Finding Surface Lyapunov Functions through Sum-of-Squares Programming

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***Abstract*— This paper presents a Sum-of-Squares method to construct polynomial Surface Lyapunov Functions (SuLF) of arbitrary order for the Impact maps of limit cycles in Piecewise Linear Systems (PLS). This work extends previous results on stability analysis of such limit cycles, which utilized quadratic SuLFs.**

I. Background and Motivation

Piecewise Linear Systems are a special class of hybrid systems. The series of results in [1], [2] demonstrated a novel constructive approach for stability analysis of equilibria and limit-cycles of PLS. The method focuses on the behvaior of the dynamical system along switching surfaces. The analytical tool for this study is the Impact Map (see [2]), which is a Poincaré-like mapping from one switching surface to another. It was proved in [2] that a sufficient condition for global stability is that these maps be contractive.

A constructive method of determining contractiveness Impact maps is that of *Surface Lyapunov Functions* (SuLF), i.e. a Lyapunov function for the Impact Map defined along the switching surface. *A priori*, it does not seem that global analysis with SuLFs and Impact maps would be any easier than standard global analysis with Lyapunov functions (as in [3] or [4]). The work in [2] shows that Impact Maps can be parametrized (by switching times) as members of a family of linear transformations. This "reduces" the global nonlinear stability problem to a continuum of linear stability problems.

The analysis in [2] addressed this problem using Linear Matrix Inequalities to construct quadratic SuLFs. However, the existence of a quadratic SuLF is merely sufficient, so no stability conclusion could be drawn in the case where no quadratic SuLF could be found. In this paper, we use Sum-of-Squares programming to find polynomial SuLFs of arbitrary order.

II. High-Order Polynomial SuLFs using Sum-of-Squares Programming

Consider a Piecewise Linear time-invariant dynamical system with an n-dimensional state-space. We are interested in finding a polynomial SuLF, say V, of arbitrary but fixed order m, on a particular switching surface S, centered at a fixed point s of the Impact Map[1]. That is, we are looking for the coefficients α_i of each term in a generic m order polynomial in $n-1$ variables (switching surfaces are $n-1$-dimensional manifolds in the state space). Thus, in the language of mathematical programming, the α_i's are the decision variables.

In order for the polynomial in question to be a legitimate SuLF, it must satisfy a particular contractiveness condition. As in previous work, we accomplish this using the parametrization of the Impact map. We will require that the SuLF decrease on successive iterates of the map, which can be formulated as non-negativity of a polynomial constraint equation.

Let $H_t, t \in [t_{min}, t_{max}]$ be the aforementioned parametrization of the Impact map (we forgo discussion for brevity, but see [2] for details). Also, let P_{sos} be an arbitrary positive definite sum-of-squares polynomial (e.g. $\epsilon \sum x_i^2, \epsilon > 0$). Then, the decrease condition can be written as:

$$V(x) - V(H_t(x)) - P_{sos} \geq 0 \ \ \forall t \in [t_{min}, t_{max}].$$

Note that the addition of the positive definite polynomial serves to allow the use of an inequality constraint as a strict decrease condition. Also note that in the quadratic SuLF case, where $V(x) = x^T Q x, Q > 0$, the above condition becomes:

$$Q - H_t^T Q H_t > 0.$$

which is precisely the LMI formulation pursued in [2]

[1] We restrict our attention to one Impact Map for clarity, but the results hold for multiple maps.

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With the notation provided above, and some finite number of sample points $t_j, j = 1, 2, \ldots, N$ in the interval $[t_{min}, t_{max}]$:

Find a polynomial V of order n such that:

$$\begin{aligned} V &\geq 0 \\ V(x) - V(H_{t_j}(x)) &\geq 0 \quad \forall x \in S, j = 1, 2, \ldots, N. \end{aligned}$$

As a measure of the "validity" as a SuLF of the polynomial V found as above, we consider some refinement of the previous mesh, say $t_k, k = 1, 2, \ldots, M$, and compute the following quantity:

$$\begin{aligned} &\max(\gamma_k) \\ s.t. \quad &V(x) - V(H_{t_k}(x)) - \gamma P_{sos} \geq 0 \end{aligned}$$

This quantity is computed using SOSP methods, and is computationally inexpensive to produce. This allows one to evaluate the validity of the function V at a large number of points in the interval of interest[2].

III. An Example

In this section we present a stability study of limit cycle of a third-order Relay Feedback System. Such systems have been studied for several decades (see, for example, [6]). The system below does not admit a global quadratic Lyapunov function or SuLF. Using the above SOS formulation, we readily obtain a fourth-order SuLF.

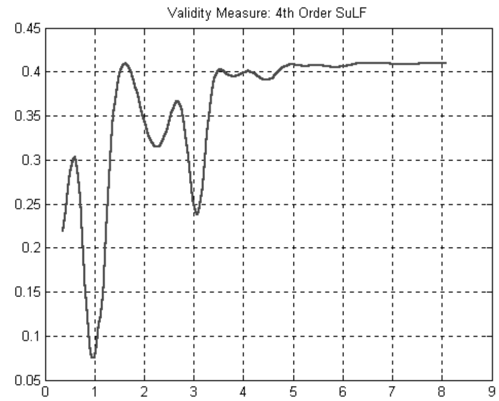
The linear plant of the RFS is as follows:

$$G = \frac{1}{s^3 + 1.1s^2 + 20.1s + 2} \tag{1}$$

This system is arranged in feedback with a simple relay, with hysteresis parameter $d = 0.05$. This system exhibits a locally stable limit cycle, which we would like to prove to be globally stable. As was shown in [1], the stability of the limit cycle can be determined by studying a single Impact map.

We begin by seeking a quadratic SuLF. This turns out to be infeasible (i.e. we obtain a proof through the SOS procedure that no such SuLF exists). Turning to our SOS formulation, we seek a fourth-order SuLF. The second plot above illustrates the behavior of the validity measure γ for the resulting SuLF. We see that our fourth-order polynomial is indeed a SuLF.

[2]Rigorous lower bounds on the mesh resolution can be obtained, but are not included here.



IV. Summary and Conclusions

In this paper we have provided a simple extension of previous work on construction of Surface Lyapunov Functions for global stability analysis of Piecewise Linear Systems. With the development of this new methodology for constructing high-order polynomial SuLFs, we can analyze the global stability properties of systems which do not admit quadratic SuLFs.

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