Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

LMI based Output Feedback Control of Discrete Linear Repetitive Processes

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***Abstract*— Repetitive processes are a distinct class of 2D systems (i.e. information propagation in two independent directions) of both systems theoretic and applications interest. They cannot be controlled by direct extension of existing techniques from either standard (termed 1D here) or 2D systems theory. Here we give new results on the relatively open problem of the design of physically based control laws. These results are for the sub-class of so-called discrete linear repetitive processes which arise in applications areas such as iterative learning control.**

I. INTRODUCTION

Repetitive processes are a distinct class of 2D systems of both system theoretic and applications interest. The essential unique characteristic of such a process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics next pass profile. This, in turn, leads to the unique control problem for these processes in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction.

To introduce a formal definition, let $\alpha < +\infty$ denote the pass length (assumed constant). Then in a repetitive process the pass profile $y_k(p),\ 0 \leq p \leq \alpha$, generated on pass k acts as a forcing function on, and hence contributes to, the dynamics of next pass profile $y_{k+1}(p),\ 0 \leq p \leq \alpha,\ k \geq 0$.

Physical examples of repetitive processes include longwall coal cutting and metal rolling operations (see, for example, [1]). Also in recent years applications have arisen where adopting a repetitive process setting for analysis has distinct advantages over alternatives. Examples of these so-called algorithmic applications include classes of iterative learning control (ILC) schemes [2] and iterative algorithms for solving nonlinear dynamic optimal control problems based on the maximum principle [3]. In the case of ILC for the linear dynamics case, the stability theory for so-called differential and discrete linear repetitive processes is the essential basis for a rigorous stability/convergence theory for a powerful class of such algorithms.

One unique feature of repetitive processes is that it is possible to define physically meaningful control laws for their dynamics. For example, in the ILC application, one such family of control laws is composed of state feedback control action on the current pass combined with information 'feedforward' from the previous pass (or trial in the ILC context) which, of course, has already been generated and is therefore available for use. In the general case of repetitive processes it is clearly highly desirable to have an analysis setting where such control laws can be designed for stability and/or performance. Also previous work has shown that an LMI re-formulation of the stability conditions for discrete linear repetitive processes leads naturally to design algorithms to ensure closed loop stability along the pass under a control law which explicitly makes use of the current pass state vector — see, for example, [4], [5].

To implement such a control law will, in general, require an observer to provide the current pass state vector component. As an alternative, this paper shows how to use the LMI setting to design control laws which only require pass profile information (which has already been generated and hence is available control action) for implementation. Note here that LMI based methods have also been investigated as a means of stability analysis and controller design for 2D discrete linear systems described by the well known Roesser [6] and Fornasini Marchesini [7] state space models, see, for example, [8]. Discrete linear repetitive processes have strong structural links with such systems class of systems and some results can be exchanged between these classes of linear systems. The key novelty in this paper is the use of physically motivated output feedback control laws.

Throughout this paper, the null matrix and the identity matrix with the required dimensions are denoted by 0 and I, respectively. Moreover, $M > 0$ (< 0) denotes a real symmetric positive (negative) definite matrix.

II. BACKGROUND

Following [9], the state-space model of a discrete linear repetitive process has the following form over $0 \leq p \leq \alpha,\ k \geq 0$

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p) + B_0 y_k(p) \\ y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) + D_0 y_k(p) \end{aligned} \qquad (1)$$

Here on pass k, $x_k(p)$ is the $n \times 1$ state vector, $y_k(p)$ is the $m \times 1$ pass profile vector and $u_k(p)$ is the $l \times 1$ vector of control inputs.

To complete the process description, it is necessary to specify the boundary conditions, i.e. the state initial vector on each pass and the initial pass profile. Here no loss of generality arises from assuming $x_{k+1}(0) = d_{k+1},\ k \geq 0,$

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and $y_0(p) = f(p)$, where the $n \times 1$ vector d_{k+1} has known constant entries and $f(p)$ is an $m \times 1$ vector whose entries are known functions of p over $[0, \alpha]$. (For ease of presentation, we will make no further explicit reference to the boundary conditions in this paper.)

