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Towards Stability Analysis of Jump Linear Systems with State-Dependent and Stochastic Switching

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***Abstract*— This paper analyzes the stability of hierarchical jump linear systems where the supervisor is driven by a Markovian stochastic process and by the values of the supervised jump linear system's states. The stability framework for this class of systems is developed over infinite and finite time horizons. The framework is then used to derive sufficient stability conditions for a specific class of hybrid jump linear systems with performance supervision. New sufficient stochastic stability conditions for discrete-time jump linear systems are also presented.**

I. INTRODUCTION

A general modeling framework for embedded control systems is a hybrid system representation that allows the inclusion of analog and digital subsystems' dynamics together with a model for the interfaces between them and the inherent constraints on their states and independent variables. An appropriate representation with sufficient fidelity is most important in the analysis and design of safety critical systems such as reliable, fault-tolerant control systems that can affect human lives. Other complex real-time systems will also benefit from the higher fidelity representation. A particular advantage of hybrid systems is that they can directly represent hierarchical control systems, including the dynamics of a decision making supervisor present in most embedded control systems (cf. [2], [5], [8]). Figure 1 shows such a hierarchical embedded control system where the low-level closed-loop dynamics are represented by a jump linear system (JLS) that is switched by $(\boldsymbol{\theta}(k), \boldsymbol{N}_l(k))$, namely the decisions $\boldsymbol{\theta}(k)$ of the supervisor and the states of a stochastic process $\boldsymbol{N}_l(k)$. The switching is used to model, for example, the change in closed-loop dynamics for different operating regions and the change in closed-loop dynamics between normal and recovery operation during an upset or critical failure. This model is useful for studying the effects of the high-level supervisor on the low-level control loop dynamics where the supervisor could be implemented, for example, with a deterministic automaton that depends on both the states of the JLS and a stochastic process $\boldsymbol{N}_h(k)$. The state dependency allows the supervisor to command a control law switch or to select among a set of fault-tolerant recovery techniques. In addition, the supervisor can monitor and make decisions based on the performance of the low-level control system. The stochastic process $\boldsymbol{N}_h(k)$ typically models the status of environmental sensors that indicate the presence of harsh environmental exogenous conditions (e.g. high intensity radiated fields or lightning) or even the states of fault detectors of internal events generated by $\boldsymbol{N}_l(k)$. The inclusion of the stochastic processes enhances the fidelity of hybrid system models by taking into account stochastic uncertainty.

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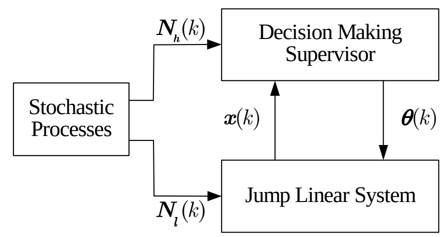

Fig. 1. Block diagram of a simplified hierarchical embedded control system with a jump linear system.

This paper sets up a framework for stability analysis of a general class of discrete-time, hierarchical, embedded control systems. A completely general analysis is complicated since the decisions made by the supervisor are state-dependent. The initial study presented in this paper considers a particular class of systems, where the supervisor only models the change in the jump linear system parameters due to either an external event or degradation of closed-loop performance. Using a worst case analysis, sufficient stability conditions are developed for this system. In addition, two associated concepts of finite-time stability are also introduced and analyzed in this context.

The rest of the paper is organized as follows. Section 2 presents the stability framework for a general class of hybrid jump linear systems, including new sufficient stability conditions for switched systems. Section 3 introduces a particular class of hybrid jump linear systems and presents sufficient stability conditions. This section also analyzes the stability of systems with a finite-time horizon defined by a performance condition being met. The paper's conclusions are given in the final section.

II. STABILITY FRAMEWORK FOR A HYBRID JUMP LINEAR SYSTEM

For qualitative and quantitative analysis, a particular type of decision making supervisor will be considered as shown in Fig. 2. The discrete-time sequence that drives the supervised jump linear system is computed by a finite state machine (FSM). The supervisor is driven by the state of the JLS and a stochastic process $\boldsymbol{N}(k)$ that is assumed to be a homogeneous, finite-state, discrete-time Markov chain, taking on symbols from the set $\Sigma_{I_N} = \{\eta_{N1}, \eta_{N2}, \ldots, \eta_{Nl_N}\}$. The process $\boldsymbol{\nu}(k)$ is a function of the JLS's n-dimensional state vector. It is the output of a memoryless analog amplitude to symbol (A/S) map defined by $\psi : \mathbb{R}^n \to \Sigma_{I_\nu}$, where $\Sigma_{I_\nu} = \{\eta_{\nu 1}, \eta_{\nu 2}, \ldots, \eta_{\nu l_\nu}\}$. The two processes $\boldsymbol{N}(k)$ and $\boldsymbol{\nu}(k)$ drive the FSM. The FSM's states, $\boldsymbol{z}(k)$, take on values in $\Sigma_S = \{e_1, e_2, \ldots, e_{l_s}\}$.

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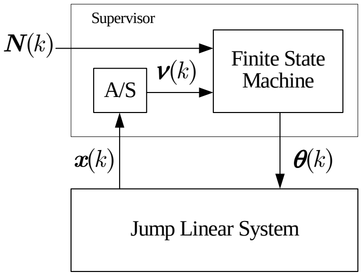

Fig. 2. A hybrid jump linear system.

