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General interpolation for input-affine nonlinear systems

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***Abstract*— This paper considers interpolation between non-linear control laws and their pre-computed invariant sets. It is shown that the resulting control law ensures the invariance and feasibility of the convex hull of the individual invariant sets. The method presented here is the extension of [2] for the case of input-affine nonlinear systems.**

I. INTRODUCTION

It is now widely accepted in the research literature that dual mode predictions in model predictive control(MPC) provide a practicable alternative to the infinite horizon model predictive control [7]. The scheme consists of two modes: MODE I (near future) and MODE II (far future). Infinite horizon costs are split into finite horizon costs (MODE I costs) and a terminal costs(MODE II costs). The finite horizon cost is given in terms of free control moves in MODE I whereas the terminal cost is the true or upper bound on the infinite horizon cost for the unconstrained terminal control law (MODE II). Terminal control laws are associated with terminal invariant sets inside which a terminal control law is feasible, therefore guaranteeing feasibility of MODE II. Performance is optimized over the control moves in MODE I, subject to a stability constraint that requires the predicted state at the end of MODE I to lie inside the terminal set (where the existence of the feasible terminal law ensures feasibility of the complete trajectory). Implementations of the idea use linear terminal laws and ellipsoidal sets (e.g.[5]).

The overall performance of dual mode predicton MPC depends on the choice of the terminal control law and the associated terminal invariant set. Clearly the size of stabilizable set of the scheme depends on the size of the terminal invariant and feasible set and length of MODE I horizon. Increasing the size of stabilizable set through larger terminal invariant and feasible set usually requires the use of "detuned" control laws which would compromise the optimality of the scheme. Alternatively one can use longer MODE I horizon but at a heavy computational cost.

The problem of choosing a terminal control law that gives both good size terminal set whilst not significantly compromising on optimality has been addressed by Bacic *et al* [2] in the context of linear systems. There the idea was to construct several ellipsoidal invariant and feasible sets corresponding to different controllers. One of these controllers was a chosen to be optimal with respect to given cost whilst others were chosen sub-optimally. It was then shown that it is possible to stabilize any state inside the convex hull of the ellipsoids by means of interpolation between the control laws associated with the corresponding ellipsoids sets.

This paper extends the results of [2] to the case of input-affine *non-linear* systems. The paper is organized as follows. We first describe briefly the class of systems under consideration in this paper. We then review conditions for the ellipsoidal invariance and feasibility using affine difference inclusion. It is then shown how to extend the results in [2] to the non-linear case by making use of affine difference inclusion. Finally, a case study demonstrates the efficacy of the approach.

II. SYSTEM DESCRIPTION

Consider, the class of systems described by nonlinear models of the form

$$\begin{aligned} x(k+1) &= f(x(k)) + g(x(k))u(k) \\ y(k) &= Cx(k) \end{aligned} \tag{1}$$

with $x \in \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^n$, $g : \Omega \to \mathbb{R}^n$ are smooth functions on a subset Ω of $\mathbb{R}^n$. Here, for simplicity we will consider SISO systems but the results are trivially extendable to MIMO case. The system is subject to input constraints

$$u(k) \in \mathbb{U}, \quad U = \{u : |u| \leq \overline{u}\} \tag{2}$$

and infinite horizon LQ cost

$$J = \sum_{k=0}^{\infty} x^T(k)Qx(k) + u^T(k)Ru(k) \tag{3}$$

where, for simplicity, it will be assumed that the origin is one of the equilibrium states for the system in (1).

III. ELLIPSOIDAL INVARIANCE AND FEASIBILITY FOR NON-LINEAR SYSTEMS

Computing invariant and feasible ellipsoids was previously addressed in [4],[6]. Here the theory presented in [4] is summarized in the context of non-linear systems through the appropriate use of linear difference inclusion(LDI). Consider therefore an ellipsoidal set E

$$E = \{x : x^T Px \leq 1\}, \quad P \succeq 0 \tag{4}$$

and the associated general non-linear control law

$$u(k) = K(x(k)) \tag{5}$$

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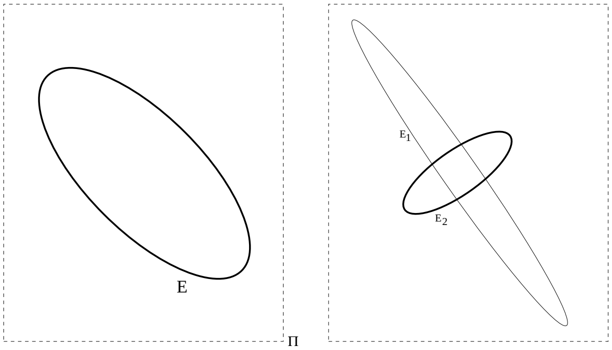