The stability theory [9] for linear repetitive processes consists of two distinct concepts but here it is the stronger of these which is required. This is termed stability along the pass and several equivalent sets of necessary and sufficient conditions for processes described by (1) to have this property are known, but here it is the sufficient condition of Theorem 1 below which will be used. A central feature of the results in this paper is that they will show that this sufficient condition allows us to design control laws in a straightforward manner whereas the currently available necessary and sufficient conditions only really allow us to obtain conditions for stability under control action.

Define the delay operators $z_1,\ z_2$ in the along the pass (p) and pass-to-pass (k) directions respectively as

$$x_k(p) := z_1 x_k(p+1), \ \ x_k(p) := z_2 x_{k+1}(p) \qquad (2)$$

and define the following matrices from (1)

$$\widehat{A}_1 = \begin{bmatrix} A & B_0 \\ 0 & 0 \end{bmatrix}, \ \widehat{B}_1 = \begin{bmatrix} B \\ 0 \end{bmatrix}, \qquad (3)$$
$$\widehat{A}_2 = \begin{bmatrix} 0 & 0 \\ C & D_0 \end{bmatrix}, \ \widehat{B}_2 = \begin{bmatrix} 0 \\ D \end{bmatrix}.$$

Then stability along the pass (see [9] for the details) holds if, and only if, the so-called 2D characteristic polynomial

$$\rho(z_1, z_2) := \det(I - z_1\widehat{A}_1 - z_2\widehat{A}_2) \qquad (4)$$

satisfies

$$\rho(z_1, z_2) \neq 0, \ \forall\, (z_1, z_2) : |z_1| \leq 1, \ |z_2| \leq 1 \qquad (5)$$

Theorem 1: [4] A discrete linear repetitive processes described by (1) is stable along the pass if there exists matrices $P > 0$ and $Q > 0$ satisfying the following LMI

$$\begin{bmatrix} \widehat{A}_1^T P \widehat{A}_1 + Q - P & \widehat{A}_1^T P \widehat{A}_2 \\ \widehat{A}_2^T P \widehat{A}_1 & \widehat{A}_2^T P \widehat{A}_2 - Q \end{bmatrix} < 0 \qquad (6)$$

III. LMI based Controller Design — Previous Work

As noted in the introduction of this paper, the design of control laws for 2D discrete linear systems described by the Roesser [6] and Fornasini Marchesini [7] state space models has received attention in the literature over the years. A valid criticism of such work, however, is that the structure of the control algorithms are not well founded physically due to the fact that, for example, the concept of a state of these systems is not uniquely defined. For example, it is possible to define a state feedback law based on the local or global state vectors. Also in the absence of generalizations of well defined and understood 1D concepts, e. g. the pole assignment problem and error actuated output feedback control action, it has not been really possible to formulate a control design problem beyond that of obtaining conditions for stabilization under the control action.

The first difficulty above does not arise with discrete linear repetitive processes. For example, it is physically meaningful to define the current pass error as the difference, at each point along the pass, between a specified reference trajectory for that pass, which in most cases will be the same on each pass, and the actual pass profile produced. Then we can define a so-called current pass error actuated controller which uses the generated error vector to construct the current pass control input vector. In which context, preliminary work, see, for example, [1], has shown that, except in a few very restrictive special cases, the controller used must be actuated by a combination of current trial information and ‘feedforward’ information from the previous trial to guarantee even stability along the pass closed-loop. Note also here that in the ILC application area the previous trial output vector is an obvious signal to use as feedforward action.

In the case of the second difficulty above, previous work [4] considered a control law of the following form over $0 \leq p \leq \alpha,\ k \geq 0$

$$u_{k+1}(p) = K_1 x_{k+1}(p) + K_2 y_k(p) := K \begin{bmatrix} x_{k+1}(p) \\ y_k(p) \end{bmatrix} \qquad (7)$$

where K_1 and K_2 are appropriately dimensioned matrices to be designed. In effect, this control law uses feedback of the current trial state vector, which is assumed to be available for use. The following is one of the main results of this previous work and establishes that (at the very least) the LMI approach also allows controller design as opposed to other approaches where it is only possible to get existence type results for stability along the pass under control action.

