Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

The static output feedback stabilization problem as a concave-convex programming problem

Alessandro Astolfi and Patrizio Colaneri

Abstract— **It is shown that the static output feedback stabilization problem for linear multi-input single-output (MISO) systems can be posed as a concave-convex programming problem. This allows the potential design of minimization algorithms yielding a stabilizing static output feedback gain, if it exists, or showing that the problem is not solvable at all.**

I. Introduction and preliminary results

The static output feedback (SOF) stabilization problem is probably one of the most known puzzle in systems and control.

The fundamental question of the existence (in the general case) of a stabilizing static output feedback control law is still open. Many attempts have been made in the last years, so that at this stage we can count several nontrivial contributions to the problem, both numerical and speculative, see the paper [8], where the state of the art is presented and the existing methods are surveyed and compared. Finally a few algorithms have been recently proposed. Among them the most interesting seem to be the min/max procedure proposed in [6], the cone complementary algorithm of [5], and the ILMI approach of [9].

In the series of papers [1], [2], [4], [3], a new necessary and sufficient condition, which is a modification of a known characterization (see [7]), has been developed. This novel characterization allows to study in-depth the structure of the SOF stabilization problem for multi-input single-output (MISO) systems (or equivalently for single-input multi-output (SIMO) systems) and to show that the problem can be posed as an LMI problem subject to a set of quadratic inequality constraints.

Goal of this paper is to show that such a constrained LMI problem can be reformulated as a concave-convex programming problem, *i.e.* the SOF stabilization problem can be formulated as a minimization problem for a concave function on a convex set. For, we first recall a few preliminary results. Consider the continuous-time linear system

$$\dot{x} = Ax + Bu \tag{1}$$
$$y = Cx \tag{2}$$

where $x \in I\!\!R^n$, $u \in I\!\!R^m$, $y \in I\!\!R^p$ are the state, input and output vectors, respectively, and A, B, C are matrices with constant real coefficients and appropriate dimensions.

A. Astolfi is with the Electrical Engineering Dept., Imperial College London, Exhibition Road, London SW7 2AZ, UK a.astolfi@imperial.ac.uk

P. Colaneri is with Dipartimento di Elettronica e Informazione, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133, Milano, Italy colaneri@elet.polimi.it

The *Static Output Feedback stabilization problem* for system (1)-(2) consists in finding, if possible, a static control law described by

$$u = Fy \tag{3}$$

such that the closed-loop system is asymptotically stable, *i.e.* the matrix $A+BFC$ has all its eigenvalues with negative real parts. If such an output feedback does exist, we say that the system (1)-(2) is *SOF stabilizable* and that F is a solution of the problem[1]. In what follows, whenever we deal with the linear system (1)-(2) we make the following standing assumptions, which are trivially without loss of generality.

(A1) The pair $\{A, B\}$ is controllable and B has full column rank.
(A2) The pair $\{A, C\}$ is observable and C has full row rank.

A simple necessary and sufficient condition for the solvability of the problem is summarized in the following statement, see [2].

Theorem 1: Consider the system (1)-(2) with Assumptions (A1) and (A2). The system is output feedback stabilizable if and only if there exists a symmetric positive definite matrix P such that

$$0 \geq A'P + PA - PBB'P + C'C \tag{4}$$
$$0 = V(A'P + PA)V \tag{5}$$

where $V = I - C'(CC')^{-1}C$. Moreover, a stabilizing gain is given by

$$F = (T'G - B'P)C'(CC')^{-1}$$

where T is any orthogonal matrix satisfying

$$GV = TB'PV$$

and G is such that

$$A'P + PA - PBB'P + C'C + G'G = 0.$$

$\triangleleft$

II. MISO linear systems

In this section we restrict our interest to linear systems described by equations of the form (1)-(2) with $p = 1$ (MISO systems) and satisfying Assumption (A1) and (A2).

