1}{2}\|\bar{x}_0\|^2_{P_0^{-1}}+\frac{1}{2}\sum_{k=1}^{N}\|y_k\|^2_{R^{-1}}.\end{aligned} \quad \text{(IV.30)}$$

The formulation of the dual problem $\mathcal{D}_e$ in (IV.1) follows from (III.2)–(III.3), and the fact that $\max\theta = -\min(-\theta)$ and the optimisers are the same. Also, from Theorem 3.1, we conclude that there is no duality gap, that is, the minimum achieved in (II.5) is equal to minus the minimum achieved in (IV.1). ■

A particular case of Theorem 4.1 is the following well known result for the *unconstrained* case.

Corollary 4.2: In the case in which the variables $\hat{w}_k$ in the primal problem $\mathcal{P}_e$ are unconstrained (i.e., $\Omega = R^m$), the dual problem becomes

$$\mathcal{D}_e : \quad \min_{\lambda_k, u_k} \frac{1}{2}\Big\{\|A^T\lambda_0 + P_0^{-1}\bar{x}_0\|^2_{P_0} + \sum_{k=1}^{N} \|u_k - R^{-1}y_k\|^2_R + \sum_{k=0}^{N-1} \|B^T\lambda_k\|^2_Q\Big\} + \gamma\,, \tag{IV.31}$$

subject to:

$$\lambda_{k-1} = A^T\lambda_k + C^T u_k, \quad k = 1, \cdots, N, \tag{IV.32}$$

$$\lambda_N = 0, \tag{IV.33}$$

where $\gamma \triangleq -\frac{1}{2}\|\bar{x}_0\|^2_{P_0^{-1}} - \frac{1}{2}\sum_{k=1}^{N}\|y_k\|^2_{R^{-1}}$ is a constant.

Proof: In the case where $\hat{w}_k$ is unconstrained, it is easily seen that the minimiser of (IV.10) is:

$$\hat{w}_k^* = QB^T\lambda_k = Q\zeta_k, \quad k = 0, \cdots, N-1. \tag{IV.34}$$

Also note that $\bar{\zeta}_k = \zeta_k$ in (IV.21) since the projection (IV.6) reduces to the identity mapping in the unconstrained case. The result then follows upon substituting $\bar{\zeta}_k = \zeta_k = B^T\lambda_k$ in expression (IV.1). ■

V. An Equivalent Formulation of the Primal Problem

In the previous section we have shown that problem $\mathcal{D}_e$ is dual to problem $\mathcal{P}_e$ in (II.5)–(II.9). We can gain further insight by expressing $\mathcal{P}_e$ in a different way. This is facilitated by the following results.

Lemma 5.1: Let $\tilde{\Omega} \subset \boldsymbol{R}^m$ be a closed convex set that contains zero in its interior. Let $v \in \boldsymbol{R}^m$ such that $v \notin \tilde{\Omega}$. Then there exists a unique point $\bar{v} \in \tilde{\Omega}$ with minimum Euclidean distance from v. Furthermore, v and $\bar{v}$ satisfy the following inequality

$$(v - \bar{v})^T\,\bar{v} > 0\,. \tag{V.1}$$

Proof: Since $0 \in \text{int}\,\tilde{\Omega}$ (the interior of $\tilde{\Omega}$) then $\tilde{\Omega}$ is nonempty. From Theorem 2.4.1 of [1] we have that there exists a unique $\bar{v} \in \tilde{\Omega}$ with minimum Euclidean distance from v, and $\bar{v}$ is the minimiser if and only if

$$(v - \bar{v})^T\,(z - \bar{v}) \leq 0\,, \qquad \forall z \in \tilde{\Omega}\,. \tag{V.2}$$