Theorem 2: Suppose that a discrete linear repetitive process of the form described by (1) is subject to a control law of the form (7). Then the resulting closed loop process is stable along the pass if there exists matrices $Y > 0,\ Z > 0$, and N such that the following LMI holds

$$\begin{bmatrix} Z - Y & 0 & Y\widehat{A}_1^T + N^T\widehat{B}_1^T \\ 0 & -Z & Y\widehat{A}_2^T + N^T\widehat{B}_2^T \\ \widehat{A}_1 Y + \widehat{B}_1 N & \widehat{A}_2 Y + \widehat{B}_2 N & -Y \end{bmatrix} < 0 \qquad (8)$$

If (8) holds, then a stabilizing K in the control law (7) is given by

$$K = NY^{-1} \qquad (9)$$

IV. Pass Profile Only Based Feedback Control

In many cases the state vector $x_{k+1}(p)$ may not be available (or at best only some of its entries are directly measurable). As in 1D systems theory, two options immediately arise. These are the development of an observer theory to estimate $x_{k+1}(p)$ or to consider the use of output information only, i.e. only activate the control law with the current and previous pass profile vectors. Of these options, we focus on the second here as it is the one which has more physical relevance in terms of applications.

The control law considered in this section has the following form over $0 \le p \le \alpha,\ k \ge 0$

$$u_{k+1}(p) = \widetilde{K}_1 y_{k+1}(p) + \widetilde{K}_2 y_k(p) \tag{10}$$

This law is, in general, weaker than that of (7) which uses the current pass state vector and examples are easily given where stability along the pass can be achieved using (7) but not (10). For example, suppose that the matrix A has at least one eigenvalue with modulus greater than or equal to unity. Then such an example of (1) is unstable along the pass under a control law of the form (10) since a necessary condition for this property (all eigenvalues of the matrix A must have modulus strictly less than unity) cannot hold. To show this, first note that under the application of (10) the matrix A maps to $A + B\widetilde{K}_1 C$ and then make use of a simple property of the trace of a square matrix.

Despite this drawback, it is clear that cases will arise where (10) (or, more generally, only pass profile data) is all that is available and in the remainder of this section we develop an LMI based approach to the design of control laws of the form of (10) for stability along the pass closed loop.

Substituting the pass profile updating equation in (1) into (10) yields (assuming the required matrix inverses exist)

$$\begin{aligned} u_{k+1}(p) = &(I - \widetilde{K}_1 D)^{-1} C x_{k+1}(p) + \\ &+(I - \widetilde{K}_1 D)^{-1}[\widetilde{K}_2 + \widetilde{K}_1 D_0] y_k(p) \end{aligned} \tag{11}$$

and hence (11) can be treated as a particular case of (7) with

$$\begin{aligned} K_1 &= (I - \widetilde{K}_1 D)^{-1} \widetilde{K}_1 C \\ K_2 &= (I - \widetilde{K}_1 D)^{-1} (\widetilde{K}_2 + \widetilde{K}_1 D_0) \end{aligned} \tag{12}$$

This route may, however, encounter serious numerical difficulties (arising from the fact that (12) defines two matrix nonlinear algebraic equations) and hence we proceed by first rewriting these last equations to obtain

$$\begin{aligned} (I - \widetilde{K}_1 D) K_1 &= \widetilde{K}_1 C \\ (I - \widetilde{K}_1 D) K_2 &= \widetilde{K}_2 + \widetilde{K}_1 D_0 \end{aligned} \tag{13}$$

and assume that

$$K_1 = L_1 C \tag{14}$$

Then

$$\begin{aligned} \widetilde{K}_1 &= L_1 (I + D L_1)^{-1} \\ \widetilde{K}_2 &= [I - L_1 (I + D L_1)^{-1} D] K_2 - L_1 (I + D L_1)^{-1} D_0 \end{aligned} \tag{15}$$

for any L_1 such that $I + DL_1$ is nonsingular, and we have the following result.

