Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Robust Adaptive Control for a Class of Perturbed Strict-Feedback Nonlinear Systems

S. S. Ge, Fan Hong, Tong Heng Lee

***Abstract*— In this paper, robust adaptive control is presented for a class of perturbed strict-feedback nonlinear systems with both completely unknown control coef£cients and parametric uncertainties. The proposed design method does not require the *a priori* knowledge of the signs of the unknown control coef£cients. It is proved that the proposed robust adaptive scheme can guarantee the global uniform ultimate boundedness of the closed-loop system signals.**

I. INTRODUCTION

With the exciting development of adaptive control for parametric uncertain nonlinear systems [1], much attention has been paid to the application-motivated problem of robust adaptive control for nonlinear systems in the presence of time-varying disturbances, as described by the following class of single-input-single-output (SISO) nonlinear uncertain systems in the perturbed strict-feedback form

$$\begin{aligned}\dot{x}_i &= g_i x_{i+1} + \theta_i^T \psi_i(\bar{x}_i) + \Delta_i(t,x), i = 1, ..., n-1 \\ \dot{x}_n &= g_n u + \theta_n^T \psi_n(x) + \Delta_n(t,x)\end{aligned} \tag{1}$$

where $x = [x_1, ..., x_n]^T \in R^n$, $\bar{x}_i = [x_1, ..., x_i]^T$, $i = 1, ..., n-1$ are the state vectors, $u \in R$ is the control, $\theta_i \in R^{p_i}$, $i = 1, ..., n$ are the unknown constant parameter vectors, p_i's are positive integers, $\psi_i(\bar{x}_i)$, $i = 1, ..., n$ are known nonlinear functions which are continuous and satisfy $\psi_i(0) = 0$, unknown constants g_i, $i = 1, ..., n-1$ are referred to as virtual control coef£cients [1], g_n is referred to as the high-frequency gain, and Δ_i's are unknown Lipschitz continuous functions.

When $g_i = 1$, robust adaptive control algorithms for system (1) have been developed in [2][3][4], and [5] for systems with inverse dynamics. When g_i's are unknown with known signs, several excellent adaptive control algorithms are also developed in the literature for nonlinear systems. For unknown constant g_i's, an adaptive control solution was presented in [1] for strict-feedback nonlinear systems without disturbance Δ_i. When g_i's are functions of states, adaptive control schemes were proposed for uncertain strict-feedback and pure-feedback nonlinear systems with the aid of neural network parameterization in [6], [7]. When g_i's are completely unknown, i.e., with unknown signs, the £rst solution was given in [8] for a class of £rst-order linear systems using Nussbaum functions, adaptive control was given in [9] for £rst-order nonlinear systems, adaptive output feedback control is proposed in [10] for general nonlinear system, and adaptive control was investigated for a class of high-order nonlinear systems in the parameter-strict-feedback form in [11], [12]. A recent work on output feedback control of uncertain systems with unknown control direction was reported in [13].

In [14], a class of uncertain nonlinear systems with completely unknown time-varying g_i's, uncertain time-varying parameters and unknown time-varying bounded disturbances. The exponentially decaying terms have been introduced in the controller design to handle the bounded disturbances. The nice properties of Nussbaum functions are dif£cult to be utilized directly in the stability analysis due to the presence of the exponential terms. In addition, the stability proof has to be function-dependent by fully exploiting the speci£c Nussbaum functions being chosen. Though a much neater proof was provided in [14] for the choice of $N(\zeta) = \exp(\zeta^2)\cos(\frac{\pi}{2}\zeta)$, it is not the case for $N(\zeta) = \zeta^2\cos(\zeta)$ as chosen in this paper. The proof cannot be straightforwardly extended and the speci£c properties of the function need to be investigated fully in the derivation. Due to the different problem formulations and methodologies used (e.g., projection algorithm has to be utilized for on-line tuning of the time-varying unknown parameters in [14]), the proposed design in this paper is much more tighter and the controller is composed of smooth functions, which is a must in backstepping design.

In this paper, robust adaptive control is proposed for systems in strict feedback form with disturbances. The main contributions are: (i) a new technical lemma is introduced, which plays a fundamental role in solving the proposed problem, (ii) the controller does not require the *a priori* knowledge of the signs of the unknown control coef£cients, and the unknown bounds of the disturbance terms are estimated on-line for improving performance, and (iii) the proposed design method expands the class of nonlinear systems for which robust adaptive control approaches have been studied through the introduction of exponential decaying terms in stability analysis.

II. PROBLEM FORMULATION AND PRELIMINARIES

The control objective is to construct a robust adaptive nonlinear control law so that the state x_1 of system (1) is driven to a small neighborhood of the origin, while keep internal Lagrange stability.

In system (1), the unknown nonlinear functions $\Delta_i(t,x)$ could be due to many factors [3], such as measurement noise, modeling errors, external time-varying disturbances, modeling simpli£cations or changes due to time variations. The occurrence of virtual control coef£cients g_i's is also

The authors are with Department of Electrical & Computer Engineering, National University of Singapore, Singapore 117576. elegesz@nus.edu.sg

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quite common in practice. The examples range from electric motors and robotic manipulators to ¤ight dynamics [1].

Assumption 1: There exist unknown positive constants p_i^*, $1 \leq i \leq n$, such that $\forall(t, x) \in R_+ \times R^n$, $|\Delta_i(t, x)| \leq p_i^* \phi_i(x_1, \cdots, x_i)$, where ϕ_i is a known nonnegative smooth function.