Now, let $\xi \in \text{int}\,\tilde{\Omega}$. This implies that there exists an $\varepsilon > 0$ such that the ball $N_\varepsilon(\xi) \triangleq \{z : \|z - \xi\| < \varepsilon\}$ is contained in $\tilde{\Omega}$. We will show that

$$(v - \bar{v})^T\,(\xi - \bar{v}) < 0\,. \tag{V.3}$$

Since $\xi \in \tilde{\Omega}$, (V.2) holds for $z = \xi$. Thus we only need to show that (V.2) for $z = \xi \in \text{int}\,\tilde{\Omega}$ can never be an equality. Suppose, by contradiction, that

$$(v - \bar{v})^T\,(\xi - \bar{v}) = 0\,. \tag{V.4}$$

Note that $\|v - \bar{v}\| > 0$ since $\tilde{\Omega}$ is closed, and $v \notin \tilde{\Omega}$, $\bar{v} \in \tilde{\Omega}$. Define

$$\tilde{\xi} = \xi + \alpha\frac{v - \bar{v}}{\|v - \bar{v}\|}, \qquad 0 < \alpha < \varepsilon\,, \tag{V.5}$$

hence, $\|\tilde{\xi} - \xi\| = \alpha < \varepsilon$ and $\tilde{\xi} \in N_\varepsilon(\xi)$. We then have, using (V.4) and (V.5), that

$$\begin{aligned}(v - \bar{v})^T(\tilde{\xi} - \bar{v}) &= (v - \bar{v})^T(\xi - \bar{v}) + \alpha\frac{(v-\bar{v})^T(v-\bar{v})}{\|v - \bar{v}\|}\\ &= \alpha\|v - \bar{v}\| > 0.\end{aligned}$$

Thus, we have found a point $\tilde{\xi} \in \tilde{\Omega}$ (since $N_\varepsilon(\xi)$ is contained in $\tilde{\Omega}$) such that $(v - \bar{v})^T(\tilde{\xi} - \bar{v}) > 0$, which contradicts (V.2). Thus, (V.3) must be true. Inequality (V.1) then follows taking $\xi = 0$, which is in the interior of $\tilde{\Omega}$ by assumption. ■

Lemma 5.2: Let $f : \boldsymbol{R}^m \to \boldsymbol{R}$ be any function and let $\Omega \subset \boldsymbol{R}^m$ be a closed convex set that contains zero in its interior. Consider the optimisation problem

$$\mathcal{P}_1' : \quad \min_w J(w)\,, \tag{V.6}$$

with

$$J(w) \triangleq f(\bar{w}) + (w - \bar{w})^T Q^{-1}\bar{w}\,, \tag{V.7}$$

$$\bar{w} \triangleq Q^{1/2}\,\Pi_{\tilde{\Omega}}\,Q^{-1/2}\,w\,, \tag{V.8}$$

where $\Pi_{\tilde{\Omega}}$ is the mapping that assigns to any vector v in $\boldsymbol{R}^m$ the vector $\bar{v}$ in $\tilde{\Omega}$ that is closest to v in Euclidean distance, that is

$$\begin{aligned}\Pi_{\tilde{\Omega}} : \boldsymbol{R}^m &\longrightarrow \tilde{\Omega}\\ v &\longmapsto \bar{v} = \Pi_{\tilde{\Omega}}\,v \triangleq \arg\min_{z\in\tilde{\Omega}} \|z - v\|\,.\end{aligned} \tag{V.9}$$

The set $\tilde{\Omega}$ is defined as

$$\tilde{\Omega} \triangleq \{z = Q^{-1/2}w : w \in \Omega\}\,. \tag{V.10}$$

Then any solution w^* to (V.6)–(V.10) satisfies $w^* \in \Omega$.

Proof: Suppose, by contradiction, that $w^* \notin \Omega$. Let,

$$\bar{w}^* \triangleq Q^{1/2}\,\Pi_{\tilde{\Omega}}\,Q^{-1/2}\,w^*. \tag{V.11}$$

Notice that $\bar{w}^* \in \Omega$ since (V.8), with $\Pi_{\tilde{\Omega}}$ and $\tilde{\Omega}$ defined in (V.9) and (V.10), respectively, defines a projection of $\boldsymbol{R}^m$ onto Ω.