The evolution of $z(k)$ is given by the next state map $\zeta : \Sigma_{I_N} \times \Sigma_{I_\nu} \times \Sigma_S \rightarrow \Sigma_S$ of the form

$$\begin{aligned} z(k+1) &= \zeta(N(k), \nu(k), z(k)) \\ &= \zeta(N(k), \psi(x(k)), z(k)). \end{aligned} \tag{1}$$

The output of the FSM, $\theta(k)$, is determined by an isomorphism $\varpi : \Sigma_S \rightarrow \Sigma_O$ that assigns an output symbol in $\Sigma_O = \{\xi_1, \xi_2, \ldots, \xi_{l_s}\}$ according to the following relation

$$\theta(k) = \varpi(z(k)). \tag{2}$$

Specifically, the unique isomorphic mapping between the finite state machine's output and states is given by $\xi_j = \varpi(e_j),\ \ j = 1, \ldots, l_s$. This output then drives the JLS with state vector $x(k) \in \mathbb{R}^n$, according to

$$x(k+1) = A_{\theta(k)} x(k). \tag{3}$$

The system in Fig. 2 is a special case of a hybrid system. It will be referred to as a hybrid jump linear system (HJLS). To better describe the jump linear nature of the closed-loop system, let the symbols for the states of the FSM be the elementary vectors $e_j = \begin{bmatrix} 0 \cdots 0\ 1\ 0 \cdots 0 \end{bmatrix}'$ with a 1 in the j-th position. A matrix representation of the next state map ζ in (1) is then

$$z(k+1) = S_{(N(k), \nu(k))} z(k), \tag{4}$$

where each of the $l_n \cdot l_\nu$ matrices $S_{(\eta_N, \eta_\nu)} \in \mathbb{R}^{l_s \times l_s}$, $(\eta_N, \eta_\nu) \in \Sigma_{I_N} \times \Sigma_{I_\nu}$, is a deterministic transition matrix, i.e., a matrix where each column contains exactly a single one and $l_s - 1$ zeros. The state of the hybrid jump linear system is given by $[x'(k), z'(k)]'$, which yields the following jump linear system representation of a HJLS

$$\begin{bmatrix} x(k+1) \\ z(k+1) \end{bmatrix} = \begin{bmatrix} A_{\theta(k)} & 0 \\ 0 & S_{(N(k), \nu(k))} \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix}. \tag{5}$$

The state-dependence is made evident by noting that in (5) $A_{\theta(k)} = A_{\varpi(z(k))}$ (by (2)) and $S_{(N(k), \nu(k))} = S_{(N(k), \psi(x(k)))}$ (by the A/S map definition). So a HJLS is a jump linear system driven by the direct product of three finite-state, stochastic processes, that is, $\varrho(k) = (\theta(k), N(k), \nu(k))$. Note that $\varrho(k)$ is not necessarily a Markovian process. Since $\theta(k)$ and $z(k)$ are related via an isomorphism, with a slight abuse of notation, the state of a HJLS can be written as $q(k) = [x'(k), \theta(k)]'$. Thus, a HJLS can now be succinctly represented by

$$q(k+1) = F_{\varrho(k)} q(k), \tag{6}$$

where $F_{\varrho(k)} = \begin{bmatrix} A_{\theta(k)} & 0 \\ 0 & S_{(N(k), \nu(k))} \end{bmatrix}$. It should be noted that even though (6) has been made to look like a typical JLS, new analysis tools are needed to study its behavior since $\varrho(k)$ depends on $x(k)$ and $q(0)$, and $\varrho(k)$, in general, will *not* be Markovian. The stability analysis of jump linear systems as in (3), however, is well-known when $\theta(k)$ is Markovian (cf. [1], [3], [7]). Necessary and sufficient stability conditions are also known when $\theta(k)$ is the output of a FSM driven by a Markovian input, and the FSM is not dependent on the JLS states [4], [12].

From a control systems point of view it would be simpler if the stability of the HJLS in (6) were a property of the equilibrium point $x_e = 0$ of the JLS in (3), regardless of the values of the FSM output $\theta(k)$. In practice, however, stability is a property of not only the equilibrium point of the JLS but also of a subset of the states of the FSM. So, it is important to study stability of a subset that includes the equilibrium point of the JLS and all or some of the states of the FSM. A formal framework to analyze stability with respect to such a subset is given in the remainder of this section. This framework will also make it easier to analyze a sampled-data version of the JLS in Fig. 2, consisting of a continuous-time plant interfaced via A/D and D/A converters to a discrete-time jump linear controller. The desired stability definitions will be adapted from [6], generalizing some of the standard JLS stability definitions to hybrid stochastic systems. These definitions need to be given in terms of an appropriate metric over $\mathbb{R}^n \times \Sigma_O$. To simplify the stability analysis, let the output symbols of the FSM be $\Sigma_O = \{0, 1, \ldots, l_s - 1\}$.

Consider sample solutions of a HJLS generated by starting at a common initial time $k_0 = 0$ and a fixed initial state $q_0 = [x'(0), \theta(0)]' \in X \triangleq \mathbb{R}^n \times \Sigma_O$, when $N(k)$ has initial distribution $\pi(0)$ and transition probability matrix Π. The only source of randomness in the HJLS is assumed to be $N(k)$, a sequence of random variables $\{N(k) \in \Sigma_{I_N}, k \in \mathbb{Z}^+\}$ defined over the underlying probability space $(\Omega, \mathcal{F}_N, P_N)$. At time k, the events are described by the σ-field of the k random variables $\{N(0), \ldots, N(k-1)\}$. Since $N(k)$ is a discrete-time process, the sequences in the HJLS, $\theta(k)$, $x(k)$, and $\nu(k)$ will also be discrete-time stochastic processes defined over the same sample space and their probability measures can be computed in terms of the probability measure for $N(k)$.