[1] It is obvious that, if a solution exists, this is not unique.

For MISO observable systems, and without loss of generality, it is possible to write the system in the observability canonical form, *i.e.* with

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix}, \qquad (6)$$

and

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \qquad (7)$$

We now study the structure of the set of solutions of equations (5) enforced by the form of the matrices A and C given in (6) and (7).

Lemma 1: Let A and C be as in (6) and (7). All symmetric matrices $P = \{P_{i,j}\}$ satisfying (5) are such that

$$0 = P_{i,j}, \qquad \text{if } i+j = \text{ odd} \qquad (8)$$

$$0 = P_{i,j} + P_{i+1,j-1}, \qquad \text{if } i \neq j \text{ and } i+j = \text{ even} \qquad (9)$$

◁

Definition 1: The set of all symmetric matrices satisfying (8) will be denoted by $\mathcal{P}$. The set of all symmetric matrices satisfying (8) and (9) will be denoted by Π. ◁

It is important to point out that the set Π does not depend on the system matrices. In fact its structure is inherited from the particular parameterization selected, namely the observability canonical form. We now investigate the properties of the solutions of the equations (4) and (5) noting that these can be equivalently written in terms of coupled LMI's, namely

$$0 \leq \begin{bmatrix} -AX - XA' + BB' & XC' \\ CX & I \end{bmatrix} \qquad (10)$$

$$0 = V(A'P + PA)V \qquad (11)$$

$$I = XP. \qquad (12)$$

Note now that the set of all X satisfying the condition (10) is convex, and it is also convex the set of all P satisfying condition (11). Unfortunately, the coupling condition (12) is not convex, hence we conclude that the problem is in general non-convex. In general, equation (11) cannot be easily recast in terms of the unknown $X = P^{-1}$. However, as shown below, for MISO systems with A and C as in equations (6) and (7), it is possible to characterize the structure of the solutions of equation (11) in terms of property of the matrix X. More precisely, starting from the properties of the sets $\mathcal{P}$ and Π it is possible to work out the properties and the structure of the sets

$$\mathcal{X} = \{X = P^{-1} \mid P \in \mathcal{P}\} \quad \Xi = \{X = P^{-1} \mid P \in \Pi\}.$$

In what follows, we observe that using the set Ξ, and the LMI (10) it is possible to give further characterizations for the solution of the SOF stabilization problem for MISO systems, *i.e.* it will be noted that the SOF stabilization problem can be recast as an LMI problem with quadratic constraints, and this in turn, can be reformulated as a convex-concave programming problem.

III. Characterization of the set Ξ

In this section, which is based on the results in [3], we provide some properties of the set Ξ. Note that, the corresponding set Π can be described by linear equations, however this is not the case for Ξ, which can be however described by a set of quadratic constraints.

Lemma 2: The set Ξ of all positive matrices X such that $P = X^{-1} \in \Pi$ can be described by a set of quadratic equations (constrains) in the elements X_{ij} of X. ◁

Corollary 1: Consider the system (1)-(2) and assume that (A1) and (A2) hold. Moreover assume that $p = 1$ and that the system is in the observability canonical form. Then, the SOF stabilization problem is solvable if and only if there exists a positive definite matrix $X \in \Xi$ satisfying the LMI (10). ◁

Example 1: If $n = 5$, then a matrix X is in the set $\mathcal{X}$ if it is of the form

$$X = \begin{bmatrix} X_{11} & 0 & X_{13} & 0 & X_{15} \\ 0 & X_{22} & 0 & X_{24} & 0 \\ X_{13} & 0 & X_{33} & 0 & X_{35} \\ 0 & X_{24} & 0 & X_{44} & 0 \\ X_{15} & 0 & X_{35} & 0 & X_{55} \end{bmatrix}$$

moreover it belongs to the set Ξ if in addition the following quadratic constraints hold