Define,

$$v^* \triangleq Q^{-1/2}w^*\,, \qquad \bar{v}^* \triangleq Q^{-1/2}\bar{w}^*\,. \tag{V.12}$$

Then, by construction, v^* and $\bar{v}^*$ satisfy,

$$\bar{v}^* = \Pi_{\tilde{\Omega}}v^*\,, \tag{V.13}$$

and, in particular, $\bar{v}^* \in \tilde{\Omega}$. Using (V.8), (V.11) and (V.12) in (V.7) we obtain,

$$\begin{aligned}J(w^*) &= f(\bar{w}^*) + (w^* - \bar{w}^*)^T Q^{-1}\bar{w}^*\\ &= f(Q^{1/2}\bar{v}^*) + (v^* - \bar{v}^*)^T\bar{v}^*\,.\end{aligned} \tag{V.14}$$

Also, since $\bar{v}^* \in \tilde{\Omega}$, we have $\overline{\bar{w}^*} \triangleq Q^{1/2}\,\Pi_{\tilde{\Omega}}\,Q^{-1/2}\,\bar{w}^* = \bar{w}^*$. Thus, similarly to what was done in (V.14), we obtain,

$$J(\bar{w}^*) = f(\bar{w}^*) + (\bar{w}^* - \bar{w}^*)^T Q^{-1}\bar{w}^* = f(Q^{1/2}\bar{v}^*). \tag{V.15}$$

It is easy to see that $\tilde{\Omega}$ in (V.10) is a closed convex set and $0 \in \text{int}\, \tilde{\Omega}$ since $Q^{1/2} > 0$. From Lemma 5.1, equation (V.13), and the definition of $\Pi_{\tilde{\Omega}}$ in (V.9), we conclude that

$$(v^* - \bar{v}^*)^T \bar{v}^* > 0\,.$$

Hence, from (V.14) and (V.15), we have

$$J(w^*) - J(\bar{w}^*) = (v^* - \bar{v}^*)^T \bar{v}^* > 0\,.$$

We have thus found a point $\bar{w}^* \in \Omega$ that yields a strictly lower value for the cost, which contradicts the fact that w^* is a solution of (V.6)–(V.10). It follows that w^* must be in Ω, and the proof is then completed. ■

Corollary 5.3: Under the conditions of Lemma 5.2, problem $\mathcal{P}_1'$ defined by (V.6)–(V.10) is equivalent to the following problem

$$\mathcal{P}_1 : \quad \min_{w \in \Omega} f(w)\,. \tag{V.16}$$

Proof: Lemma 5.2 shows that any solution to (V.6)–(V.10) belongs to Ω, and hence we can perform the minimisation of (V.7) in Ω without losing optimal solutions. Since the mapping $Q^{1/2}\,\Pi_{\tilde{\Omega}}\,Q^{-1/2}$ used in (V.8) reduces to the identity mapping in Ω, we conclude that (V.7) is equal to the cost function in (V.16) for all $w \in \Omega$, and thus both problems are equivalent. ■

Corollary 5.4: Let $f : \boldsymbol{R}^n \times \boldsymbol{R}^m \times \cdots \times \boldsymbol{R}^m \to \boldsymbol{R}$ be any function and let $\Omega \subset \boldsymbol{R}^m$ be a closed convex set that contains zero in its interior. Consider the optimisation problem

$$\mathcal{P}_2' : \quad \min_{x_0, w_0, \ldots, w_{N-1}} J(x_0, w_0, \ldots, w_i, \ldots, w_{N-1})\,, \tag{V.17}$$

with

$$\begin{aligned} J(x_0, w_0, \ldots, w_i, \ldots, w_{N-1}) & \\ \triangleq f(x_0, \bar{w}_0, \ldots, \bar{w}_i, \ldots, \bar{w}_{N-1}) & \\ + \sum_{k=0}^{N-1} (w_k - \bar{w}_k)^T Q^{-1} \bar{w}_k\,, & \end{aligned} \tag{V.18}$$

$$\bar{w}_i = Q^{1/2}\,\Pi_{\tilde{\Omega}}\,Q^{-1/2}\,w_i\,, \quad i = 0, \ldots, N-1\,, \tag{V.19}$$

where $\Pi_{\tilde{\Omega}}$ and $\tilde{\Omega}$ are defined in (V.9) and (V.10), respectively.