(a) Inclusion polytope Π for E (b) Two invariant/feasible ellipsoids

Fig. 1. Inclusion polytope Π and relevant ellipsoids

The closed loop dynamics of (1) under (5) are therefore governed by

$$x(k+1) = \Phi(x(k)), \quad \Phi(x(k)) = F(x(k), K(x(k))) \qquad (6)$$

It is well known[4], [3] that E will be positively invariant under (6) if and only if

$$(\forall x(k) \in E)\left(\Phi(x(k))^T P \Phi(x(k)) - x^T(k) P x(k) \leq 0\right) \qquad (7)$$

Computing P using directly definition above is not practicable and practicable alternatives all use linear difference inclusion. Consider therefore a low complexity polytope Π with vertices w_i

$$\Pi = \{x : \|Vx\|_\infty \leq 1\} \qquad (8)$$

which defines the region in which the linear difference inclusions(LDIs) for $\Phi(x), K(x)$ are valid and for which $\Pi \supset E$ (see Figure 1(a)). The inclusion polytope is required out of neccessity, as it is much easier to compute LDIs by using simple gridding technique over a polytope than over the ellipsoid. It will be assumed, for simplicity, that the system in (1) has zero equilibrium state $(x,u) = (0,0)$ which implies $F(0,0) = 0$ and $K(0) = 0$. The linear difference inclusions for $F(x,u)$ and $K(x)$, centred about the origin are given by

$$(\forall x \in \Pi)\left(\begin{array}{l} F(x,u) = \sum_{i=1}^{2^{n^2}} \zeta_i A_i^F x + \sum_{j=1}^{2^n} \xi_j B_j^F u, \\ \sum_{i=1}^{2^{n^2}} \zeta_i = 1, \sum_{j=1}^{2^n} \xi_j = 1 \quad \zeta_i, \xi_j \geq 0 \end{array}\right) \qquad (9a)$$

$$(\forall x \in \Pi)\left(K(x) = \sum_{i=1}^{2^n} \gamma_i A_i^K x, \quad \sum_{i=1}^{2^n} \gamma_i = 1, \quad \gamma_i \geq 0\right) \qquad (9b)$$

where A_i^F, B_j^F, A_i^K can be computed using a simple griding strategy (see [1] for example). These can be used together with the definition of ellipsoidal invariance to establish the following result.

Theorem III.1 (Invariance and feasibility of E). *The ellipsoid E is positively invariant and feasible under the closed loop dynamics of (6) if*

$$\begin{bmatrix} S & S(A_i^F + B_j^F A_l^K)^T \\ (A_i^F + B_j^F A_l^K)S & S \end{bmatrix} > 0, i = 1, \ldots, 2^{n^2} \qquad (10)$$

$$\begin{bmatrix} \bar{u}^2 & A_l^K S \\ S A_l^{K\,T} & S \end{bmatrix} \geq 0, j = 1, \ldots, 2^n, \quad l = 1, \ldots, 2^n \qquad (11)$$

where $S = P^{-1}$.

Proof: From (10) it follows that

$$\sum_{i=1}^{2^{n^2}} \zeta_i \begin{bmatrix} S & S(A_i^F + B_j^F A_l^K)^T \\ (A_i^F + B_j^F A_l^K)S & S \end{bmatrix} > 0$$

$$\Rightarrow \begin{bmatrix} S & S(\sum_{i=1}^{2^{n^2}} \zeta_i A_i^F + B_j^F A_l^K)^T \\ (\sum_{i=1}^{2^{n^2}} \zeta_i A_i^F + B_j^F A_l^K)S & S \end{bmatrix} > 0$$