$$0 = X_{15}X_{44} + X_{15}X_{35} - X_{55}X_{13}$$

$$0 = X_{15}X_{33} - X_{13}X_{35} + X_{15}X_{24}$$

$$0 = X_{11}X_{55} - X_{15}X_{24} - X_{15}^2$$

$$0 = X_{15}X_{22} + X_{15}X_{13} - X_{35}X_{11}.$$

The above quadratic constrains can be (uniquely) solved for X_{11}, X_{22}, X_{33} and X_{44} and the solution can be substituted in the matrix X. As a result, in this example, the solution of the SOF stabilization problem reduces to an LMI condition for a matrix X which is parameterized through rational functions of five unknowns. ◁

Remark 1: From the example above, it is possible to draw a general conclusion: the SOF stabilization problem can be always recast as an LMI feasibility condition for a matrix X which is parameterized through rational functions of n unknowns. ◁

IV. The SOF stabilization problem as a concave-convex programming problem

In this section we show that the SOF stabilization problem for MISO systems can be recast as a concave minimization problem on a convex set. For, note that in view of Lemma 2 the condition $X \in \Xi$ can be written as[2]

$$(\text{vec}(X))' M_i \text{vec}(X) = (\text{vec}(X))' N_i \text{vec}(X) \qquad (13)$$

[2]We denote with $\text{vec}(X)$ the vector composed of all non-zero entries of the matrix $X \in \mathcal{X}$.

for $i = 1, 2, \cdots, \nu$, where

$$\nu = \begin{cases} \frac{(n-1)^2}{4} & \text{if } n \text{ odd} \\ \frac{n(n-2)}{4} & \text{if } n \text{ even,} \end{cases} \qquad (14)$$

and M_i and N_i are suitable positive semi-definite matrices.

As a result, by Corollary 1, we obtain the following characterization of the SOF stabilization problem.

Theorem 2: Consider the system (1)-(2) with A and C as in equations (6) and (7). The system is output feedback stabilizable if and only only if the following constraint optimization problem

$$\min_{X, \rho_1, \cdots, \rho_\nu} J(X, \rho_1, \cdots, \rho_\nu)$$

with

$$J(X, \rho_1, \cdots, \rho_\nu) =$$

$$\sum_{i=1}^{\nu} [2\rho_i - (\text{vec}(X))' M_i \text{vec}(X) - (\text{vec}(X))' N_i \text{vec}(X)]$$

subject to

$$\begin{array}{rcl} 0 & < & X \leq X_0 = P_0^{-1} \\ 0 & \leq & \rho_i \\ 0 & \leq & \begin{bmatrix} -AX - XA' + BB' & XC' \\ CX & I \end{bmatrix} \\ \rho_i & \geq & (\text{vec}(X))' M_i \text{vec}(X) \\ \rho_i & \geq & (\text{vec}(X))' N_i \text{vec}(X) \end{array} \qquad (15)$$

where P_0 is the unique positive definite solution of the Riccati equation

$$A' P_0 + P_0 A - P_0 B B' P_0 + C' C = 0, \qquad (16)$$

has a solution $X^\star, \rho_1^\star, \cdots, \rho_\nu^\star$ such that $J(X^\star, \rho_1^\star, \cdots, \rho_\nu^\star) = 0$. ◁

Note that the function $J(\bar{X}, \rho_1^\star, \cdots, \rho_\nu^\star) = 0$ is (non-strictly) concave, whereas the set of all X and ρ_i such that conditions (15) hold is convex and bounded. Therefore the considered optimization problem has always a (global, possibly non-unique) solution which belongs to the closure of the feasible set. If this solution is such that $X > 0$ and $J(X, \rho_1^\star, \cdots, \rho_\nu^\star) = 0$ then the SOF stabilization problem is solvable. Otherwise, if all global minima are such that $X \not> 0$ the problem has no solution. Therefore, the difficulty in applying the result outlined in Theorem 2 lies in the fact that there is no algorithm which is guaranteed to converge to the global solution.