Theorem 3: Suppose that a discrete linear repetitive process of the form described by (1) is subject to a control law of the form (10) and that (14) holds. Then the resulting closed loop process is stable along the pass if there exists matrices $Y > 0,\ Z > 0, X > 0$ and N such that the following LMI holds

$$\begin{bmatrix} Z - Y & 0 & Y\widehat{A}_1^T + \widehat{C}^T N^T \widehat{B}_1^T \\ 0 & -Z & Y\widehat{A}_2^T + \widehat{C}^T N^T \widehat{B}_2^T \\ \widehat{A}_1 Y + \widehat{B}_1 N \widehat{C} & \widehat{A}_2 Y + \widehat{B}_2 N \widehat{C} & -Y \end{bmatrix} < 0$$
$$X\widehat{C} = \widehat{C} Y \tag{16}$$

where $\widehat{A}_1,\ \widehat{A}_2,\ \widehat{B}_1,\ \widehat{B}_2$ are defined as in Theorem 2, and

$$\widehat{C} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix},\quad N = [N_1\ \ N_2] \tag{17}$$

Also if this condition holds, the controller matrices $\widetilde{K}_1$ and $\widetilde{K}_2$ are obtained from (15), where

$$[L_1\ \ K_2] = N X^{-1} \tag{18}$$

and $I + DL_1$ is nonsingular.

Proof: From (18) we have that $N = LX,\ L := \begin{bmatrix} L_1 & K_2 \end{bmatrix}$ and substitution into the LMI of (16) together with $X\widehat{C} = \widehat{C}Y$ yields

$$\begin{bmatrix} Z - Y & 0 & Y\widehat{A}_1^T + Y\widehat{C}^T L^T \widehat{B}_1^T \\ 0 & -Z & Y\widehat{A}_2^T + Y\widehat{C}^T L^T \widehat{B}_2^T \\ \widehat{A}_1 Y + \widehat{B}_1 L \widehat{C} Y & \widehat{A}_2 Y + \widehat{B}_2 L \widehat{C} Y & -Y \end{bmatrix} < 0$$

Finally, set $L\widehat{C} = K$ to obtain the following LMI stabilization condition (i.e the result of Theorem 1 applied to the closed loop process)

$$\begin{bmatrix} Z - Y & 0 & Y(\widehat{A}_1^T + K^T \widehat{B}_1^T) \\ 0 & -Z & Y(\widehat{A}_2^T + K^T \widehat{B}_2^T) \\ (\widehat{A}_1 + \widehat{B}_1 K) Y & (\widehat{A}_2 + \widehat{B}_2 K) Y & -Y \end{bmatrix} < 0$$

and the proof is complete.

V. Extended Pass Profile Based Control

The control law design algorithm given in the previous section can be applied in the case when $m \le n$, i.e. only when the information content in the $m \times 1$ current pass profile vector is sufficient to stabilize the $n \times 1$ current pass state vector. This is a somewhat restrictive limitation and hence the question; what can be done using additional (e.g. from more than just the previous pass) pass profile vector information?

In the case of repetitive processes, the pass profiles generated on all passes before the current one constitute causal information and hence are available for control purposes. Clearly, however, if information from more than the previous pass profile is to be used then an obvious objective is to use only the minimum required since this will limit the storage required for implementation.

As a first stage of analysis in this area, we consider the following control law which at any point p along a given pass is (10) augmented by an additive contribution from the same point on the last but one pass

$$u_{k+1}(p) = \widetilde{K}_1 y_{k+1}(p) + \widetilde{K}_2 y_k(p) + \widetilde{K}_3 y_{k-1}(p) \tag{19}$$

which can be rewritten as

$$\begin{aligned} u_{k+1}(p) =& (I_l - \widetilde{K}_1 D)^{-1}\Big(\widetilde{K}_1 C x_{k+1}(p) \\ &+ [\widetilde{K}_2 + \widetilde{K}_1 D_0] y_k(p) + \widetilde{K}_3 y_{k-1}(p)\Big) \end{aligned} \tag{20}$$

