Lyapunov-like Stability of Switched Stochastic Systems

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***Abstract*— In this paper, three theorems regarding stability of switched stochastic systems are stated and proved. Lyapunov techniques are used to derive sufficient conditions for stability in probability of the overall system and we distinguish between the cases of a common Lyapunov function and multiple Lyapunov functions. An application to distributed Air Traffic Management is discussed as a future goal.**

I. INTRODUCTION

In the past few years the introduction of randomness in the hybrid system formalism has led to the concept of stochastic hybrid and switched systems. Many models of stochastic hybrid systems have been proposed (see [13] for an overview). The most important difference between these models lies in where to introduce the randomness. A first choice is to add a stochastic component to the deterministic continuous control law that governs each discrete state so that the dynamics at each mode are now described by a stochastic differential equation. This approach was adopted in [10], for example. Another choice is to replace the deterministic transitions between discrete states by stochastic ones, governed by some prescribed probabilistic rule. This approach was adopted in [3]. More general models can be proposed by mixing the above approaches and keeping in mind the application in hand. In this paper, we adopt the first approach.

The extension of Lyapunov stability theory to deterministic switched and hybrid systems has been studied in detail by many authors during the last past years (see [1],[12] and [4] for an overall review of the topic). A common feature in the various Lyapunov-like theorems that were presented was that the decrease of the energy function at the switching instants-or at each time a subsystem is activated- is a sufficient condition for the stability of the hybrid system. On the other hand, there exists a solid theory on stability in probability of stochastic differential equations and the corresponding Lyapunov theorems (see [2],[7],[9] and the references therein). It is therefore natural and appealing to extend the Lyapunov-like theorems for stochastic differential equations to the case of switched stochastic systems.

In this paper, three theorems regarding stability in probability of switched stochastic systems are stated and proved. We combine the formulation of [1] with the Lyapunov theory for stochastic differential equations of [9] to obtain the aforementioned results. The motivation of our work comes from the field of Decentralized Air Traffic Management,the formulation of which is a system that combines continuous, discrete and stochastic dynamics. A preliminary discussion on this application is included in section IV.

The rest of the paper is organized as follows: In section II we recall some definitions and results regarding Lyapunov stability of stochastic differential equations. In section III the system model is defined, three theorems that guarantee stability in probability are presented and we make some remarks on the aforementioned theorems. In section IV we discuss some examples. Section V summarizes the conclusions and indicates our current research.

II. STOCHASTIC STABILITY IN LYAPUNOV'S VEIN

We consider the following *stochastic system*

$$\begin{aligned} &dX(t) = b(X(t))dt + G(X(t))d\xi(t) \\ &X(0) = X_0 \end{aligned} \tag{1}$$

where $X(t)$ a *n*-dimensional random process, $b(X(t)), G(X(t))$ a vector and a matrix valued second order random process respectively of appropriate dimensions, $\xi(t)$ a standard *n*-dimensional Wiener process, and X_0 a second order random vector independent of the σ-algebra $F\ (\xi(\tau),\ \tau \geq t)$. We assume that the processes $b(X(t)), G(X(t))$ are non-anticipating, so that the corresponding Itô integrals are well defined. The stochastic differential equation (1) admits a unique solution $X(t)$ if there exist constants $K_1, K_2 > 0$ such that $\forall x \in \mathcal{R}^n$ and $\forall\ t \geq\ 0$, the following hold [11]:

$$\begin{aligned} &|b(x_1) - b(x_2)| + |G(x_1) - G(x_2)| \leq K_1|x_1 - x_2| \\ &|b(x)| + |G(x)| \leq K_2(1 + |x|) \end{aligned} \tag{2}$$

We examine the stability of the trivial solution $X(t) = 0$ of equation (1), therefore we make the following assumption: $b(0) = 0, G(0) = 0$.

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Definition 1 [9]: *The solution $X(t)$ of equation (1) is said to be stable in probability for $t \geq 0$ if for any $s \geq 0$ and $\varepsilon > 0$*

$$\lim_{x \to 0} P\{\sup_{t>s} \mid X^{s,x}(t) \mid > \varepsilon\} = 0 \tag{3}$$

Here, $X^{s,x}(t)$ denotes the sample path of the solution of equation (1) starting from a point x at time s. Intuitively, the definition implies that for a stable stochastic system, the probability of escape from a spherical region around the origin should be small for a small deviation from equilibrium state.