Similarly summing over j and l independently leads to

$$\begin{bmatrix} S & S\Psi^T \\ \Psi S & S \end{bmatrix} > 0, \quad \Psi = \sum_{i=1}^{2^{n^2}} \zeta_i A_i^F + \sum_{j=1}^{j=2^n} \xi_j B_j^F \sum_{l=1}^{l=2^n} \gamma_l A_l^K$$

$$\Rightarrow \Psi^T P \Psi - P < 0$$

where

$$\sum_{i=1}^{2^{n^2}} \zeta_i = 1, \quad \sum_{j=1}^{2^n} \xi_j = 1, \quad \sum_{l=1}^{l=2^n} \gamma_l = 1, \quad \zeta_i, \xi_j, \gamma_l \geq 0 \qquad (12)$$

Pre- and post-multiplying the above inequality by x^T and x respectively, leads to

$$(\Psi x)^T P (\Psi x) - x^T P x < 0 \qquad (13)$$

The condition (13) implies that for any $x \in \Pi$ and **any** ζ_i, ξ_j, γ_l satisfying (12), the next state governed by the *LDI* dynamics will be inside E. Therefore out of all possible values of ζ_i, ξ_j, γ_l there exist a triple for any particular x for which it follows that

$$F(x, K(x)) = \sum_{i=1}^{2^{n^2}} \zeta_i A_i^F x + \sum_{j=1}^{j=2^n} \xi_j B_j^F \sum_{l=1}^{l=2^n} \gamma_l A_l^K x$$

and therefore

$$\begin{array}{l} (\forall x \in \Pi)\left(F(x,K(x))^T P F(x,K(x)) - x^T P x < 0\right) \\ \Rightarrow (\forall x \in E)\left(\Phi(x)^T P \Phi(x) - x^T P x < 0\right) \end{array} \qquad (14)$$

which proves that (10) is a sufficient condition for the positive invariance of E under closed loop dynamics of (6). Similar argument can be applied to the feasibility condition of (11).□

The conditions of the Theorem III.1 can be utilized to maximize the area of E using semi-definite programming. Arguably the size of E will be limited partly by the input constraints but predominantly by the conservativeness of the affine difference inclusion employed in the conditions of Theorem III.1. The larger the inclusion polytope the more conservative the LDI becomes which therefore leads to smaller E.

IV. ENLARGEMENT OF INVARIANT SETS THROUGH INTERPOLATION

Previous work [2] considered the use of interpolation to expand the size of stabilizable set for linear systems. Here the idea is extended to the case of non-linear systems. Consider therefore ellipsoidal invariant and feasible sets E_i which have been constructed using the common Π for the affine difference inclusion (see Figure 1(b)). Mathematical description of the membership of the convex hull was first derived in [2] and is given here for completeness.

Lemma IV.1. *A vector $x(k)$ lies in the convex hull of ellipsoids given by*

$$E_i = \{x \in \mathbb{R}^n : x^T P_i x \leq 1\}, \quad P_i \succeq 0, \quad i = 1, \ldots, \nu \quad (15)$$

if and only if there exist vectors $\hat{x}_i$ for $i = 1, \ldots, \nu$ such that

$$\begin{bmatrix} \lambda_i(k) & \hat{x}_i^T(k) \\ \hat{x}_i(k) & \lambda_i(k) S_i \end{bmatrix} \geq 0 \quad (16a)$$

$$x(k) = \sum_{i=1}^{\nu} \hat{x}_i(k) \quad (16b)$$

$$\sum_{i=1}^{\nu} \lambda_i(k) = 1 \quad (16c)$$

$$0 \leq \lambda_i(k) \leq 1, \quad i = 1, \ldots, \nu \quad (16d)$$

where $S_i = P_i^{-1}$.

Proof: See [2] for the proof.

It is clear that Lemma IV.1 applies in the non-linear case, as it only relies on the current state. Assuming E_i are invariant and feasible under $K_i(x)$ and satisfy conditions of Theorem III.1, interpolating control law in [2] can be extended for the non-linear case through the theorem below.