Then any solution $\{x_0^*, w_0^*, \ldots, w_i^*, \ldots, w_{N-1}^*\}$ of (V.17)–(V.19) satisfies $w_i^* \in \Omega$ for $i = 0, \ldots, N-1$.

Proof: Let $\{x_0^*, w_0^*, \ldots, w_i^*, \ldots, w_{N-1}^*\}$ be an optimal solution of (V.17)–(V.19), and suppose $w_i^* \notin \Omega$ for some i. Via a similar argument to that used in the proof of Lemma 5.2, we can show that the sequence $\{x_0^*, w_0^*, \ldots, \bar{w}_i^*, \ldots, w_{N-1}^*\}$, with $\bar{w}_i^*$ computed from (V.19), gives a lower value of the cost (V.18). Thus w_i^* must belong to Ω. Since the same is true for all $i = 0, \ldots, N-1$, the result follows. ■

Corollary 5.5: Under the conditions of Corollary 5.4, problem $\mathcal{P}_2'$ defined by (V.17)–(V.19) is equivalent to the following problem

$$\mathcal{P}_2 : \quad \min_{w_k \in \Omega, x_0} f(x_0, w_0, \ldots, w_i, \ldots, w_{N-1})\,. \tag{V.20}$$

Proof: Similar to the proof of Corollary 5.3. ■

We are now ready to express the primal estimation problem $\mathcal{P}_e$ defined by equations (II.5)–(II.9) in an equivalent form. This is done in the following theorem.

Theorem 5.6 (Equivalent Primal Formulation): Assume that Ω is a convex set that contains zero in its interior. Then the primal estimation problem $\mathcal{P}_e$ defined by equations (II.5)–(II.9) is equivalent to the following optimisation problem

$$\begin{aligned} \mathcal{P}_e' : \quad \min_{\hat{x}_k, \hat{e}_k, \hat{w}_k} & \left\{ \frac{1}{2} \|\hat{x}_0 - \bar{x}_0\|^2_{P_0^{-1}} + \frac{1}{2} \sum_{k=1}^{N} \|\hat{e}_k\|^2_{R^{-1}} \right. \\ & \left. + \sum_{k=0}^{N-1} \left[\frac{1}{2} \|\bar{w}_k\|^2_{Q^{-1}} + (\hat{w}_k - \bar{w}_k)^T Q^{-1} \bar{w}_k \right] \right\}, \end{aligned} \tag{V.21}$$

subject to:

$$\hat{x}_{k+1} = A\hat{x}_k + B\bar{w}_k\,, \qquad k = 0, \cdots, N-1\,, \tag{V.22}$$

$$\hat{e}_k = y_k - C\hat{x}_k\,, \qquad k = 1, \cdots, N\,,$$

$$\bar{w}_k = Q^{1/2}\,\Pi_{\tilde{\Omega}}\,Q^{-1/2}\,\hat{w}_k\,, \quad k = 0, \ldots, N-1\,, \tag{V.23}$$

where $\Pi_{\tilde{\Omega}}$ and $\tilde{\Omega}$ are defined in (V.9) and (V.10), respectively.

Proof: First note that, using the equations (II.4), the cost function (II.3) can be written in the form

$$J(\{\hat{x}_k\}, \{\hat{e}_k\}, \{\hat{w}_k\}) = f(\hat{x}_0, \hat{w}_0, \ldots, \hat{w}_i, \ldots, \hat{w}_{N-1})\,.$$

Since the minimisation of the above cost function is performed for $\hat{x}_0 \in \boldsymbol{R}^n$, and for $\hat{w}_k \in \Omega$, we conclude that problem $\mathcal{P}_e$ can be written in the form (V.20). Using Corollary 5.5 we can then express $\mathcal{P}_e$ in the form of problem $\mathcal{P}_2'$ defined by (V.17)–(V.19). However, this is equivalent to (V.21)–(V.23) (note the presence of $\bar{w}_k$ in (V.22)), and the result then follows. ■

VI. Symmetry of Constrained Estimation and Control

In summary, we have shown that the two following problems are dual in the Lagrangian sense.