Many Lyapunov-like sufficient conditions for the stability in probability of equation (1) have been proposed in literature. Here, we adopt the approach of Hasminskii [2],[7],[9] which is summarized in the following theorem:

Theorem 1: *Let $U \subset \mathcal{R}^n$ be a domain which contains the origin, and assume that there exists a positive definite function $V : U \to \mathcal{R}_+$, twice continuously differentiable everywhere except possibly at the origin, that satisfies for all $x \in U \setminus \{0\}$:*

$$LV(x) = \sum_{i=1}^{n} b_i(x) \frac{\partial V}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} \alpha_{ij}(x) \frac{\partial^2 V}{\partial x_i \partial x_j} \leq 0 \tag{4}$$

*in which $\alpha_{ij} = [GG^*]_{ij}$, where * denotes the complex conjugate transformation. Then the trivial solution of equation (1) is stable in probability.*

III. Switched Stochastic Systems

In this paper, we examine stability properties of switched stochastic systems of the form:

$$\begin{aligned} dX(t) &= b^i(X(t))dt + G^i(X(t))d\xi(t) \\ X(0) &= x_0 \end{aligned} \tag{5}$$

where $i \in Q = \{ 1, \ldots, N\}$, the set of indices of each mode of the switched system. For simplicity we consider x_0 to be a constant vector. A closed-loop stochastic controller has been designed for each mode of the system, and we wish to find suitable switching conditions which ensure some desired performance for the overall system. We consider *arbitrary switching*, i.e. there are no *guards* that enforce switching between different modes, and the switching is state-independent, unlike most models of stochastic switched and hybrid systems found in literature [3],[8],[10]. Furthermore, two different switching situations are encountered : (i) there are no *reset maps* at each switching instant, and the state jumps to an arbitrary new value whenever a switch between two different modes occurs and (ii) continuity is preserved at each switching instant, i.e. the reset map is the identity operator in this case. Case (i) is dealt with in Theorems 3,4 and case (ii) in Theorem 2. In section 2.1 we make the assumption that a common Lyapunov function exists as a measure for the energy of each subsystem, while in section 2.2 we consider different Lyapunov functions for each mode. Finally, there are only finite switches in finite time, i.e. the switched system does not exhibit *Zeno* behavior.

A. Common Lyapunov Function

We make the following assumptions for each subsystem: $b^i(0) = 0, G^i(0) = 0, \forall\ i \in Q$. A *Common Lyapunov Function* for the system (5) is a function $V : \mathcal{R}^n \to \mathcal{R}_+$, twice continuously differentiable everywhere except possibly at the origin, that is positive definite, i.e. $V(x) > 0 \forall x \neq 0$ and $V(0) = 0$, and proper, i.e. $\lim_{|x| \to \infty} V(x) = \infty$ such that

$$\begin{aligned} LV(x) = &\sum_{j=1}^{n} b_j^i(x) \frac{\partial V}{\partial x_j} + \\ &+ \frac{1}{2} \sum_{j,k=1}^{n} \alpha_{jk}^i(x) \frac{\partial^2 V}{\partial x_j \partial x_k} \leq 0\ \forall t, \forall i \in Q \end{aligned} \tag{6}$$

in which $\alpha_{jk}^i = \left[G^i G^{i*}\right]_{jk}$.

Suppose switchings occur at the time instants: $t_0, t_1, ...,\ t_0 < t_1 < \ldots$ and that continuity in mean square of the state is maintained at the switching instants, i.e.

$$\lim_{h \to 0} E\|x(t_i + h) - x(t_i^-)\|^2 = 0, \forall i \tag{7}$$

We have the following theorem:

Theorem 2: *If there is a Common Lyapunov Function for the system (5) and all the previous assumptions hold, then the system (5) is stable in probability according to definition (3).*

Proof: For each $t_j \in \{t_0, t_1, ...\}$ Ito's formula results in:

$$\begin{aligned} V(t_j) = V(t_{j-1}) + &\int_{t_{j-1}}^{t_j} LV(x(s))ds + \\ &+ \int_{t_{j-1}}^{t_j} \sum_{k,l=1}^{n} \frac{\partial V}{\partial x_k} G_{kl}^q d\xi_l(s) \end{aligned}$$

where q is the subsystem which is active in the interval $[t_{j-1}, t_j)$. The expectation of the second integral in this equation is zero. Hence taking the expectation of both sides and using (6),(7) we have $EV(t_j) \leq EV(t_{j-1}) \leq \ldots \leq EV(0) = V(x_0)$. We make use of the fundamental remark of [9] that the process $V(x)$ is a supermartingale. By the supermartingale inequality we have

$$P\{\sup_{t_j \leq t \leq t_{j+1}} V(t) \geq \varepsilon\} \leq \frac{1}{\varepsilon}[EV(t_j) + EV^-(t_{j+1})]$$

where $V^-(a) = \max\{-V(a), 0\} = 0\ \forall a$ since V is positive definite. Hence the last inequality becomes

$$P\{\sup_{t_j \leq t \leq t_{j+1}} V(t) \geq \varepsilon\} \leq \frac{1}{\varepsilon} EV(t_j)$$

Using the fact that $EV(t_j) \leq EV(t_{j-1}) \leq \ldots \leq EV(0) = V(x_0)$ we finally get

$$P\{\sup_{t\geq 0} V(t) \geq \varepsilon\} \leq \frac{1}{\varepsilon}V(x_0)\ \forall \varepsilon > 0.$$