For stability analysis, the stochastic process of interest is the sequence of random variables $\{q(k) \in X, k \in \mathbb{Z}^+\}$. At each time instant k, these random variables map samples $\omega \in \Omega$ into X. A useful metric space for the desired stability analysis is (X, d), where the metric $d : X \rightarrow \mathbb{R}$ is defined by

$$d\left(\begin{bmatrix} x \\ \theta \end{bmatrix}, \begin{bmatrix} \hat{x} \\ \hat{\theta} \end{bmatrix} \right) = \|x - \hat{x}\| + \left|\theta - \hat{\theta}\right| \tag{7}$$

with $x, \hat{x} \in \mathbb{R}^n,\ \theta, \hat{\theta} \in \Sigma_0$, and $\|\cdot\|$ and $|\cdot|$ denoting the Euclidean norm in $\mathbb{R}^n$ and the absolute value, respectively. The distance between $q \in X$ and a set $M \subset X$ is defined in the usual manner as $d(q, M) = \inf\{d(q, m) : m \in M\}$.

Definition 2.1 (Stochastic Motion of a HJLS): Let (X, d) be the metric space with metric d given in (7).

Consider a HJLS in (6) with initial time $k_0 = 0$, a fixed initial state $q_0 = [x'(0), \theta(0)]' \in X$, and $\boldsymbol{N}(k)$ having initial distribution $\pi(0)$ and transition probability matrix Π. Then a stochastic process $\{\boldsymbol{\phi}(k, \omega, q_0, \pi(0)), k \geq k_0 \in \mathbb{Z}^+\}$ is called a **stochastic motion** if for each $\omega \in \Omega$, $\boldsymbol{\phi}(k, \omega, q_0, \pi(0))$ is a sample solution of (6) and $\boldsymbol{\phi}(0, \omega, q_0, \pi(0)) = q_0$ for all $\omega \in \Omega$.

Definition 2.2 (Stochastic Dynamical System): Let $\mathcal{S}$ be the collection of stochastic motions corresponding to the fixed initial states $q_0 \in X$. This collection together with the space $\mathbb{Z}^+$ for time values and the metric space (X, d) is called a **stochastic dynamical system** represented by $\{\mathbb{Z}^+, X, d, \mathcal{S}\}$.

The following definitions and lemma provide the core concepts for analyzing stability of hybrid jump linear systems. In the sequel, $\mathcal{S}$ denotes a set of stochastic motions of a HJLS.

Definition 2.3 (Invariant Set of a HJLS): A set $M \subset X$ is said to be **invariant with respect to $\mathcal{S}$**, or simply **$(\mathcal{S}, M)$ is invariant**, if $q \in M$ implies that $P\{d(\boldsymbol{\phi}(k, \omega, q, \pi(0)), M) = 0, \forall k \in \mathbb{Z}^+\}$=1.

Definition 2.4 (Equilibrium Point of a HJLS): A point $q_e \in X$ is called an **equilibrium point** of $\mathcal{S}$ if $(\mathcal{S}, \{q_e\})$ is invariant.

Definition 2.5 (Lyapunov Mean Square Stability): Let $\{\mathbb{Z}^+, X, d, \mathcal{S}\}$ be a stochastic dynamical system defined for a HJLS. A set $M \subset X$ is said to be **Lyapunov mean square stable** when $k_0 = 0$ if for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon, k_0) > 0$ such that

$$E\{d^2(\boldsymbol{\phi}(k, \omega, q_0, \pi(0)), M)\} < \epsilon, \ \ \forall k \in \mathbb{Z}^+ \tag{8}$$

for any stochastic motion in $\mathcal{S}$ whenever $d(q_0, M) < \delta$.
The following lemma shows that stability of a particular set M such that (S, M) is invariant leads to conditions only on the equilibrium point x_e of the JLS.

Lemma 2.1: Let $\{\mathbb{Z}^+, X, d, \mathcal{S}\}$ be a stochastic dynamical system defined for a HJLS. The set $M \triangleq \{[x_e', \xi]' \subset X | x_e = 0 \in \mathbb{R}^n, \xi \in \Sigma_O\}$ is Lyapunov mean square stable when $k_0 = 0$ if for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon, k_0) > 0$ such that

$$E\{\|x(k)\|^2\} = E\{\|\boldsymbol{\phi}(k, \omega, q_0, \pi(0))\|^2\} < \epsilon \tag{9}$$

for any stochastic motion whenever $\|x(0)\| < \delta$.

By Lemma 2.1 Lyapunov mean square stability of $M = \left\{\left[\begin{smallmatrix} 0 \\ \xi \end{smallmatrix}\right] \subset X | \ \xi \in \Sigma_O\right\}$, an invariant set of the HJLS, is equivalent to the Lyapunov mean square stability of $x_e = 0$, an equilibrium of the JLS. So, without loss of generality, consider the following stability definitions written in terms of $x_e = 0$ only.

Definition 2.6 (Second Moment Stability): Let $\{\mathbb{Z}^+, X, d, \mathcal{S}\}$ be a stochastic dynamical system defined for a HJLS. The equilibrium $x_e = 0$ of the JLS or simply the HJLS is said to be

- **stochastically stable** if
$$\sum_{k=0}^{\infty} E\{\|\boldsymbol{x}(k)\|^2\} < \infty, \tag{10}$$
- **mean square stable** if
$$E\{\|\boldsymbol{x}(k)\|^2\} \to 0 \text{ as } k \to \infty, \tag{11}$$
- **exponentially mean square stable** if there exist constants $0 < \lambda < 1$ and $\mu > 0$ such that for all $k \geq 0$ and initial states $x_0 \in \mathbb{R}^n$
$$E\{\|\boldsymbol{x}(k)\|^2\} \leq \mu \lambda^k \|x_0\|^2, \tag{12}$$
- **second moment stable** if the HJLS is simultaneously stochastically, exponentially mean square, and mean square stable and
- **second moment unstable** if the HJLS is simultaneously stochastically, exponentially mean square, and mean square unstable.