A function $N(\zeta)$ is called a Nussbaum-type function if it has the following properties [8]

$$\lim_{s\to+\infty} \sup \int_{s_0}^{s} N(\zeta)d\zeta = +\infty \quad (2)$$

$$\lim_{s\to+\infty} \inf \int_{s_0}^{s} N(\zeta)d\zeta = -\infty \quad (3)$$

Commonly used Nussbaum functions include: $\zeta^2\cos(\zeta)$, $\zeta^2\sin(\zeta)$ and $\exp(\zeta^2)\cos(\frac{\pi}{2}\zeta)$ [15]. In this paper, the even Nussbaum function, $N(\zeta) = \zeta^2\cos(\zeta)$, $\zeta \in R$, is used for analysis. In comparison with the de£nition for Nussbaum functions in [12], the de£nition given by (2) and (3) gives a much larger set of functions, though the example functions satisfy both de£nitions.

Lemma 1: Let $V(\cdot)$ and $\zeta(\cdot)$ be smooth functions de£ned on $[0, t_f)$ with $V(t) \geq 0$, $\forall t \in [0, t_f)$, and $N(\zeta) = \zeta^2\cos(\zeta)$. If the following inequality holds:

$$V(t) \leq c_0 + e^{-c_1 t}\int_0^t [g_0 N(\zeta) + 1]\dot{\zeta}e^{c_1\tau}d\tau, \forall t \in [0, t_f) \quad (4)$$

where constant $c_1 > 0$, g_0 is a nonzero constant, and c_0 represents some suitable constant, then $V(t)$, $\zeta(t)$ and $\int_0^t g_0 N(\zeta)\dot{\zeta}d\tau$ must be bounded on $[0, t_f)$.

Proof: See Appendix. ■

Though the proof is not trivial even for £nite t_f already, it is the case that $t_f \to \infty$ is of interest. This can be easily extended due to Proposition 1 below. Consider

$$\dot{x}(t) \in F(x(t)),\ x(0) = x^0 \quad (5)$$

where $z \mapsto F(z) \subset R^N$ is upper semicontinuous on R^n with non-empty convex and compact values. It is well known that the initial-value problem has a solution and that every solution can be maximally extended.

Proposition 1: [16] If $x : [0, t_f) \to R^N$ is a bounded maximal solution of (5), then $t_f = \infty$.

Remark 1: From our understanding, we can make a conjecture that Lemma 1 is true for all the Nussbaum functions. Because of the presence of $e^{c_1\tau}$ in (4), the proof is function-dependent. We hope that interested reader can prove the Lemma for general Nussbaum functions. In addition, we would like to point out that $N(\cdot)$ is not necessarily to be an even function, which is only made for the convenience of proof. If $N(\cdot)$ is chosen as an odd function, e.g., $N(\zeta) = \zeta^2\sin(\zeta)$, the Lemma can be easily proven by following the same procedure.

III. Robust Adaptive Control and Main Results

In this section, the robust adaptive control design procedure for nonlinear system (1) is presented. The design of both the control law and the adaptive laws is based on a change of coordinates $z_1 = x_1$, $z_i = x_i - \alpha_{i-1}$, $i = 1, ..., n$, where the functions α_i, $i = 1, ..., n-1$ are referred to as intermediate control functions which will be designed using backstepping technique, $\hat{b}_i$ is the parameter estimate for b_i^* which is the grouped unknown bound for p_i^*, $\hat{\theta}_{a,i}$ represents the estimate of unknown parameter $\theta_{a,i}^*$ which is an augmented parameter and consists of $g_j, j = 1, \cdots, i-1$ and $\theta_j, j = 1, \cdots, i$ as will be clari£ed later, and ζ_i is the argument of the Nussbaum function. At each intermediate step i, we design the intermediate control function α_i using an appropriate Lyapunov function V_i, and give the updating laws $\dot{\hat{b}}_i$, $\dot{\hat{\theta}}_{a,i}$ and $\dot{\zeta}_i$. At the nth step, the actual control u appears and the design is completed. For clarity and conciseness, let us de£ne $\tilde{\theta}_{a,i} = \hat{\theta}_{a,i} - \theta_{a,i}^*$, $\tilde{b}_i = \hat{b}_i - b_i^*$, constants

$$c_{i1} := \min\{2k_{i0}, \frac{\sigma_{\theta_i}}{\lambda_{\min}(\Gamma_i^{-1})}, \sigma_{b_i}\gamma_i\} \quad (6)$$

$$c_{i2} := b_i^* 0.2785\epsilon_i + \frac{1}{2}\sigma_{\theta_i}\|\theta_{a,i}^* - \theta_{a,i}^0\|^2 + \frac{1}{2}\sigma_{b_i}(b_i^* - b_i^0)^2 \quad (7)$$

the Lyapunov function candidate

$$V_i = \frac{1}{2}z_i^2 + \frac{1}{2}\tilde{\theta}_{a,i}^T\Gamma_i^{-1}\tilde{\theta}_{a,i} + \frac{1}{2}\gamma_i^{-1}\tilde{b}_i^2 \quad (8)$$

and the intermediate variables including the control functions and adaptive laws

$$\eta_i = k_i z_i + \hat{\theta}_{a,i}^T\psi_{a,i} + \hat{b}_i\bar{\phi}_i\tanh\left(\frac{z_i\bar{\phi}_i}{\epsilon_i}\right) \quad (9)$$