or

$$u_{k+1}(p) = K_1 x_{k+1}(p) + K_2 y_k(p) + K_3 y_{k-1}(p) \tag{21}$$

It is also possible to 'swap' between these two forms of the control law considered here. In particular, if $\widetilde{K}_1,\ \widetilde{K}_2$ and $\widetilde{K}_3$ are known then K_1 and K_2 can be obtained from (12) and K_3 from

$$K_3 = (I - \widetilde{K}_1 D)^{-1} \widetilde{K}_3 \tag{22}$$

Conversely, if $K_1,\ K_2$ and K_3 are given then, assuming (14) holds, $\widetilde{K}_1$ and $\widetilde{K}_2$ can again be computed using (15) and $\widetilde{K}_3$ from

$$\widetilde{K}_3 = [I - L_1(I + DL_1)^{-1} D] K_3 \tag{23}$$

Using controller matrices $K_1,\ K_2$ and K_3, the closed loop process state space model under this form of control law is described by

$$\begin{aligned} x_{k+1}(p+1) &= (A + BK_1) x_{k+1}(p) \\ &+ (B_0 + BK_2) y_k(p) + BK_3 y_{k-1}(p) \\ y_{k+1}(p) &= (C + DK_1) x_{k+1}(p) \\ &+ (D_0 + DK_2) y_k(p) + DK_3 y_{k-1}(p) \end{aligned} \tag{24}$$

This last description is not in the form to which Theorem 1 can be applied but it is possible to obtain an equivalent state space model for which this is the case. Here the route is by using the delay operators of (2) and the 2D characteristic polynomial. In particular, apply (2) to (24) and introduce

$$\rho_c(z_1, z_2) := det \begin{bmatrix} I - z_1 \widetilde{A} & -z_1 \widetilde{B}_0 - z_1 z_2 F_1 \\ -z_2 \widetilde{C} & I - z_2 \widetilde{D}_0 - z_2^2 F_2 \end{bmatrix} \tag{25}$$

Application of appropriate elementary operations (which leave the determinant invariant) to the right-hand side of this last expression now yields that it can be replaced by

$$det \begin{bmatrix} I & 0 & z_2 I \\ z_1 F_1 & I - z_1 \widetilde{A} & -z_1 \widetilde{B}_0 \\ z_2 F_2 & -z_2 \widetilde{C} & I - z_2 \widetilde{D}_0 \end{bmatrix} \tag{26}$$

where

$$\begin{aligned} \widetilde{A} = A + BK_1,&\ \widetilde{B}_0 = B_0 + BK_2 \\ \widetilde{C} = C + DK_1,&\ \widetilde{D}_0 = D_0 + DK_2 \\ F_1 = BK_3,&\ F_2 = DK_3 \end{aligned}$$

At this stage, the closed loop state space model has a 2D characteristic polynomial which is of the form (4) (and therefore the result of Theorem 1 can be applied) with

$$\widetilde{A}_1 = \begin{bmatrix} 0 & 0 & 0 \\ -F_1 & \widetilde{A} & \widetilde{B}_0 \\ 0 & 0 & 0 \end{bmatrix},\ \widetilde{A}_2 = \begin{bmatrix} 0 & 0 & -I \\ 0 & 0 & 0 \\ -F_2 & \widetilde{C} & \widetilde{D}_0 \end{bmatrix}$$

Now we have the following result.