The relations between these definitions are well-known. For example, it is easy to see that stochastic and exponential mean square stability imply mean square stability. Hence, mean square instability implies that the HJLS is second moment unstable. Also, notice that exponential stability implies stochastic stability since $\sum_{k=0}^{\infty} E\{\|\boldsymbol{x}(k)\|^2\} \leq \sum_{k=0}^{\infty} \mu \lambda^k \|x_0\|^2 < \infty$. Hence, exponential mean square stability implies that the equilibrium $x_e = 0$ is second moment stable. Finally, notice that when a HJLS represents a Markovian jump linear system, it is stochastically, exponentially or mean square stable if and only if it is second moment stable [7].

The framework for stability analysis of a HJLS is now formally established and it has been shown that a HJLS is essentially a JLS as given in (6). A sufficient test for second moment stability and instability is given next in terms of the maximum and minimum singular values of A_ξ, $\xi \in \Sigma_O$.

Theorem 2.1: Let $\{\mathbb{Z}^+, X, d, \mathcal{S}\}$ be a stochastic dynamical system defined for a HJLS. If $\bar{\sigma}(A_\xi) < 1, \forall \xi \in \Sigma_O$ then the HJLS is second moment stable. If $\underline{\sigma}(A_\xi) > 1, \forall \xi \in \Sigma_O$ then the HJLS is second moment unstable.

Proof : Consider the sample solutions of the JLS in (3) that are embedded in $\mathcal{S}$. Note that for every fixed initial state $x_0 \in \mathbb{R}^n$, the sample solutions at time k yield $\|\boldsymbol{x}(k)\| = \|A_{\boldsymbol{\theta}(k-1)} A_{\boldsymbol{\theta}(k-2)} \ldots A_{\theta(0)} x_0\|$. This expression can be bounded using singular values as follows:

$$\prod_{i=0}^{k-1} \underline{\sigma}(A_{\boldsymbol{\theta}(i)}) \|x_0\| \leq \|\boldsymbol{x}(k)\| \leq \prod_{i=0}^{k-1} \bar{\sigma}(A_{\boldsymbol{\theta}(i)}) \|x_0\| \tag{13}$$
$$\min_{\xi \in \Sigma_O} \{\underline{\sigma}(A_\xi)\}^k \|x_0\| \leq \|\boldsymbol{x}(k)\| \leq \max_{\xi \in \Sigma_O} \{\bar{\sigma}(A_\xi)\}^k \|x_0\|.$$

Suppose that $\bar{\sigma}(A_\xi) < 1$ for all $\xi \in \Sigma_O$ and let $\mu = 1$ and $\lambda = \max_{\xi \in \Sigma_O} \{\bar{\sigma}(A_\xi)\}^2 < 1$ in (12). Thus, $\|\boldsymbol{x}(k)\|^2 \leq \mu \lambda^k \|x_0\|^2$ and $E\{\|\boldsymbol{x}(k)\|^2\} \leq \mu \lambda^k \|x_0\|^2$, showing that the HJLS is exponentially mean square stable; hence, it is second moment stable.
Now suppose that $\underline{\sigma}(A_\xi) > 1$ for all $\xi \in \Sigma_O$. Using the same reasoning as in part (a), it follows that $\min_{\xi \in \Sigma_O} \{\underline{\sigma}(A_\xi)\}^{2k} \|x_0\|^2 \leq E\{\|\boldsymbol{x}(k)\|^2\}$. Since $\min_{\xi \in \Sigma_O} \{\underline{\sigma}(A_\xi)\}^{2k} \|x_0\|^2$ is a strictly increasing sequence, $E\{\|\boldsymbol{x}(k)\|^2\} \to \infty$ as $k \to \infty$ and the result follows. ∎

This theorem is a discrete-time generalization of Theorem 1 for switched systems in [11]. The stability of a particular type of HJLS is analyzed in the next section.

III. Stability Analysis of a HJLS with Performance Supervision

In this section a specific class of hybrid jump linear systems which makes decisions based on the performance of the low-level jump linear system is defined, and sufficient stability conditions are developed. This class, represented in Fig. 2, consists of a particular discrete-time Markov chain $\boldsymbol{N}(k)$, an A/S converter, and a finite state machine. The performance of interest is a bound on the norm of the state vector $x(k)$ for a sample path. A motivation for this performance bound is, for example, a safety critical system where the saturation of certain signals or a critical event is detected by a threshold test on the state's norm. If the events are such that the low-level jump linear system is known to become unstable, the role of the supervisor is to command a switch to a fail-safe operation, if possible. The focus here, however, is on the case where the supervisor does not command such a fail-safe operation. The effect of the state-dependent input when the performance boundary is reached is simply to reflect the change of models for the low-level jump linear system. Under these conditions, it is not expected that the HJLS will perform satisfactorily once the performance boundary is reached. This is formally shown in Theorem 3.2, where this specific HJLS class is shown to be second moment unstable. Despite the limited application of this HJLS, this is a useful initial problem to analyze with the stability framework developed in Section 2. Since stability over an infinite-time horizon is not expected, it is useful to consider the concept of stability over a finite-time horizon as first proposed in [9] and used in, for example, [10], [13]–[15]. Two such definitions and a sufficient stability condition are given. All the sufficient stability and instability conditions are given when the $A_\xi, \xi \in \Sigma_O$ satisfy specific singular value conditions.