$$\alpha_i = N(\zeta_i)\eta_i \quad (10)$$

$$\dot{\zeta}_i = z_i\eta_i \quad (11)$$

$$\dot{\hat{\theta}}_{a,i} = \Gamma_i\left[z_i\psi_{a,i} - \sigma_{\theta_i}(\hat{\theta}_{a,i} - \theta_{a,i}^0)\right] \quad (12)$$

$$\dot{\hat{b}}_i = \gamma_i\left[z_i\bar{\phi}_i\tanh\left(\frac{z_i\bar{\phi}_i}{\epsilon_i}\right) - \sigma_{b_i}(\hat{b}_i - b_i^0)\right] \quad (13)$$

where constants $\hat{b}_i$ and $\hat{\theta}_{a,i}$, functions $\psi_{a,i}$ and $\bar{\phi}_i$ are de£ned by

$$b_i^* = \max\{p_1^*, \cdots, p_i^*\} \quad (14)$$

$$\theta_{a,i}^* = [1, g_1, ..., g_{i-1}, \theta_i^T, \theta_1^T, ..., \theta_{i-1}^T]^T \quad (15)$$

$$\bar{\phi}_i(\bar{x}_i) = \phi_i + \sum_{j=1}^{i-1}\left|\frac{\partial\alpha_{i-1}}{\partial x_j}\right|\phi_j \quad (16)$$

$$\psi_{a,i} = [\beta_i, -\frac{\partial\alpha_{i-1}}{\partial x_1}x_2, ..., -\frac{\partial\alpha_{i-1}}{\partial x_{i-1}}x_i, \psi_i^T, -\frac{\partial\alpha_{i-1}}{\partial x_1}\psi_1^T, ..., -\frac{\partial\alpha_{i-1}}{\partial x_{i-1}}\psi_{i-1}^T]^T \quad (17)$$

with $\alpha_n = u$, $b_1^* = p_1^*$, $\bar{\phi}_1 = \phi_1$, $\theta_{a,1}^* = \theta_1$, $\psi_{a,1} = \psi_1$, and β_i being de£ned by

$$\beta_i = -\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_{a,j}}\dot{\hat{\theta}}_{a,j} - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{b}_j}\dot{\hat{b}}_j - \frac{\partial\alpha_{i-1}}{\partial\zeta_{i-1}}\dot{\zeta}_{i-1} \quad (18)$$

and $\Gamma_i = \Gamma_i^T > 0$, $\gamma_i > 0$, $\epsilon_i > 0$, $\hat{\theta}_{a,i}$ and $\hat{b}_i$ are the estimates of $\theta_{a,i}^*$ and b_i^*, constant $k_{10} = k_i - \frac{1}{4} > 0$, and σ_{θ_i}, σ_{b_i}, $\theta_{a,i}^0$, and b_i^0 are positive design constants.

In this paper, the following inequalities play an important role

$$0 \le |x| - x\tanh(\frac{x}{\epsilon}) \le 0.2785\epsilon, \text{for } \epsilon > 0,\ x \in R[3] \quad (19)$$

$$-\tilde{\theta}_{a,i}^T(\hat{\theta}_{a,i} - \theta_{a,i}^0) \le -\frac{1}{2}\|\tilde{\theta}_{a,i}\|^2 + \frac{1}{2}\|\theta_{a,i}^* - \theta_{a,i}^0\|^2 \quad (20)$$

$$-\tilde{b}_i(\hat{b}_i - b_i^0) \le -\frac{1}{2}\tilde{b}_i^2 + \frac{1}{2}(b_i^* - b_i^0)^2 \quad (21)$$

Step 1: To start, let us study the z_1-subsystem of (1):

$$\dot{x}_1 = g_1 x_2 + \theta_1^T \psi_1(x_1) + \Delta_1(t,x) \quad (22)$$

where x_2 is taken for a virtual control input. In light of Assumption 1, we have

$$\begin{aligned} z_1\dot{z}_1 &= z_1(g_1x_2 + \theta_1^T\psi_1(x_1) + \Delta_1(t,x)) \\ &\le z_1(g_1x_2 + \theta_1^T\psi_1) + b_1^*|z_1|\bar{\phi}_1 \quad (23) \end{aligned}$$

Consider the Lyapunov function candidate given in (8). The time derivative of V_1 along (23) is

$$\begin{aligned} \dot{V}_1 &\le z_1(g_1x_2 + \theta_{a,1}^{*T}\psi_{a,1}) + b_1^*|x_1|\bar{\phi}_1 \\ &\quad + \tilde{\theta}_{a,1}^T\Gamma_1^{-1}\dot{\hat{\theta}}_{a,1} + \gamma_1^{-1}\tilde{b}_1\dot{\hat{b}}_1 \quad (24) \end{aligned}$$

Since $x_2 = z_2 + \alpha_1$, substituting (9)-(11) with $i = 1$ into (24) yields

$$\begin{aligned} \dot{V}_1 &\le g_1z_1z_2 + g_1N(\zeta_1)\dot{\zeta}_1 + z_1\theta_{a,1}^{*T}\psi_{a,1} + b_1^*|x_1|\bar{\phi}_1 \\ &\quad + \tilde{\theta}_{a,1}^T\Gamma_1^{-1}\dot{\hat{\theta}}_{a,1} + \gamma_1^{-1}\tilde{b}_1\dot{\hat{b}}_1 \quad (25) \end{aligned}$$