Theorem 4: Suppose that a discrete linear repetitive process of the form described by (1) is subject to a control law defined by (21) with K_1 satisfying (14). Then the resulting closed loop process is stable along the pass if there exist matrices $Y > 0$, $X > 0$ and $Z > 0$ such that

$$\begin{aligned} &\begin{bmatrix} Z - Y & 0 & Y\widehat{A}_1^T + \widehat{C}^T N^T \widehat{B}_1^T \\ 0 & -Z & Y\widehat{A}_2^T + \widehat{C}^T N^T \widehat{B}_2^T \\ \widehat{A}_1 Y + \widehat{B}_1 N \widehat{C} & \widehat{A}_2 Y + \widehat{B}_2 N \widehat{C} & -Y \end{bmatrix} < 0 \\ &X\widehat{C} = \widehat{C} Y \end{aligned} \tag{27}$$

where

$$\widehat{A}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A & B_0 \\ 0 & 0 & 0 \end{bmatrix},\ \widehat{A}_2 = \begin{bmatrix} 0 & 0 & -I \\ 0 & 0 & 0 \\ 0 & C & D_0 \end{bmatrix},$$

$$\widehat{B}_1 = \begin{bmatrix} 0 \\ B \\ 0 \end{bmatrix},\ \widehat{B}_2 = \begin{bmatrix} 0 \\ 0 \\ D \end{bmatrix},\ \widehat{C} = \begin{bmatrix} I & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & I \end{bmatrix},$$

$$N = \begin{bmatrix} -N_3 & N_1 & N_2 \end{bmatrix}$$

and it is assumed that $I + DL_1$ is nonsingular. Also if (27) holds then

$$[-K_3\ \ L_1\ \ K_2] = NX^{-1} \tag{28}$$

Proof. Substituting $N = LX$, $L := [-K_3\ \ L_1\ \ K_2]$ and $X\widehat{C} = \widehat{C}Y$ into the LMI of (27) yields

$$\begin{bmatrix} Z - Y & 0 & Y\widehat{A}_1^T + Y\widehat{C}^T L^T \widehat{B}_1^T \\ 0 & -Z & Y\widehat{A}_2^T + Y\widehat{C}^T L^T \widehat{B}_2^T \\ \widehat{A}_1 Y + \widehat{B}_1 L \widehat{C} Y & \widehat{A}_2 Y + \widehat{B}_2 L \widehat{C} Y & -Y \end{bmatrix} < 0$$

Now set $L\widehat{C} = K$ to obtain the following LMI stabilization condition (i.e. the result of Theorem 1 applied to the closed loop process)

$$\begin{bmatrix} Z - Y & 0 & Y(\widehat{A}_1^T + K^T \widehat{B}_1^T) \\ 0 & -Z & Y(\widehat{A}_2^T + K^T \widehat{B}_2^T) \\ (\widehat{A}_1 + \widehat{B}_1 K) Y & (\widehat{A}_2 + \widehat{B}_2 K) Y & -Y \end{bmatrix} < 0$$

and the proof is complete.

Given the controller matrices K_1, K_2 and K_3 from this last result, the corresponding $\widetilde{K}_1$ and $\widetilde{K}_2$ can be computed using (15) and $\widetilde{K}_3$ using (23).

VI. Alternative Pass Profile based Control

In the previous section we used a control law which included a contribution from the last but one pass profile. In this section, we investigate the use of delayed current pass profile information in the control law. The particular control law investigated is given by

$$u_{k+1}(p) = \widetilde{K}_1 y_{k+1}(p) + \widetilde{K}_2 y_k(p) + \widetilde{K}_3 y_{k+1}(p-1) \quad (29)$$

or

$$u_{k+1}(p) = K_1 x_{k+1}(p) + K_2 y_k(p) + K_3 y_{k+1}(p-1) \quad (30)$$

where

$$\begin{aligned} K_1 &= (I - \widetilde{K}_1 D)^{-1} \widetilde{K}_1 C \\ K_2 &= (I - \widetilde{K}_1 D)^{-1} (\widetilde{K}_2 + \widetilde{K}_1 D_0) \\ K_3 &= (I - \widetilde{K}_1 D)^{-1} \widetilde{K}_3 \end{aligned} \quad (31)$$

By again making the assumption that $K_1 = L_1 C$ we have that

$$\begin{aligned} &\widetilde{K}_1 = L_1 (I + D L_1)^{-1} \quad (32) \\ &\widetilde{K}_2 = [I - L_1 (I + D L_1)^{-1} D] K_2 - L_1 (I + D L_1)^{-1} D_0 \\ &\widetilde{K}_3 = [I - L_1 (I + D L_1)^{-1} D] K_3 \end{aligned}$$