The stochastic process $\boldsymbol{N}(k)$ is assumed to be a two-state, time-homogeneous, Markov chain with initial distribution $\pi(0) = [\pi_0, \pi_1]$ and transition probability matrix $\Pi = [\pi_{ij}],\ i,j \in \{0,1\}$. For simplicity, let $\boldsymbol{N}(k)$ be aperiodic with no absorbing states, that is, let $\pi_{ii} \in (0,1),\ i,j \in \{0,1\}$. This stochastic process could model the output of a digital upset detector, where the upsets may or may not manifest themselves as errors in the dynamical system evolution. The analog to symbol converter is used to detect whether a function of the sample solutions have reached a performance boundary given by a scalar $\alpha \in \mathbb{R}$. This is implemented by a composition of a piece-wise linear function and the Euclidean norm of the JLS states defined by the mapping $\psi : \mathbb{R}^n \to \Sigma_{I_\nu}$ where $\Sigma_{I_\nu} = \{0,1\}$. The output of the mapping is given by

$$\boldsymbol{\nu}(k) = \psi(\boldsymbol{x}(k)) = \begin{cases} 1 : \ \|x(k)\| \geq \alpha > 0 \\ 0 : \ \|x(k)\| < \alpha\ . \end{cases} \tag{14}$$

The role of the detector is to output a 1 when the performance boundary is exceeded. If $\boldsymbol{N}(k) = 1$ or if $\boldsymbol{\nu}(k) = 1$ always results in the same failure mode of operation, the change can be represented by the FSM in Fig. 3. The arcs in the transition diagram are labeled with the values of the direct product of the input processes $(\boldsymbol{\nu}(k), \boldsymbol{N}(k))$ or simply by $\eta_\nu \eta_N$ where $\eta_\nu \in \Sigma_{I_\nu}$ and $\eta_N \in \Sigma_{I_N}$.

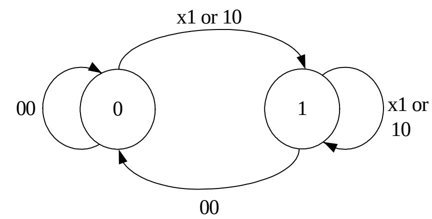

Fig. 3. Transition diagram for a finite state machine representation of a logical OR operation.

(An x in the transition diagram represents a don't care condition.) Let the outputs of the FSM be equal to its states, so that the nodes in Fig. 3 correspond to the outputs $\boldsymbol{\theta}(k) \in \Sigma_O = \{0,1\}$. The dynamical equation for the FSM outputs can thus be written as follows

$$\begin{aligned} \boldsymbol{\theta}(k+1) &= \begin{cases} 1 & : \ \boldsymbol{\nu}(k) = 1 \\ \boldsymbol{N}(k) & : \ \boldsymbol{\nu}(k) = 0 \end{cases} \\ &= \boldsymbol{N}(k) + \boldsymbol{\nu}(k)(1 - \boldsymbol{N}(k)). \end{aligned} \tag{15}$$

The proofs in this section rely on the following concept.

Definition 3.1: An **auxiliary scalar Markovian jump linear system** (MJLS) for the HJLS in Figure 2 is any scalar jump linear system driven by the same stochastic process $\boldsymbol{N}(k)$ and represented by

$$\tilde{\boldsymbol{x}}(k+1) = \tilde{a}_{\boldsymbol{N}(k)} \tilde{\boldsymbol{x}}(k), \tag{16}$$

where $\tilde{x}(0) = \tilde{x}_0$ and $\tilde{a}_0, \tilde{a}_1 \in \mathbb{R}$.

Notice that the sample solutions of (16) are of the form

$$\tilde{\boldsymbol{x}}(k) = (\tilde{a}_1/\tilde{a}_0)^{\boldsymbol{n}(k)} (\tilde{a}_0)^k \tilde{x}_0 \tag{17}$$

where $\boldsymbol{n}(k) \triangleq \sum_{i=0}^{k-1} \boldsymbol{N}(i),\ k \geq 1$ is the stochastic process that counts the number of ones in the sequence $\{\boldsymbol{N}(0), \ldots, \boldsymbol{N}(k-1)\}$. This auxiliary system will be used to prove the following results.

Theorem 3.1: Let $\{\mathbb{Z}^+, X, d, \mathcal{S}\}$ be a stochastic dynamical system defined for a HJLS in (6) with initial state $q_0 = [x_0', 0]'$, $x_0 \in \mathbb{R}^n$, where $\|x_0\| < \alpha$.

(a) If $\underline{\sigma}(A_1)/\underline{\sigma}(A_0) > 1$ then the HJLS is second moment unstable whenever its auxiliary scalar MJLS in (16) with $\tilde{a}_0 = \underline{\sigma}(A_0)$, $\tilde{a}_1 = \underline{\sigma}(A_1)$ and $\tilde{x}_0 = \underline{\sigma}(A_0)\|x_0\|$ is second moment unstable.

(b) If $\bar{\sigma}(A_0)/\bar{\sigma}(A_1) > 1$ then the HJLS is second moment stable whenever its auxiliary scalar MJLS in (16) with $\tilde{a}_0 = \bar{\sigma}(A_0)$, $\tilde{a}_1 = \bar{\sigma}(A_1)$ and $\tilde{x}_0 = \bar{\sigma}(A_0)\|x_0\|$ is second moment stable.