Adding and subtracting $\dot{\zeta}_1$ on the right hand side of (25), and noting (12) and (13), we have

$$\begin{aligned} \dot{V}_1 &\le -k_1z_1^2 + g_1z_1z_2 + g_1N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 + b_1^*|x_1|\bar{\phi}_1 \\ &\quad - b_1^*x_1\bar{\phi}_1\tanh(\frac{x_1\bar{\phi}_1}{\epsilon_1}) - \sigma_{\theta_1}\tilde{\theta}_{a,1}^T(\hat{\theta}_{a,1} - \theta_{a,1}^0) \\ &\quad - \sigma_{b_1}\tilde{b}_1(\hat{b}_1 - b_1^0) \quad (26) \end{aligned}$$

Using the inequalities (19)-(21), (26) becomes

$$\dot{V}_1 \le -c_{11}V_1 + c_{12} + g_1N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 + g_1^2z_2^2 \quad (27)$$

with constants $k_{10} = k_1 - \frac{1}{4} > 0$, and c_{11}, c_{12} being de£ned in (6) and (7) respectively.

Let $\rho_1 := \frac{c_{12}}{c_{11}}$. Multiplying (27) by $e^{c_{11}t}$ leads to

$$\begin{aligned} \frac{d}{dt}(V_1e^{c_{11}t}) &\le c_{12}e^{c_{11}t} + g_1N(\zeta_1)\dot{\zeta}_1e^{c_{11}t} \\ &\quad + \dot{\zeta}_1e^{c_{11}t} + g_1^2z_2^2e^{c_{11}t} \quad (28) \end{aligned}$$

Integrating (28) over $[0,t]$, we have

$$\begin{aligned} V_1(t) &\le \rho_1 + V_1(0) + e^{-c_{11}t}\int_0^t [g_1N(\zeta_1) + 1]\dot{\zeta}_1e^{c_{11}\tau}d\tau \\ &\quad + \int_0^t g_1^2z_2^2e^{-c_{11}(t-\tau)}d\tau \quad (29) \end{aligned}$$

Remark 2: If there was no uncertain term Δ_1 as in [11][12], where the uncertainty is from unknown parameters only, adaptive control can be used to solve the problem elegantly and the asymptotic stability can be guaranteed. However, it is not the case here due to the presence of the uncertainty terms Δ_1 in system (1). For illustration, integrating (27) over $[0,t]$ leads to

$$V_1(t) \le V_1(0) + c_{12}t + \int_0^t (g_1N(\zeta_1) + 1)\dot{\zeta}_1 d\tau + \int_0^t g_1^2z_2^2 d\tau$$

from which, no conclusion on the boundedness of $V_1(t)$ or $\zeta_1(t)$ can be drawn by applying Lemma 1 in [12] due to the extra term $c_{12}t$. The problem can be successfully solved by multiplying the exponential term $e^{c_{11}t}$ to both sides of (27) as in the paper. From (29), the stability results can be drawn by invoking Lemma 1 if $\int_0^t g_1^2z_2^2e^{-c_{11}(t-\tau)}d\tau$ is upper bounded.

Remark 3: In equation (29), if there is no extra term $\int_0^t g_1^2z_2^2e^{-c_{11}(t-\tau)}d\tau$ within the inequality, we can conclude that $V_1(t), \zeta_1$ and $z_1, \hat{\theta}_{a,1}, \hat{b}_1$ are all bounded on $[0, t_f)$ according to Lemma 1. Thus, from Proposition 1, $t_f = \infty$, and we claim that $z_1, \hat{\theta}_{a,1}, \hat{b}_1$ are globally uniformly ultimately bounded. Due to the presence of term $\int_0^t g_1^2z_2^2e^{-c_{11}(t-\tau)}d\tau$ in (29), Lemma 1 cannot be applied directly. By noting that

$$\begin{aligned} e^{-c_{11}t}\int_0^t g_1^2z_2^2e^{c_{11}\tau}d\tau &\le e^{-c_{11}t}g_1^2\sup_{\tau\in[0,t]} z_2^2\int_0^t e^{c_{11}\tau}d\tau \\ &\le \frac{g_1^2\sup_{\tau\in[0,t]} z_2^2}{c_{11}} \end{aligned}$$

we know that if z_2 can be regulated as bounded, the boundedness of $\int_0^t g_1^2z_2^2e^{-c_{11}(t-\tau)}d\tau$ is obvious. Then, according to Lemma 1, the boundedness of $z_1(t)$ can be guaranteed. The effect of $\int_0^t g_1^2z_2^2e^{-c_{11}(t-\tau)}d\tau$ will be dealt with at the following steps.

Step i ($2 \le i \le n-1$): In view of Assumption 1, we have

$$z_i\dot{z}_i \le z_i(g_ix_{i+1} + \theta_{a,i}^{*T}\psi_{a,i}) + b_i^*|z_i|\bar{\phi}_i$$

where b_i^*, $\theta_{a,i}^*$, $\bar{\phi}_i$, $\psi_{a,i}$ and β_i are de£ned in (14), (15), (16), (17) and (18) respectively.

Consider the Lyapunov function candidate V_i given in (8). Selecting α_i and parameters adaptation laws as in (10)-(13), we can similarly obtain

$$\begin{aligned} V_i(t) &\le \rho_i + V_i(0) + e^{-c_{i1}t}\int_0^t [g_iN(\zeta_i) + 1]\dot{\zeta}_ie^{c_{i1}\tau}d\tau \\ &\quad + \int_0^t g_i^2z_{i+1}^2e^{-c_{i1}(t-\tau)}d\tau \end{aligned}$$

with $\rho_i := \frac{c_{i2}}{c_{i1}}$, constants $k_{i0} = k_i - \frac{1}{4} > 0$, and c_{i1}, c_{i2} being de£ned in (6) and (7) respectively.