The closed loop process now is given by

$$\begin{aligned} x_{k+1}(p+1) &= (A + B L_1 C) x_{k+1}(p) \\ &\quad + (B_0 + B K_2) y_k(p) + B K_3 y_{k+1}(p-1) \\ y_{k+1}(p) &= (C + D L_1 C) x_{k+1}(p) \\ &\quad + (D_0 + D K_2) y_k(p) + D K_3 y_{k+1}(p-1) \end{aligned} \quad (33)$$

This last description is again not in the form to which Theorem 1 can be applied but, by following the method of the previous section, it is possible to obtain an equivalent state space model for which this is the case. In particular, apply (2) to (24) to obtain

$$\begin{aligned} x_k(p) &= z_1 (A + B L_1 C) x_k(p) \\ &\quad + z_1 z_2 (B_0 + B K_2) y_k(p) + z_1^2 B K_3 y_k(p) \\ y_k(p) &= (C + D L_1 C) x_k(p) \\ &\quad + z_2 (D_0 + D K_2) y_k(p) + z_1^2 D K_3 y_k(p) \end{aligned} \quad (34)$$

and introduce

$$\rho_c(z_1, z_2) := det \begin{bmatrix} I - z_1 \widetilde{A} & -z_1 z_2 \widetilde{B}_0 - z_1^2 B K_3 \\ -\widetilde{C} & I - z_2 \widetilde{D}_0 - z_1^2 D K_3 \end{bmatrix} \quad (35)$$

which is obviously equivalent to replacing the right-hand side by

$$det \begin{bmatrix} I - z_1 \widetilde{A} & -z_2 \widetilde{B}_0 - z_1 B K_3 \\ -z_1 \widetilde{C} & I - z_2 \widetilde{D}_0 - z_1^2 D K_3 \end{bmatrix} \quad (36)$$

Application of appropriate elementary operations (which leave the determinant invariant) to the right-hand side of this last expression now yields that it can be replaced by

$$det \begin{bmatrix} I - z_1 \widetilde{A} & 0 & -z_2 \widetilde{B}_0 - z_1 B K_3 \\ 0 & I & -z_1 D K_3 \\ -z_1 \widetilde{C} & -z_1 I & I - z_2 \widetilde{D}_0 \end{bmatrix} \quad (37)$$

where

$$\widetilde{A} = A + B L_1 C, \ \widetilde{B}_0 = B_0 + B K_2$$
$$\widetilde{C} = C + D L_1 C, \ \widetilde{D}_0 = D_0 + D K_2$$

and

$$\widetilde{A}_1 = \begin{bmatrix} \widetilde{A} & 0 & B K_3 \\ 0 & 0 & D K_3 \\ \widetilde{C} & I & 0 \end{bmatrix}, \ \widetilde{A}_2 = \begin{bmatrix} 0 & 0 & \widetilde{B}_0 \\ 0 & 0 & 0 \\ 0 & 0 & \widetilde{D}_0 \end{bmatrix} \quad (38)$$

Now we can replace the right-hand side of the expression defining $\rho_c(z_1, z_2)$ by

$$\det(I - z_1 \widetilde{A}_1 - z_2 \widetilde{A}_2)$$

where

$$\begin{aligned} \widetilde{A}_1 &= A_1 + B_1 K \\ \widetilde{A}_2 &= A_2 + B_2 K \end{aligned} \quad (39)$$

or

$$A_1 = \begin{bmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ C & I & 0 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & 0 & B_0 \\ 0 & 0 & 0 \\ 0 & 0 & D_0 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} B & 0 & B \\ 0 & 0 & D \\ D & 0 & 0 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 & B & 0 \\ 0 & 0 & 0 \\ 0 & D & 0 \end{bmatrix} \quad (40)$$

and

$$K = \begin{bmatrix} K_1 & 0 & 0 \\ 0 & 0 & K_2 \\ 0 & 0 & K_3 \end{bmatrix} \quad (41)$$

Theorem 1 is now applicable and we have the following result.