Primal Problem (Equivalent Formulation):

$$\begin{aligned} \mathcal{P}_e' : \quad \min_{\hat{x}_k, \hat{e}_k, \hat{w}_k} & \left\{ \frac{1}{2} \|\hat{x}_0 - \bar{x}_0\|^2_{P_0^{-1}} + \frac{1}{2} \sum_{k=1}^{N} \|\hat{e}_k\|^2_{R^{-1}} \right. \\ & \left. + \sum_{k=0}^{N-1} \left[\frac{1}{2} \|\bar{w}_k\|^2_{Q^{-1}} + (\hat{w}_k - \bar{w}_k)^T Q^{-1} \bar{w}_k \right] \right\}, \end{aligned}$$

subject to:

$$\hat{x}_{k+1} = A\hat{x}_k + B\bar{w}_k\,, \qquad k = 0, \cdots, N-1\,,$$

$$\hat{e}_k = y_k - C\hat{x}_k\,, \qquad k = 1, \cdots, N\,,$$

$$\bar{w}_k = Q^{1/2}\,\Pi_{\tilde{\Omega}}\,Q^{-1/2}\,\hat{w}_k\,, \quad k = 0, \ldots, N-1\,.$$

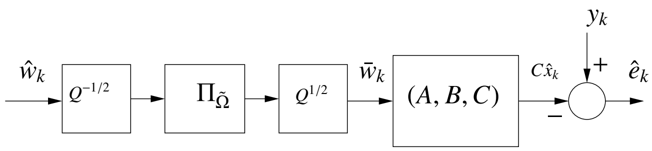

Fig. 1. Configuration for the Primal Problem (Equivalent Formulation)

Dual Problem:

$$\mathcal{D}_e : \quad \min_{\lambda_k, u_k} \left\{ \frac{1}{2} \|A^T \lambda_0 + P_0^{-1} \bar{x}_0\|^2_{P_0} + \frac{1}{2} \sum_{k=1}^{N} \|u_k - R^{-1} y_k\|^2_R + \sum_{k=0}^{N-1} \left[\frac{1}{2} \|\bar{\zeta}_k\|^2_Q + (\zeta_k - \bar{\zeta}_k)^T Q \bar{\zeta}_k \right] \right\} + \gamma$$

subject to:

$$\lambda_{k-1} = A^T \lambda_k + C^T u_k, \qquad k = 1, \cdots, N,$$
$$\lambda_N = 0,$$
$$\zeta_k = B^T \lambda_k, \qquad k = 0, \cdots, N-1,$$
$$\bar{\zeta}_k = Q^{-1/2} \Pi_{\tilde{\Omega}} Q^{1/2} \zeta_k, \qquad k = 0, \cdots, N-1,$$

where $\gamma = -\frac{1}{2}\|\bar{x}_0\|^2_{P_0^{-1}} - \frac{1}{2}\sum_{k=1}^{N} \|y_k\|^2_{R^{-1}}$.

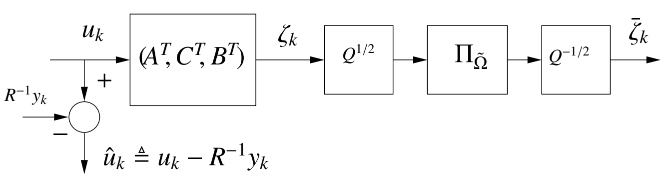

Fig. 2. Configuration for the Dual Problem

In the above two problems, $\Pi_{\tilde{\Omega}}$ is the minimum Euclidean distance projection defined in (IV.6).