Proof: Let $\boldsymbol{m}(k) \triangleq \sum_{i=1}^{k-1} \boldsymbol{\theta}(i)$. Since $\boldsymbol{\theta}(0) = 0$, (13) reduces to

$$\|\boldsymbol{x}(k)\| \geq (\underline{\sigma}(A_1)/\underline{\sigma}(A_0))^{\boldsymbol{m}(k)} \underline{\sigma}(A_0)^k \|x_0\| \tag{18}$$

$$\|\boldsymbol{x}(k)\| \leq (\bar{\sigma}(A_1)/\bar{\sigma}(A_0))^{\boldsymbol{m}(k)} \bar{\sigma}(A_0)^k \|x_0\|. \tag{19}$$

To prove (a), assume $\underline{\sigma}(A_1)/\underline{\sigma}(A_0) > 1$ and let $\tilde{a}_0 = \underline{\sigma}(A_0)$, $\tilde{a}_1 = \underline{\sigma}(A_1)$, and $\tilde{x}_0 = \underline{\sigma}(A_0)\|x_0\|$ in (17). Thus,

$$\tilde{\boldsymbol{x}}(k-1) = (\underline{\sigma}(A_1)/\underline{\sigma}(A_0))^{\boldsymbol{n}(k-1)} \underline{\sigma}(A_0)^k \|x_0\|.$$

But $\boldsymbol{m}(k) \geq \boldsymbol{n}(k-1)$ for $k \geq 1$, since $\boldsymbol{\theta}(k+1) \geq \boldsymbol{N}(k)$ by (15). Thus, it follows that

$$\|\boldsymbol{x}(k)\| \geq \tilde{\boldsymbol{x}}(k-1), \tag{20}$$

which implies that $E\{\|\boldsymbol{x}(k)\|^2\} \geq E\{\tilde{\boldsymbol{x}}^2(k-1)\}$. Now, if (16) is mean square unstable then $E\{\tilde{\boldsymbol{x}}^2(k-1)\} \to \infty$ as $k \to \infty$. Therefore $E\{\|\boldsymbol{x}(k)\|^2\} \to \infty$ as $k \to \infty$ and the HJLS is mean square unstable, making it also second moment unstable.

A similar argument proves (b). Assume $\bar{\sigma}(A_0)/\bar{\sigma}(A_1) > 1$ and let $\tilde{a}_0 = \bar{\sigma}(A_0)$, $\tilde{a}_1 = \bar{\sigma}(A_1)$ and $\tilde{x}_0 = \bar{\sigma}(A_0)\|x_0\|$ in (17). Then (19) reduces to

$$\|\boldsymbol{x}(k)\| \leq \tilde{\boldsymbol{x}}(k-1),$$

which implies that $E\{\|\boldsymbol{x}(k)\|^2\} \leq E\{\tilde{\boldsymbol{x}}^2(k-1)\}$. Now observe that if the MJLS in (16) is second moment stable then $E\{\tilde{\boldsymbol{x}}^2(k-1)\} \leq \mu\lambda^{k-1}\|x_0\|^2$ for some $\mu > 0$ and $0 < \lambda < 1$. So it follows that $E\{\|\boldsymbol{x}(k)\|^2\} \leq (\mu/\lambda)\lambda^k\|x_0\|^2$. Thus, the HJLS is exponentially mean square stable and consequently it is also second moment stable. ∎

The following lemmas are used to prove the next main result. The ceil operator $\lceil y \rceil$ represents the smallest integer greater than or equal to y for $y \in \mathbb{R}$.

Lemma 3.1: Consider the auxiliary scalar Markovian jump linear system in (16) with $0 < \tilde{x}_0 < \alpha$, $\tilde{a}_0 < 1$, $\tilde{a}_1 > 1$ and $\tilde{a}_0\tilde{a}_1 > 1$. Then the first sample time for which $\tilde{\boldsymbol{x}}(k)$ may reach or exceed α is $k = k^*$, where

$$k^* = \left\lceil \frac{\log(\alpha/\tilde{x}_0)}{\log(\tilde{a}_1)} \right\rceil \geq 1. \tag{21}$$

The probability that the worst case sample solutions, that is, those for which $\boldsymbol{N}(k)$ is 1 for $k > 0$, reach α at $k = k^*$ is $P\{|\tilde{\boldsymbol{x}}(k^*)| \geq \alpha\} = \pi_1(\pi_{11})^{k^*-1} > 0$. Furthermore, for every $k < k^*$, $P\{|\tilde{\boldsymbol{x}}(k)| \geq \alpha\} = 0$.

Proof: First, assume without loss of generality that $\pi_1 > 0$. If $\pi_1 = 0$ then analyze the system as if it were starting at $k = 1$ with initial state $\tilde{a}_0 x_0 < \alpha$ and $\pi_1 = \pi_{01} > 0$, which is not zero by assumption. Since $\tilde{x}_0 < \alpha$, $\tilde{\boldsymbol{x}}(k)$ can reach α only from below. Since the worst case sample solution is of the form

$$\max_{\omega\in\Omega}\{\tilde{\boldsymbol{x}}(k)\} = (\tilde{a}_1)^k\tilde{x}_0, \tag{22}$$

the first time at which $\tilde{\boldsymbol{x}}(k)$ could reach or exceed α is $k = k^*$. Setting (22) equal to α and solving for $k = k^*$ gives equation (21). Now, observe from (17) that $|\boldsymbol{x}(k^*)| \geq \alpha$ only if $\boldsymbol{n}(k^*) = k^*$. Then, it follows that $P\{|\boldsymbol{x}(k^*)| \geq \alpha\} = P\{\boldsymbol{N}(i) = 1,\ \forall i = 0, 1, \ldots, k^*-1\} = \pi_1(\pi_{11})^{k^*-1} > 0$. Clearly, for every $k < k^*$, $|\tilde{\boldsymbol{x}}(k)| < \alpha$ and $P\{|\tilde{\boldsymbol{x}}(k)| \geq \alpha\} = 0$. ∎

The long term behavior of sample solutions for the multidimensional generalization of the conditions in Lemma 3.1 is considered next.