Theorem IV.2 (Interpolation law). *Let the invariant and feasible ellipsoidal sets $E_i \subset \Pi$, $K_i(x)$ satisfy conditions of Theorem III.1 using common polytope Π for the linear difference inclusion of the (1) and respective control laws $u(k) = K_i(x(k))$. Then for state vectors $x_i(k)$ satisfying (16), the interpolation law*

$$u(k) = \sum_{i=1}^{\nu} \lambda_i(k) K_i(x_i(k)) \quad (17)$$

renders the convex hull of the ellipsoids E_i invariant under dynamics of (1) and feasible under the constraints of (2).

Proof: Under the control law of (17) and the system dynamics of (1), the state $x(k)$ of the theorem is steered to

$$\begin{aligned} x(k+1) &= F\left(x(k), \sum_{i=1}^{\nu} \lambda_i(k) K_i(x_i(k))\right) \\ &= F\left(\sum_{i=1}^{\nu} \lambda_i(k) x_i(k), + \sum_{i=1}^{\nu} \lambda_i(k) K_i(x_i(k))\right) \\ &= \sum_{j=1}^{2^{n^2}} \zeta_j A_j^F \sum_{i=1}^{\nu} \lambda_i(k) x_i(k) + \sum_{l=1}^{2^n} \xi_l B_l^F \sum_{i=1}^{\nu} \lambda_i(k) K_i(x_i(k)) \\ x(k+1) &= \sum_{i=1}^{\nu} \lambda_i(k) \left(\sum_{j=1}^{2^{n^2}} \zeta_j A_j^F x_i(k) + \sum_{l=1}^{2^n} \xi_l B_l^F K_i(x_i(k)) \right) \end{aligned} \quad (18)$$

Expanding above using linear difference inclusion for $K_i(x)$ leads

$$\begin{aligned} x(k+1) &= \sum_{i=1}^{\nu} \lambda_i(k) \left(\sum_{j=1}^{2^{n^2}} \zeta_j A_j^F x_i(k) + \sum_{j=1}^{2^n} \xi_j B_j^F \sum_{m=1}^{2^n} \gamma_m A_m^{K_i} x_i(k) \right) \\ &= \sum_{i=1}^{\nu} \lambda_i(k) x_i'(k) \end{aligned} \quad (19)$$

where $x_i'(k) \in E_i$ due to (13) of the proof of Theorem III.1 and the fact that the affine difference inclusion $\{A_j, B_j, A_l^{K_i}\}$ is derived using the common inclusion polytope Π. Consequently it follows that $x(k+1) \in Co\{E_1, E_2, \ldots, E_\nu\}$ which completes the invariance proof of the theorem. Feasibility can be proved using triangle inequality in a similar fashion to the proof of Theorem IV.2.

Remark IV.3. *It is important to note the requirement of Theorem IV.2 that all ellipsoids E_i have been calculated using the common linear difference inclusion. This is essential in establishing the invariance of the convex hull. Constructing E_i in any other way would not guarantee stability.*

Although results of Theorem IV.2 imply stability of the interpolating control law, this was proved through the explicit use of the linear difference inclusion arguments. It is not possible to separate control trajectory in terms of the original system model (1), like it was done for the linear case (see [2] for details). However the state-trajectory can be expressed in terms of parameters of the linear difference inclusion. It is clear from equation (19) of the proof of Theorem IV.2 that the set of predicted states $x_i(k+1)$ is given by

$$\begin{aligned} x_i(k+1) &= \sum_{j=1}^{2^{n^2}} \zeta_j A_j^F x_i(k) + \sum_{l=1}^{2^n} \xi_l B_l^F \sum_{m=1}^{2^n} \gamma_m A_m^{K_i} x_i(k) \\ &= \Psi_i^k(\zeta_j, \xi_l, \gamma_m) x_i(k) \end{aligned} \quad (20)$$

where $\Psi_i^k(\zeta_j, \xi_l, \gamma_m) = \sum_{j=1}^{2^{n^2}} \zeta_j A_j^F + \sum_{l=1}^{2^n} \xi_l B_l^F \sum_{m=1}^{2^n} \gamma_m A_m^{K_i}$ and superscript k indicates that triple ζ_j, ξ_l, γ_m corresponds to parameters of the LDI representation of F at $x(k)$. Therefore the state and input trajectories are simply given by