Remark 4: Similarly, if z_{i+1} can be regulated as bounded, and therefore $\int_0^t g_i^2z_{i+1}^2e^{-c_{i1}(t-\tau)}d\tau$ is bounded at the following steps, then according to Lemma 1, the boundedness of $z_i(t)$ can be guaranteed.

Step n: In this £nal step, the actual control u appears. Similarly, we have

$$z_n\dot{z}_n \le z_n(g_nu + \theta_{a,n}^{*T}\psi_{a,n}) + b_n^*|z_n|\bar{\phi}_n$$

where b_n^*, $\theta_{a,n}^*$, $\bar{\phi}_n$, $\psi_{a,n}$ and β_n are de£ned in (14), (15), (16), (17) and (18) respectively.

Consider the Lyapunov function candidate V_n given in (8). Selecting u and parameters adaptation laws as in (10)-(13), we can similarly obtain

$$V_n(t) \leq \rho_n + V_n(0) + e^{-c_{n1}t} \int_0^t [g_n N(\zeta_n) + 1]\dot{\zeta}_n e^{c_{n1}\tau} d\tau$$

with $\rho_n := \frac{c_{n2}}{c_{n1}}$, constants $c_{n1}, c_{n2} > 0$ being de£ned in (6) and (7) respectively.

Using Lemma 1, we can conclude that $\zeta_n(t)$ and $V_n(t)$, hence $z_n(t)$, $\hat{\theta}_{a,n}(t)$, $\hat{b}_{a,n}(t)$ are bounded on $[0, t_f)$. From the boundedness of $z_n(t)$, the boundedness of the extra term $\int_0^t g_{n-1}^2 z_n^2 e^{-c_{n-1,1}(t-\tau)} d\tau$ at Step $(n-1)$ is readily obtained. Applying Lemma 1 backward $(n-1)$ times, it can be seen from the above design procedures that $V_i(t)$, $z_i(t)$, $\hat{\theta}_{a,i}(t)$, $\hat{b}_{a,i}(t)$, and hence $x_i(t)$ are bounded on $[0, t_f)$.

Theorem 1: For the perturbed strict-feedback nonlinear system (1) with completely unknown control coef£cients g_i, under Assumption 1, if we apply the controller (10)-(13), the solutions of the resulting closed-loop adaptive system are globally uniformly ultimately bounded. Furthermore, given any $\mu > \mu^* = \sqrt{\sum_{i=1}^n 2(\rho_i + c_i)}$, there exists T such that, for all $t \geq T$, we have $\|z(t)\| \leq \mu$, where $z(t) := [z_1, \cdots, z_n]^T \in R^n$, $\rho_i := \frac{c_{i2}}{c_{i1}}$, $i = 1, \cdots, n$, constants $c_{i1} > 0$ and $c_{i2} > 0$ are de£ned by (6) and (7) respectively, and c_i is the upper bound of $\int_0^t (g_i N(\zeta_i)\dot{\zeta}_i + \dot{\zeta}_i + g_i^2 z_{i+1}^2) e^{-c_{i1}(t-\tau)} d\tau$, $i = 1, \cdots, n-1$ and c_n is the upper bound of $\int_0^t (g_n N(\zeta_n)\dot{\zeta}_n + \dot{\zeta}_n) e^{-c_{n1}(t-\tau)} d\tau$. The compact set $\Omega_z = \{z \in R^n | \|z(t)\| \leq \mu\}$ can be made as small as desired by appropriately choosing the design constants. Furthermore, the output $y(t)$ satis£es the following property:

$$|y(t)| \leq \sqrt{2V_1(0)e^{-c_{11}t} + 2(\rho_1 + c_1)}, \forall t \geq 0. \qquad (30)$$

Proof: The proof can be easily completed by following the above design procedures from Step 1 to Step n. According to Proposition 1, if the solution of the closed-loop system is bounded, then $t_f = \infty$. Therefore, we can obtain the globally uniformly ultimately boundedness of all the signals in the closed-loop system. Since $x_1(t) = z_1(t)$, from the de£nition of V_1 and (29), the property (30) can be readily obtained. Thus, by appropriately choosing the design constants, we can achieve the regulation of the state $x_1(t)$ to any prescribed accuracy while keeping the boundedness of all the signals and states of the close-loop system. ■

IV. Conclusion

In this paper, a robust adaptive control approach for a class of perturbed uncertain strict-feedback nonlinear systems with unknown control coef£cients has been presented. The design method does not require the *a priori* knowledge of the signs of the unknown control coef£cients due to the incorporation of Nussbaum gain in the controller design. It has been proved that the proposed robust adaptive scheme can guarantee the global uniform ultimate boundedness of the closed-loop system signals.

Acknowledgement

The authors thank E. P. Ryan for the many insightful comments.

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Appendix

Proof of Lemma 1: To start with, re-write (4) as

$$V(t) \leq c_0 + e^{-c_1 t} \int_0^t [g_0 N(\zeta) + 1]\dot{\zeta} e^{c_1\tau} d\tau, \forall t \in [0, t_f) \quad (31)$$

We £rst show that $\zeta(t)$ is bounded on $[0, t_f)$ by seeking a contradiction. Suppose that $\zeta(t)$ is unbounded and two cases should be considered: (i) $\zeta(t)$ has no upper bound and (ii) $\zeta(t)$ has no lower bound.