Theorem 5: Suppose that a discrete linear repetitive process of the form described by (1) is subject to a control law defined by (30) with K_1 satisfying (14). Then the resulting closed loop process is stable along the pass if there exist matrices $Y > 0$, $X = diag(X_1, X_2, X_3) > 0$ and $Z > 0$ such that

$$\begin{bmatrix} Y - Z & 0 & \widehat{C}^T N^T B_1^T + Y A_1^T \\ 0 & -Z & \widehat{C}^T N^T B_2^T + Y A_2^T \\ A_1 Y + B_1 N \widehat{C} & A_2 Y + B_2 N \widehat{C} & -Y \end{bmatrix} < 0, \quad (42)$$
$$X \widehat{C} = \widehat{C} Y$$

where $A_1,\ A_2,\ B_1,\ B_2$ are given by (40)

$$\widehat{C} = \begin{bmatrix} C & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \ N = \begin{bmatrix} N_1 & 0 & 0 \\ 0 & 0 & N_2 \\ 0 & 0 & N_3 \end{bmatrix} \quad (43)$$

and

$$\begin{bmatrix} L_1 & 0 & 0 \\ 0 & 0 & K_2 \\ 0 & 0 & K_3 \end{bmatrix} = N X^{-1} \quad (44)$$

Proof. This is virtually identical to that of Theorem 4 and hence the details omitted here.

VII. Numerical example

Consider the case of (1) defined by

$$A=\begin{bmatrix} 0.06 & -1.62 & 0.0 \\ -0.98 & 0.28 & -2.89 \\ 0.03 & 2.66 & 2.63 \end{bmatrix},\ B=\begin{bmatrix} -1.43 & -2.13 \\ 1.23 & 1.48 \\ 2.91 & -2.18 \end{bmatrix},$$

$$B_0=\begin{bmatrix} 0.04 & 0.0 \\ 0.0 & 0.0 \\ 0.0 & 1.04 \end{bmatrix},\ C=\begin{bmatrix} -1.40 & -0.03 & -2.70 \\ 0.52 & 0.0 & -2.15 \end{bmatrix},$$

$$D=\begin{bmatrix} -1.64 & -0.52 \\ -0.71 & 0.11 \end{bmatrix},\ D_0=\begin{bmatrix} -0.28 & -0.31 \\ 1.15 & -0.31 \end{bmatrix}$$

In this case, the design algorithm of Theorem 3 is successful with $X = diag(X_1, X_2)$, where

$$X_1 = \begin{bmatrix} 606209.8877 & -862471.6775 \\ -862471.6775 & 1346539.1315 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 5077595.6297 & -2067002.4926 \\ -2067002.4926 & 11684609.2832 \end{bmatrix}$$

and

$$N=\begin{bmatrix} -178963.77 & 260104.7 & 439963.13 & -2530866.6 \\ 4116.45 & -35496.08 & -360858.98 & 2093081.41 \end{bmatrix}$$

where Y and Z are omitted due to space limitations. Then the matrices L_1 and K_2 of (18) are

$$L_1 = \begin{bmatrix} -0.2299 & 0.0459 \\ -0.3462 & -0.2481 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} -0.0016 & -0.2169 \\ 0.002 & 0.1795 \end{bmatrix}$$

which yield the controller (10) matrices

$$\widetilde{K}_1 = \begin{bmatrix} -0.1523 & 0.0576 \\ -0.2020 & -0.2524 \end{bmatrix},$$
$$\widetilde{K}_2 = \begin{bmatrix} -0.1103 & -0.2163 \\ 0.2363 & 0.1355 \end{bmatrix}$$

VIII. Conclusions

One unique feature of repetitive processes in comparison to other classes of 2D systems is that it is possible define physically meaningful control laws for them. It is hence essential to have an analysis setting where such control laws can be designed for stability and/or performance.

Previous work has shown that, of the currently available tools, it is only an LMI based setting can meet this last specification. In this paper we have continued the development of control laws based on this analysis setting which critically remove the need to use current pass state feedback information.

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