Lemma 3.2: Let $\{\mathbb{Z}^+, X, d, \mathcal{S}\}$ be a stochastic dynamical system defined for a HJLS in (6) with $\bar{\sigma}(A_0) < 1$, $\underline{\sigma}(A_1) > 1$ and $\underline{\sigma}(A_0)\underline{\sigma}(A_1) > 1$. If $\|\boldsymbol{x}(k')\| \geq \alpha$ for some $k' \geq 0$ then there exists a finite $k'' \geq k'$ such that $\|\boldsymbol{x}(k+1)\| > \|\boldsymbol{x}(k)\| > \alpha$ for all $k \geq k''$.

Proof : Note that $\boldsymbol{\theta}(k'+1) = 1$, since by assumption $\|\boldsymbol{x}(k')\| \geq \alpha$. Thus, it follows from (3) that

$$\|\boldsymbol{x}(k'+2)\| = \|A_1 A_{\boldsymbol{\theta}(k')}\boldsymbol{x}(k')\| \geq \underline{\sigma}(A_1 A_{\boldsymbol{\theta}(k')})\|\boldsymbol{x}(k')\|.$$

But $\underline{\sigma}(A_1 A_{\boldsymbol{\theta}(k')}) \geq \underline{\sigma}(A_1)\underline{\sigma}(A_{\boldsymbol{\theta}(k')})$, $\underline{\sigma}(A_{\boldsymbol{\theta}(k')}) \geq \underline{\sigma}(A_0)$, and $\underline{\sigma}(A_0)\underline{\sigma}(A_1) > 1$ (by assumption), so

$$\|\boldsymbol{x}(k'+2)\| \geq \underline{\sigma}(A_1)\underline{\sigma}(A_0)\|\boldsymbol{x}(k')\| > \|\boldsymbol{x}(k')\|. \tag{23}$$

In the same way, it can be shown that $\|\boldsymbol{x}(k'+2m)\| > \|\boldsymbol{x}(k'+2m-2)\|$ for all $m \geq 1$. Even though initially nothing can be said about $\|\boldsymbol{x}(k'+m)\|$ when m is odd, notice that $\|\boldsymbol{x}(k'+2m)\|$ is a strictly increasing subsequence. Thus, there exists $m' \geq 1$ such that $\|\boldsymbol{x}(k'+2m')\| > \alpha/\underline{\sigma}(A_0) > \alpha$ and $\boldsymbol{\theta}(k'+2m'+1) = 1$. This in turn implies that

$$\begin{aligned}
\|\boldsymbol{x}(k'+2m'+1)\| &= \|A_{\boldsymbol{\theta}(k'+2m')}\boldsymbol{x}(k'+2m')\| \\
&\geq \underline{\sigma}(A_{\boldsymbol{\theta}(k'+2m')})\|\boldsymbol{x}(k'+2m')\| \\
&\geq \underline{\sigma}(A_0)\|\boldsymbol{x}(k'+2m')\| > \alpha.
\end{aligned}$$

Hence, for $k \geq k'' = k'+2m'+1$, $\boldsymbol{\theta}(k) = 1$ so $\|\boldsymbol{x}(k+1)\| > \|\boldsymbol{x}(k)\| > \alpha$ and the result follows. ∎

Theorem 3.2: Let $\{\mathbb{Z}^+, X, d, \mathcal{S}\}$ be a stochastic dynamical system defined for a HJLS in (6) with initial state $q_0 = [x_0', 0]', x_0 \in \mathbb{R}^n$, where $\|x_0\| < \alpha$ and with $\bar{\sigma}(A_0) < 1$, $\underline{\sigma}(A_1) > 1$ and $\underline{\sigma}(A_0)\underline{\sigma}(A_1) > 1$. Then the HJLS is second moment unstable.

Proof: Let $\tilde{a}_0 = \underline{\sigma}(A_0)$, $\tilde{a}_1 = \underline{\sigma}(A_1)$ and $\tilde{x}_0 = \underline{\sigma}(A_0)\|x_0\|$ in (16) and observe that $\underline{\sigma}(A_1)/\underline{\sigma}(A_0) > 1$. This scalar MJLS satisfies the conditions of Lemma 3.1, so there are sample solutions for which at time $k'-1 = k^*$, $|\tilde{\boldsymbol{x}}(k'-1)| \geq \alpha$, where k^* is the first time where sample solutions of the scalar MJLS reach or exceed α. Now, by (20) every sample solution that makes $|\tilde{\boldsymbol{x}}(k'-1)| \geq \alpha$ also makes $\|\boldsymbol{x}(k'+1)\| \geq \alpha$, which in turn implies, by Lemma 3.2, that those sample solutions make $\|\boldsymbol{x}(k)\| \geq \alpha$ for every $k \geq k''$ and a constant $k'' \geq k'$. Hence, the Markov inequality implies that for all $k \geq k'' > k^*$,

$$E\{\|\boldsymbol{x}(k)\|^2\}\frac{1}{\alpha^2} \geq P\{\|\boldsymbol{x}(k)\| \geq \alpha\} \geq P\{\tilde{\boldsymbol{x}}(k^*) \geq \alpha\} > 0,$$

which in turn implies that $\lim_{k\to\infty} E\{\|\boldsymbol{x}(k)\|^2\} > 0$ and $\sum_0^\infty E\{\|\boldsymbol{x}(k)\|^2\} = \infty$. Therefore, the HJLS is mean square and second moment unstable. ∎

Clearly, over an infinite time horizon a HJLS with the conditions in Theorem 3.2 is mean square unstable since there are sample solutions that are ultimately unbounded. But if the HJLS were to operate only over finite time horizons, then its sample solutions will be bounded and may not exceed the performance boundary. The following definition from [10] formalizes the concept of finite-time stability.