Case (i): $\zeta(t)$ has no upper bound on $[0, t_f)$. In this case, there must exist a monotone increasing sequence $\{t_i\}$, $i = 1, 2, \cdots$, such that $\{\omega_i = \zeta(t_i)\}$ is monotone increasing with $\omega_1 > |\zeta(0)|$, $\lim_{i\to+\infty} t_i = t_f$, and $\lim_{i\to+\infty} \omega_i = +\infty$.

For clarity, de£ne

$$N_g(\omega_i, \omega_j) = \int_{\omega_i}^{\omega_j} g_0 N(\zeta(\tau)) e^{-c_1(t_j-\tau)} d\zeta(\tau) \qquad (32)$$

with an understanding that $N_g(\omega_i,\omega_j) = N_g(\omega(t_i),\omega(t_j)) = N_g(t_i,t_j)$ for notation convenience, and $\omega_i \le \omega_j$, $\tau \in [t_i,t_j]$. Let $\zeta^{-1}(x)$ denote the inverse function of $\zeta(\tau)$, i.e., $\zeta(\zeta^{-1}(\tau)) = \zeta^{-1}(\zeta(\tau)) \equiv \tau$. Noting $N(\zeta) = \zeta^2\cos(\zeta)$, (32) can be re-written as

$$N_g(\omega_i,\omega_j) = \int_{\omega_i}^{\omega_j} g_0\zeta^2\cos(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}d\zeta \tag{33}$$

Integration by parts, we have

$$N_g(\omega_i,\omega_j) = g_0\zeta^2\sin(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\Big|_{\omega_i}^{\omega_j} - \int_{\omega_i}^{\omega_j} g_0\sin(\zeta)d\{\zeta^2 e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\} \tag{34}$$

Noting the fact that $d\zeta^{-1}(\zeta) = d\tau$ and $d\{\zeta^2 e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\} = 2\zeta e^{-c_1[t_j-\zeta^{-1}(\zeta)]}d\zeta + c_1\zeta^2 e^{-c_1(t_j-\tau)}d\tau$, equation (34) becomes

$$\begin{aligned} N_g(\omega_i,\omega_j) = {} & g_0\zeta^2\sin(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\Big|_{\omega_i}^{\omega_j} \\ & - \int_{\omega_i}^{\omega_j} 2g_0\zeta\sin(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}d\zeta \\ & - \int_{t_i}^{t_j} c_1g_0\zeta^2\sin(\zeta)e^{-c_1(t_j-\tau)}d\tau \end{aligned} \tag{35}$$

Integration by parts for the term $\int_{\omega_i}^{\omega_j} 2g_0\zeta\sin(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}d\zeta$ in (35), we have

$$\begin{aligned} & \int_{\omega_i}^{\omega_j} 2g_0\zeta\sin(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}d\zeta \\ = {} & -2g_0\zeta\cos(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\Big|_{\omega_i}^{\omega_j} \\ & + \int_{\omega_i}^{\omega_j} 2g_0\cos(\zeta)d\{\zeta e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\} \end{aligned} \tag{36}$$

Noting that $d\{\zeta e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\} = e^{-c_1[t_j-\zeta^{-1}(\zeta)]}d\zeta + c_1\zeta e^{-c_1[t_j-\zeta^{-1}(\zeta)]}d\tau$, equation (36) becomes

$$\begin{aligned} & \int_{\omega_i}^{\omega_j} 2g_0\zeta\sin(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}d\zeta \\ = {} & -2g_0\zeta\cos(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\Big|_{\omega_i}^{\omega_j} \\ & + \int_{\omega_i}^{\omega_j} 2g_0\cos(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}d\zeta \\ & + \int_{t_i}^{t_j} 2c_1g_0\zeta\cos(\zeta)e^{-c_1(t_j-\tau)}d\tau \end{aligned} \tag{37}$$

Substituting (37) into (35) yields

$$\begin{aligned} N_g(\omega_i,\omega_j) = {} & g_0\zeta^2\sin(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\Big|_{\omega_i}^{\omega_j} \\ & +2g_0\zeta\cos(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\Big|_{\omega_i}^{\omega_j} \\ & - \int_{\omega_i}^{\omega_j} 2g_0\cos(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}d\zeta \\ & - \int_{t_i}^{t_j} 2c_1g_0\zeta\cos(\zeta)e^{-c_1(t_j-\tau)}d\tau \\ & - \int_{t_i}^{t_j} c_1g_0\zeta^2\sin(\zeta)e^{-c_1(t_j-\tau)}d\tau \end{aligned} \tag{38}$$

Similarly, integration by parts for the term $\int_{\omega_i}^{\omega_j} 2g_0\cos(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}d\zeta$ in (38) by noting that $d\{e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\} = c_1e^{-c_1[t_j-\zeta^{-1}(\zeta)]}d\tau$, we have

$$\begin{aligned} & \int_{\omega_i}^{\omega_j} 2g_0\cos(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}d\zeta \\ & = 2g_0\sin(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\Big|_{\omega_i}^{\omega_j} \\ & - \int_{t_i}^{t_j} 2c_1g_0\sin(\zeta)e^{-c_1(t_j-\tau)}d\tau \end{aligned} \tag{39}$$