$$\mathbb{X}_{LDI} = \left\{ \begin{array}{l} \sum_{i=1}^{\nu} \hat{x}_i(k), \sum_{i=1}^{\nu} \Psi_i^k(\zeta_j,\xi_l,\gamma_m)\hat{x}_i(k), \\ \sum_{i=1}^{\nu} \Psi_i^{k+1}(\zeta_j,\xi_l,\gamma_m)\Psi_i^k(\zeta_j,\xi_l,\gamma_m)\hat{x}_i(k),\dots \end{array} \right\} \quad (21a)$$

$$\mathbb{U}_{LDI} = \left\{ \begin{array}{l} \sum_{i=1}^{\nu} \lambda_i(k)K_i(x_i(k)), \\ \sum_{i=1}^{\nu} \lambda_i(k)K_i(\Psi_i^k(\zeta_j,\xi_l,\gamma_m)x_i(k)),\dots \end{array} \right\} \quad (21b)$$

Since $\Psi_i^k(\zeta_j,\xi_l,\gamma_m)$ is not known a priori, computation of the exact predicted cost in a manner similar to the linear case is not practicable. This therefore precludes specifying closed loop optimization in terms of minimization of the predicted cost. It was argued in [2] that one of the controllers $K_i(x)$ should be chosen to be optimal with respect to the cost in (3) in order to ensure a degree of optimality for the interpolating control law. Therefore this suggests that maximizing the influence of the optimal control law, say $K_{opt}(x)$ will maximize the optimality of the over-all scheme. Based on this premise the algorithm below maximizes λ_{opt} which corresponds to the optimal controller $K_{opt}(x)$.

Algorithm IV.4 (Quasi-optimal non-linear interpolation).

1) *Given $x(k)$ perform the following optimization*

$$\max_{\lambda_1,\dots,\lambda_\nu,\hat{x}_1,\dots,\hat{x}_\nu} \lambda_{opt} \quad \textit{subject to (16)} \qquad (22)$$

2) *Execute control action using (17).*
3) *Go to step (i).*

Theorem IV.5 (Stability and convergence of Algorithm IV.4). *Provided that optimization (22) is feasible at initial time, the Algorithm IV.4 is asymptotically stable.*

Proof: By assumption optimization in (22) is feasible at initial time. This therefore implies that there exist at least one control input trajectory, in the class $\mathbb{U}_{ADI}$ of (21) that will be feasible at the next time instant. Therefore by induction, this guarantees the feasibility of (22) at all times. To prove stability consider the following cost function:

$$\Theta(\lambda_{opt}(k)) = 1 - \lambda_{opt}(k) \qquad (23)$$

where clearly $\Theta(\lambda_{opt}(k)) \geq 0$. To prove stability of Algorithm IV.4 it is sufficient to prove that $\Theta(\lambda_{opt}(k))$ is *Lyapunov* function. Hence,

$$\Theta(\lambda_{opt}(k+1)) - \Theta(\lambda_{opt}(k)) = \lambda_{opt}(k) - \lambda_{opt}(k+1)$$

Noting that the feasible trajectories $\mathbb{X}_{LDI}, \mathbb{U}_{LDI}$ of (21) maintain constant $\lambda_{opt}(k)$ at all times, it is clear that optimization in (22) can do no worse than that (as $\lambda_{opt}(k+1) = \lambda_{opt}(k)$ is feasible). Consequently it is clear that $\lambda_{opt}(k) - \lambda_{opt}(k+1) \leq 0$ and therefore

$$\Theta(\lambda_{opt}(k+1)) - \Theta(\lambda_{opt}(k)) \leq 0$$

To prove asymptotic stability assume that $\lambda_{opt}(k+1) = \lambda_{opt}(k)$ for all k. In that case the assumed trajectory is given by $\mathbb{X}_{ADI}$ which is asymptotically stable due to positive invariance of the convex hull of the ellipsoids E_i. Consequently Algorithm IV.4 is asymptotically stable.