Definition 3.2: Let $I = \{k_0, k_0+1, \ldots, k_0+T\}$ for some nonnegative integers k_0 and T. Then, the stochastic dynamical system $\{I, X, d, \mathcal{S}\}$ defined for a HJLS in (6) is

(a) **finite-time stable with respect to $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{k_0}, \boldsymbol{T})$** when $0 < \alpha \leq \beta$, if $\|x(k_0)\| < \alpha$ implies that $\|\boldsymbol{x}(k)\| \leq \beta$ for $k \in I$, and it is
(b) **mean square finite-time stable with respect to $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{k_0}, \boldsymbol{T})$** when $0 < \alpha \leq \beta$, if $\|x(k_0)\| < \alpha$ implies that $E\{\|\boldsymbol{x}(k)\|^2\} \leq \beta^2$ for $k \in I$.

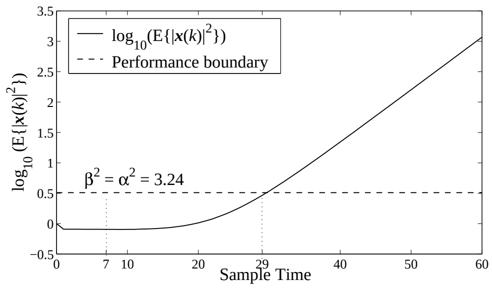

Fig. 4. Plots of $\log_{10} E\{|\boldsymbol{x}(k)|^2\}$ vs. time and the performance boundary as computed by Monte Carlo simulation.

In this definition β represents the performance boundary for all the sample solutions of a HJLS. The next result follows directly from Lemma 3.1.

Lemma 3.3: Let $\{I, X, d, \mathcal{S}\}$ be a stochastic dynamical system defined for a HJLS in (6) with initial state $q_0 = [x_0', 0]', x_0 \in \mathrm{I\!R}^n$, where $\|x_0\| < \alpha$ and with $\bar{\sigma}(A_0) < 1$, $\underline{\sigma}(A_1) > 1$ and $\underline{\sigma}(A_0)\underline{\sigma}(A_1) > 1$. The system is finite-time and mean square finite-time stable with respect to $(\alpha, \beta, 0, T^* - 1)$, where

$$T^* = \left\lceil 1 + \frac{\log(\beta/\bar{\sigma}(A_0)\|x_0\|)}{\log(\bar{\sigma}(A_1))} \right\rceil.$$

Proof : First, observe that the maximum value that the right hand side of (19) can take at time k is $\bar{\sigma}(A_1)^{k-1}\bar{\sigma}(A_0)\|x_0\|$. Since $\bar{\sigma}(A_0)\|x_0\| < \alpha$, the smallest time at which the right hand side (19) can reach or exceed β is $k = T^*$. Hence, $\max_{\omega \in \Omega}\{\|\boldsymbol{x}(k)\|\} < \beta$ for all $k \in I = \{0, 1, \ldots, T^* - 1\}$. Also, notice that $E\{\|\boldsymbol{x}(k)\|^2\} \leq \max_{\omega\in\Omega}\{\|\boldsymbol{x}(k)\|^2\} < \beta^2$ for all $k \in I$. These two facts directly prove that the HJLS is finite-time and mean square finite-time stable with respect to $(\alpha, \beta, 0, T^* - 1)$. ■

Notice that if $\beta = \alpha$, Lemma 3.3 shows that the performance boundary will not be reached for at least $T^* + 1$ samples. Until this time, the output of the analog to signal converter is zero and the finite state machine evolution is simply described by $\boldsymbol{\theta}(k) = \boldsymbol{N}(k-1)$ for $k \leq T^*$. In addition, if the JLS has only one state, by Lemma 3.1, $T^* + 1$ represents the maximum time over which all sample solutions behave as a Markovian jump linear system. Lemma 3.3 also provides a method to estimate the system's worst case sample solution given a desired mission time. An example is used to illustrate these concepts.

Example

The theory was exercised on a simple HJLS with parameters $A_0 = 0.9$, $A_1 = 1.05$, $x_0 = 1$, $\alpha = 1.8$, $\theta_0 = 0$, $\pi(0) = \begin{bmatrix} 0.54 & 0.46 \end{bmatrix}$, and $\Pi = [\pi'(0), \pi'(0)]'$. Ten million Monte Carlo runs of 100 samples each were performed. An estimate of $E\{|\boldsymbol{x}(k)|^2\}$ from sample averages of $|\boldsymbol{x}(k)|^2$ is shown in Figure 4, which shows that the HJLS is mean square unstable. This agrees with the results from Theorem 3.1. The mean square finite-time stability of this example, with respect to $(\alpha, \alpha, 0, T^* - 1)$, was also tested. By Lemma 3.3 $T^* = 8$, that is, the mean square response should be below the performance boundary $E\{|\boldsymbol{x}(k)|^2\} < \beta^2$ until at least $T^* - 1 = 7$. Fig. 4 shows that this result is conservative since the performance boundary is not reached until $k = 29$.

IV. Conclusions

An initial study of hierarchical jump linear systems where the switching is driven by feedback of the low-level dynamical system states and a Markovian process was presented. It was shown that the overall system has a switched system representation called a hybrid jump linear system. The mathematical framework to analyze stability of these systems over infinite and finite time horizons was presented and sufficient conditions for stability and instability were presented for hybrid jump linear systems that monitor the control performance. Future research to reduce the conservativeness of the sufficient conditions and to develop necessary and sufficient stability conditions is ongoing.

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