Substituting (39) into (38), we have

$$\begin{aligned} N_g(\omega_i,\omega_j) = {} & g_0\zeta^2\sin(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\Big|_{\omega_i}^{\omega_j} \\ & +2g_0\zeta\cos(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\Big|_{\omega_i}^{\omega_j} \\ & -2g_0\sin(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\Big|_{\omega_i}^{\omega_j} \\ & + \int_{t_i}^{t_j} 2c_1g_0\sin(\zeta)e^{-c_1(t_j-\tau)}d\tau \\ & - \int_{t_i}^{t_j} 2c_1g_0\zeta\cos(\zeta)e^{-c_1(t_j-\tau)}d\tau \\ & - \int_{t_i}^{t_j} c_1g_0\zeta^2\sin(\zeta)e^{-c_1(t_j-\tau)}d\tau \end{aligned} \tag{40}$$

Let us £rst consider the term $\int_{t_i}^{t_j} 2c_1g_0\sin(\zeta)e^{-c_1(t_j-\tau)}d\tau$ on the right side of (40). Using integral inequality $(b-a)m_{f1} \le \int_a^b f(x)dx \le (b-a)m_{f2}$ with $m_{f1} = \inf_{a\le x\le b} f(x)$ and $m_{f2} = \sup_{a\le x\le b} f(x)$, and noting that $0 < e^{-c_1(t_j-\tau)} \le 1$ for $\tau \in [t_i,t_j]$, we have

$$\Big|\int_{t_i}^{t_j} 2c_1g_0\sin(\zeta)e^{-c_1(t_j-\tau)}d\tau\Big| \le (t_j-t_i)2c_1g_0 \tag{41}$$

Next, for the term $\int_{t_i}^{t_j} 2c_1g_0\zeta\cos(\zeta)e^{-c_1(t_j-\tau)}d\tau$, applying integral inequality similarly by noting that $0 < e^{-c_1(t_j-\tau)} \le 1$ for $\tau \in [t_i,t_j]$, we have

$$\Big|\int_{t_i}^{t_j} 2c_1g_0\zeta\cos(\zeta)e^{-c_1(t_j-\tau)}d\tau\Big| \le (t_j-t_i)2c_1g_0\omega_j \tag{42}$$

Then, let us consider the term $\int_{t_i}^{t_j} c_1g_0\zeta^2\sin(\zeta)e^{-c_1(t_j-\tau)}d\tau$. Using the property that if $f(x) \le g(x)$, $\forall x \in [a,b]$, then $\int_a^b f(x)dx \le \int_a^b g(x)dx$ and noting that

$$-\omega_j^2 e^{c_1\tau} \le \zeta^2(\tau)\sin(\zeta(\tau))e^{c_1\tau} \le \omega_j^2 e^{c_1\tau}, \forall\tau \in [t_i,t_j]$$

we have

$$\begin{aligned} & e^{-c_1t_j}\int_{t_i}^{t_j} c_1g_0\zeta^2\sin(\zeta)e^{c_1\tau}d\tau \\ & \le e^{-c_1t_j}c_1g_0\omega_j^2\int_{t_i}^{t_j} e^{c_1\tau}d\tau = g_0\omega_j^2[1-e^{-c_1(t_j-t_i)}] \\ & e^{-c_1t_j}\int_{t_i}^{t_j} c_1g_0\zeta^2\sin(\zeta)e^{c_1\tau}d\tau \\ & \ge -e^{-c_1t_j}c_1g_0\omega_j^2\int_{t_i}^{t_j} e^{c_1\tau}d\tau = -g_0\omega_j^2[1-e^{-c_1(t_j-t_i)}] \end{aligned}$$

i.e.,

$$\left|e^{-c_1 t_j}\int_{t_i}^{t_j} c_1 g_0 \zeta^2 \sin(\zeta) e^{c_1\tau} d\tau\right| \le g_0\omega_j^2[1-e^{-c_1(t_j-t_i)}] \quad (43)$$

Noting that $\zeta^{-1}(\omega_i) = \zeta^{-1}(\zeta(t_i)) = t_i$ and $\zeta^{-1}(\omega_j) = \zeta^{-1}(\zeta(t_j)) = t_j$, from (41), (42) and (43), we have the following two inequalities

$$N_g(\omega_i,\omega_j) \le g_0\omega_j^2[\sin(\omega_j)+1-e^{-c_1(t_j-t_i)}] + f_u(\omega_i,\omega_j) \quad (44)$$

$$N_g(\omega_i,\omega_j) \ge -g_0\omega_j^2[-\sin(\omega_j)+1-e^{-c_1(t_j-t_i)}] + f_l(\omega_i,\omega_j) \quad (45)$$

where

$$\begin{aligned} f_u(\omega_i,\omega_j) &= g_0\omega_j\cos(\omega_j) - 2g_0\sin(\omega_j) \\ &+(t_j-t_i)2c_1g_0\omega_j + (t_j-t_i)2c_1g_0 \\ &-g_0e^{-c_1(t_j-t_i)}\omega_i^2\sin(\omega_i) - 2g_0e^{-c_1(t_j-t_i)}\omega_i\cos(\omega_i) \\ &+2g_0e^{-c_1(t_j-t_i)}\sin(\omega_i) \\ f_l(\omega_i,\omega_j) &= g_0\omega_j\cos(\omega_j) - 2g_0\sin(\omega_j) \\ &-(t_j-t_i)2c_1g_0\omega_j - (t_j-t_i)2c_1g_0 \\ &-g_0e^{-c_1(t_j-t_i)}\omega_i^2\sin(\omega_i) - 2g_0e^{-c_1(t_j-t_i)}\omega_i\cos(\omega_i) \\ &+2g_0e^{-c_1(t_j-t_i)}\sin(\omega_i) \end{aligned}$$