V. Case study - Continous stirred tank reactor

We consider here a two-state standard continuous stirred tank reactor which forms an essential part of any petro-chemical refinery[8]. The first order irreversible chemical reaction $A \rightarrow B$ can be described by the following state space model:

$$\begin{aligned} \frac{dC_A}{dt} &= \frac{Q_f}{V}(C_{Af} - C_A) - k_0 C_A \exp\left(-\frac{E_a}{R_1 T}\right) \\ \frac{dT}{dt} &= \frac{Q_f}{V}(T_f - T) - \frac{k_0 C_A}{C_p}(-\Delta H)\exp\left(-\frac{E_a}{R_1 T}\right) - \frac{UA_h}{VC_p}(T - T_c) \end{aligned} \qquad (24)$$

where C_A is the concentration of material A and T is the reactor temperature. Following the normalization[9] and discretization of the above model using Euler's forward difference approximation with sampling period T_s yields:

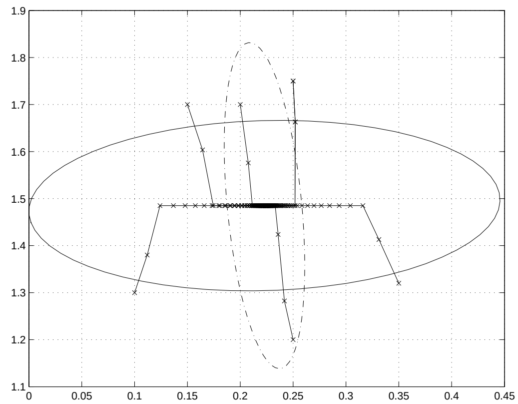
$$\begin{aligned} f(x_k) &= \left[\begin{array}{l} x_k(1) + T_s(-\alpha x_k(1) + D_a\phi(x_k)) \\ x_k(2) + T_s(-\alpha x_k(2) + BD_a\phi(x_k) - \beta x_k(2)) \end{array} \right] \\ g(x_k) &= \left[\begin{array}{c} 0 \\ \beta \end{array} \right] \end{aligned} \qquad (25)$$

where $\phi(x_k) = (1 - x_k(1))\exp\left(\frac{x_k(2)}{1 + x_k(2)/\gamma}\right)$. The process parameters have been taken from [9] and are given in the table below together with the system constraints:

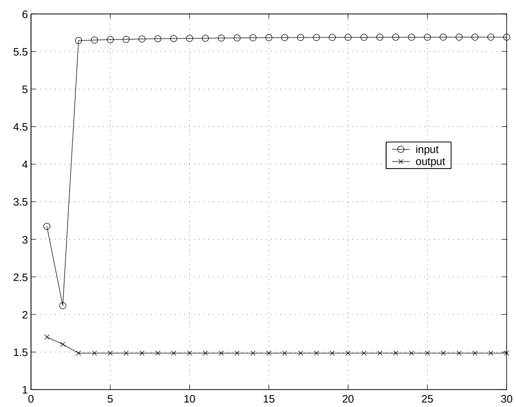
α	1.0	β	0.03	γ	20.0
B	1.0	D_a	0.072	$u \in [0,10]$	

TABLE I
System parameters

Choosing the output variable to be the second state(temperature) leads to $C = \left[\begin{array}{cc} 0 & 1 \end{array}\right]$. The equilibrium state was chosen at maximum yield(x_2/x_1), is given by $x_e = \left[\begin{array}{cc} 0.2229 & 1.4851 \end{array}\right]$ and is minimum-phase. In this example, first controller $K_1(x) = K_{opt}(x)$ is feedback linearizing controller which is optimal with respect to the output cost($Q = CC^T$,$R = 0$) and is stable due to the chosen minimum-phase equilibrium state. The second controller $K_2(x)$, is simple linear state feedback controller computed using the linearized model of (25) about requilibrium state with $Q_1 = kerC^T kerC$. Clearly the choice of $K_2(x)$ is sub-optimal with respect to the output cost. Figure 2(a) depicts the two maximum volume ellipsoidal invariant and feasible sets together with few trajectories generated using Algorithm IV.4. The dashed ellipsoid corresponds to the sub-optimal controller, $K_2(x)$. Figure 2(b) gives an example of an input/output trajectory typical for the scheme.



(a) Random trajectories



(b) Typical input/output trajectory

Fig. 2. CSTR example

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VI. CONCLUSIONS

This paper extends the results of [2] for the case of input affine non-linear systems. This is made possible through the judicious use of LDIs. The effectiveness of the approach is demonstrated on the example of continous stirred tank reactor.