Example 2: We illustrate the above theory with a simple example. Consider a SISO system with

$$A = \begin{bmatrix} 0 & 0 & 300 \\ 1 & 0 & 60 \\ 0 & 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 100 \\ 20 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

As can be easily verified, the system is SOF stabilizable. The SOF stabilization problem can be posed as the concave-convex programming problem given in Section IV, with $\nu = 1$,

$$M_1 = \begin{bmatrix} 0.25 & 0 & 0 & -0.25 \\ 0 & 1.0303 & 0.4268 & 0 \\ 0 & 0.4268 & 0.1768 & 0 \\ -0.25 & 0 & 0 & 0.25 \end{bmatrix}$$

$$N_1 = \begin{bmatrix} 0.25 & 0 & 0 & 0.25 \\ 0 & 0.0303 & -0.0732 & 0 \\ 0 & -0.0732 & 0.1768 & 0 \\ 0.25 & 0 & 0 & 0.25 \end{bmatrix}.$$

Solving the minimization problem with standard Matlab tools, yields $J(X^\star, \rho_1^\star) = 0$, with

$$X^\star = \begin{bmatrix} 0.0991 & 0 & -0.0005 \\ 0 & 0.0994 & 0 \\ -0.0005 & 0 & 0.1008 \end{bmatrix},$$

and $\rho_1^\star = .0115$. Finally, one stabilizing gain is $F = -27.1$. ◁

V. CONCLUDING REMARKS

The static output feedback stabilization problem for linear, multi-input single-output, systems has been revisited. It has been shown that the necessary and sufficient conditions derived in [3], which allow to recast the problem as an LMI problem with quadratic equality constraints, naturally lead to formulate the problem as a concave-convex programming problem. This allows to propose a novel algorithm for the solution of the SOF stabilization problem, which is illustrated by means of a simple example.

Future work will be directed toward the study of multi-input multi-output linear systems.

REFERENCES

[1] A. Astolfi and P. Colaneri. Static output feedback stabilization: from linear to nonlinear and back. In *Nonlinear control in the year 2000, A. Isidori, F. Lamnabhi-Lagarrigue, W. Respondek Ed.*, pages 49–72. Springer Verlag, 2000.

[2] A. Astolfi and P. Colaneri. Static output feedback stabilization of linear and nonlinear systems. In *39th Conference on Decision and Control, Sydney, AU*, 2000.

[3] A. Astolfi and P. Colaneri. An algebraic characterization for the static output feedback stabilization problem. In *American Control Conference, Arlington, VA*, pages 1408–1413, 2001.

[4] A. Astolfi and P. Colaneri. A novel algorithm for the solution of the static output feedback stabilization problem. In *40th Conference on Decision and Control, Orlando, FL*, pages 4875–4876, 2001.

[5] L. El Ghaoui, F. Oustry and M. AitRami, "A cone complementarity linearization algorithm for static output-feedback and related problems", *IEEE Transactions on Automatic Control*, Vol. 42, pp. 1171-1176, 1997.

[6] J.C. Geromel, C.C. de Souza and R.E. Skelton, "Static output feedback controllers: stability and convexity", *IEEE Transactions on Automatic Control*, Vol. 43, pp. 120-125, 1998.

[7] V. Kucera and C.C. De Souza, "A necessary and sufficient condition for output feedback stabilizability", *Automatica* Vol. 31, pp. 1357-1359, 1995.

[8] V.L. Syrmos, C.T. Abdallah, P. Dorato and K. Grigoriadis, "Static output feedback - A survey", *Automatica*, Vol. 33, 2, pp. 125-137, 1997.

[9] Y.-Y. Cao, J. Lam and Y.-X. Sun, "Static output feedback stabilization: an ILMI approach", *Automatica*, Vol. 34, pp. 1641-1645, 1998.