Re-write (31) as

$$V(t_i) \le c_0 + \int_{\zeta(0)}^{\zeta(t_i)} g_0 N(\zeta(\tau)) e^{-c_1(t_i-\tau)} d\zeta(\tau) + \int_{\zeta(0)}^{\zeta(t_i)} e^{-c_1(t_i-\tau)} d\zeta(\tau) \quad (46)$$

Noting (44), we have

$$\begin{aligned} V(t_i) &\le c_0 + N_g(\zeta(0),\omega_i) + (\omega_i-\zeta(0)) \sup_{\tau\in[0,t_i]} e^{-c_1(t_i-\tau)} \\ &\le c_0 + g_0\omega_i^2[\sin(\omega_i)+1-e^{-c_1t_i}] + f_u(\zeta(0),\omega_i) \\ &\quad +(\omega_i-\zeta(0)) \\ &= \omega_i^2\{g_0[\sin(\omega_i)+1-e^{-c_1t_i}] + \frac{1}{\omega_i^2}[c_0+f_u(\zeta(0),\omega_i) \\ &\quad +(\omega_i-\zeta(0))]\} \end{aligned}$$

Taking the limit as $i\to+\infty$, hence $t_i\to t_f$, $\omega_i\to+\infty$, $\frac{f_u(\zeta(0),\omega_i)}{\omega_i^2}\to 0$, we have

$$0 \le \lim_{i\to+\infty} V(t_i) \le \lim_{i\to+\infty} \omega_i^2 g_0[\sin(\omega_i)+1-e^{-c_1t_i}]$$

which, if $g_0>0$, draws a contradiction when $[\sin(\omega_i)+1-e^{-c_1t_i}]<0$, and if $g_0<0$, draws a contradictions when $[\sin(\omega_i)+1-e^{-c_1t_i}]>0$. Therefore, $\zeta(t)$ is upper bounded on $[0,t_f)$.

Case (ii): $\zeta(t)$ has no lower bound on $[0,t_f)$. There must exist a monotone increasing sequence $\{\underline{t}_i\}$, $i=1,2,\cdots$, such that $\{\underline{\omega}_i = -\zeta(\underline{t}_i)\}$ with $\omega_1 > |\zeta(0)|$, $\lim_{i\to+\infty}\underline{t}_i = t_f$, and $\lim_{i\to+\infty}\underline{\omega}_i = +\infty$.

Letting $\chi(t) = -\zeta(t)$, (31) is re-written as

$$V(\underline{t}_i) \le c_0 - \int_{\zeta(0)}^{\underline{\omega}_i} g_0 N(-\chi(\tau)) e^{-c_1(\underline{t}_i-\tau)} d\chi(\tau) - \int_{\zeta(0)}^{\underline{\omega}_i} e^{-c_1(\underline{t}_i-\tau)} d\chi(\tau) \quad (47)$$

Noting that $N(\cdot)$ is an even function, i.e., $N(\chi)=N(-\chi)$, (47) becomes

$$V(\underline{t}_i) \le c_0 - \int_{\zeta(0)}^{\underline{\omega}_i} g_0 N(\chi(\tau)) e^{-c_1(\underline{t}_i-\tau)} d\chi(\tau) - \int_{\zeta(0)}^{\underline{\omega}_i} e^{-c_1(\underline{t}_i-\tau)} d\chi(\tau)$$

Noting (45), we have

$$\begin{aligned} V(\underline{t}_i) &\le c_0 - N_g(\zeta(0),\underline{\omega}_i) - [\underline{\omega}_i-\zeta(0)] \inf_{\tau\in[0,\underline{t}_i]} e^{-c_1(\underline{t}_i-\tau)} \\ &\le c_0 + g_0\underline{\omega}_i^2[-\sin(\underline{\omega}_i)+1-e^{-c_1\underline{t}_i}] - f_l(\zeta(0),\underline{\omega}_i) \\ &\quad -(\underline{\omega}_i-\zeta(0))e^{-c_1\underline{t}_i} \\ &= \underline{\omega}_i^2\{g_0[-\sin(\underline{\omega}_i)+1-e^{-c_1\underline{t}_i}] + \frac{1}{\underline{\omega}_i^2}[c_0-f_l(\zeta(0),\underline{\omega}_i) \\ &\quad -(\underline{\omega}_i-\zeta(0))e^{-c_1\underline{t}_i}]\} \end{aligned}$$

Taking the limit as $i\to+\infty$, hence $\underline{t}_i\to t_f$, $\underline{\omega}_i\to+\infty$, $\frac{f_l(\zeta(0),\underline{\omega}_i)}{\underline{\omega}_i^2}\to 0$, we have

$$0 \le \lim_{i\to+\infty} V(\underline{t}_i) \le \lim_{i\to+\infty} \underline{\omega}_i^2 g_0[\sin(\underline{\omega}_i)+1-e^{-c_1\underline{t}_i}] \quad (48)$$

which, if $g_0>0$, draws a contradiction when $[-\sin(\underline{\omega}_i)+1-e^{-c_1\underline{t}_i}]<0$, and if $g_0<0$, draws a contradictions when $[-\sin(\underline{\omega}_i)+1-e^{-c_1\underline{t}_i}]>0$. Therefore, $\zeta(t)$ is lower bounded on $[0,t_f)$.

Therefore, $\zeta(t)$ must be bounded on $[0,t_f)$. In addition, $V(t)$ and $\int_0^t g_0N(\zeta)\dot{\zeta}d\tau$ are bounded on $[0,t_f)$. $\diamondsuit$