Figures 1 and 2 illustrate the primal equivalent problem $\mathcal{P}'_e$ and the dual problem $\mathcal{D}_e$. Note from the figures the symmetry between both problems; namely, inputs become outputs, system matrices *commute*: $A \leftrightarrow A^T$, $B \leftrightarrow C^T$, $C \leftrightarrow B^T$, time is *reversed* and input projections become output projections.

VII. Scalar Case

The above duality result takes a particularly simple form in the scalar input case, that is, when $m = 1$ in (II.1). We assume $\Omega = \{w : |w| \leq \Delta\}$, where Δ is a positive constant, and take $Q = 1$ in the cost function (II.3), without loss of generality since we can always scale by this factor.

The (equivalent) primal and dual problems are then:

Primal Problem:

$$\mathcal{P}'_e : \quad \min_{\hat{x}_k, \hat{e}_k, \hat{w}_k} \frac{1}{2} \left\{ \|\hat{x}_0 - \bar{x}_0\|^2_{P_0^{-1}} + \sum_{k=1}^{N} \|\hat{e}_k\|^2_{R^{-1}} + \sum_{k=0}^{N-1} \left[\hat{w}_k^2 - (\hat{w}_k - \mathrm{sat}_\Delta \hat{w}_k)^2 \right] \right\},$$

subject to:

$$\hat{x}_{k+1} = A\hat{x}_k + B\mathrm{sat}_\Delta \hat{w}_k, \quad k = 0, \cdots, N-1,$$
$$\hat{e}_k = y_k - C\hat{x}_k, \quad k = 1, \cdots, N.$$

Dual Problem:

$$\mathcal{D}_e : \quad \min_{\lambda_k, u_k} \frac{1}{2} \left\{ \|A^T \lambda_0 + P_0^{-1} \bar{x}_0\|^2_{P_0} + \sum_{k=1}^{N} \|u_k - R^{-1} y_k\|^2_R + \sum_{k=0}^{N-1} \left[\zeta_k^2 - (\zeta_k - \mathrm{sat}_\Delta \zeta_k)^2 \right] \right\} + \gamma$$

subject to:

$$\lambda_{k-1} = A^T \lambda_k + C^T u_k, \quad k = 1, \cdots, N,$$
$$\lambda_N = 0,$$
$$\zeta_k = B^T \lambda_k, \quad k = 0, \cdots, N-1,$$

where $\gamma = -\frac{1}{2}\|\bar{x}_0\|^2_{P_0^{-1}} - \frac{1}{2}\sum_{k=1}^{N} \|y_k\|^2_{R^{-1}}$.

In the above two problems, $\mathrm{sat}_\Delta u$ is the usual saturation function defined as $\mathrm{sat}_\Delta u = \mathrm{sign}\, u \, \min(|u|, \Delta)$.

VIII. Conclusions

This paper has established a form of strong duality between constrained estimation and control. The result has a pleasing symmetry when the primal estimation problem is expressed in an equivalent form.

References

[1] M.S. Bazaraa, H.D. Sherali, and C.M. Shetty. *Nonlinear Programming: Theory and Algorithms*. J. Wiley, New York, 2nd. edition, 1993.

[2] A.H. Jazwinski. *Stochastic processes and filtering theory*. New York, Academic Press, 1970.

[3] T. Kailath, A.H. Sayed, and B. Hassibi. *Linear Estimation*. Prentice Hall, New York, 2000.

[4] R.E. Kalman. A new approach to linear filtering and prediction problems. *Trans. ASME, J. Basic Engrg.*, 82:34–45, 1960.

[5] R.E. Kalman and R.S. Bucy. New results in linear filtering and prediction theory. *Trans. ASME, J. Basic Engrg.*, 83-D:95–107, 1961.

[6] C.V. Rao. *Moving Horizon Strategies for the Constrained Monitoring and Control of Nonlinear Discrete-Time Systems*. PhD thesis, University of Wisconsin-Madison, 2000.

[7] C.V. Rao, J.B. Rawlings, and D.Q. Mayne. Constrained state estimation for nonlinear discrete-time systems: stability and moving horizon approximations. *IEEE Trans. on Automatic Control*, 48(2):246 –258, 2003.