The properness of V implies that $\forall \varepsilon_1 > 0 \exists \varepsilon > 0$ s.t. $V(x) \geq \varepsilon$ whenever $\|x\| \geq \varepsilon_1$. The positive definiteness and continuity of V imply that $\forall \varepsilon_2 > 0 \exists \delta > 0$ s.t. $\frac{1}{\varepsilon}V(x_0) \leq \varepsilon_2$ whenever $\|x_0\| \leq \delta$. So the last equation is equivalent to

$$P\{\sup_{t\geq 0} \|x(t)\| \geq \varepsilon_1\} \leq \varepsilon_2 \forall x_0 : \|x_0\| \leq \delta.$$

Letting x_0 tend to zero we derive the desired result.$\diamondsuit$

B. Multiple Lyapunov Functions

In this section, we present some extensions of the stability theorems for deterministic switched systems ([1],[12]) to the stochastic case. Specifically, we present and prove two theorems that guarantee stability in probability of the switched stochastic system with arbitrary switches. The conditions imposed on these theorems are somewhat stronger than those of the deterministic case, however they have the advantage that the state of the system need not be continuous at the switching instants. Theorem 3 is more general than Theorem 4, but as in the deterministic case, the first theorem requires knowledge of the trajectory of the system. Theorem 4 on the other hand requires only local analysis where the switches occur and hence is more applicable.

We make the following assumptions for each subsystem: (a) $b^i(0) = 0, G^i(0) = 0, \forall\ i \in Q$, (b) $\forall\ i \in Q$, there exists a function $V^i : \mathcal{R}^n \to \mathcal{R}_+$, twice continuously differentiable everywhere except possibly at the origin, that is positive definite, i.e. $V^i(x) > 0\ \forall x \neq 0$ and $V^i(0) = 0$,and proper, i.e. $\lim_{|x|\to\infty} V(x) = \infty$ and (c) $\forall\ i \in Q$,b^i and G^i satisfy the existence and uniqueness conditions (2). The state evolution is described by the following *switching sequence* $S = x_0;\ (i_0, t_0), (i_1, t_1), \ldots$, where (i_j, t_j) means that the state evolves according to $dX(t) = b^{i_j}(X(t))dt + G^{i_j}(X(t))d\xi(t)$ for $t_j \leq t < t_{j+1}$. Define $S \mid i = \tau_0^i, \tau_1^i, \ldots$ the endpoints of the intervals in which the i^{th} subsystem is active, and $I(S \mid i) = \bigcup_{j\in\mathcal{N}} [\tau_{2j}^i, \tau_{2j+1}^i]$ the set of intervals in which the i^{th} subsystem is active.

Theorem 3: *Suppose that assumptions (a),(b),(c) hold for each subsystem. Let $\mathcal{S}$ denote the set of all switching sequences related to the system. If $\forall S \in \mathcal{S}$ and $\forall i \in Q$ the following conditions are satisfied:*

$$LV^i(x) = \sum_{j=1}^{n} b_j^i(x)\frac{\partial V^i}{\partial x_j} + \\ + \frac{1}{2}\sum_{j,k=1}^{n} \alpha_{jk}^i(x)\frac{\partial^2 V^i}{\partial x_j \partial x_k} \leq 0\ \forall t \in I(S \mid i) \quad (8)$$

$$EV^i(x(\tau_{2j}^i)) \geq EV^i(x(\tau_{2j+2}^i))\ \forall j \in \mathcal{N} \quad (9)$$

$$EV^i(x(\tau_0^i)) \leq EV^{i_0}(x(\tau_0^{i_0})) = V^{i_0}(x_0) \quad (10)$$

where $\alpha_{jk}^i = [G^i G^{i*}]_{jk}$ then the trivial solution $X(t) \equiv 0$ is stable in probability, according to definition (3) .

Proof: By the supermartingale inequality we have

$$P\{\sup_{\tau_{2j}^i \leq t \leq \tau_{2j+1}^i} V^i(t) \geq \varepsilon_i\} \leq \frac{1}{\varepsilon_i} EV^i(\tau_{2j}^i)\ \forall i, \forall \varepsilon_i > 0$$

(9),(10) imply that

$$P\{\sup_{\tau_{2j}^i \leq t \leq \tau_{2j+1}^i} V^i(t) \geq \varepsilon_i\} \leq \frac{1}{\varepsilon_i} EV^i(\tau_0^i) \leq \\ \leq \frac{1}{\varepsilon_i} V^{i_0}(x_0)\ \forall i, \forall \varepsilon_i > 0, \forall j \in \mathcal{N}$$

so that

$$P\{\sup_{t\in I(S|i)} V^i(t) \geq \varepsilon_i\} \leq \frac{1}{\varepsilon_i} V^{i_0}(x_0)\ \forall i, \forall \varepsilon_i > 0, \forall j \in \mathcal{N}$$

Pick $\varepsilon > 0$ arbitrary. The properness of V^i implies that $\forall i$ there are $\varepsilon_i(\varepsilon) > 0$ such that $V^i(t) \geq \varepsilon_i$ is implied by $\|x(t)\| \geq \varepsilon$ so that

$$P\{\sup_{t\in I(S|i)} \|x(t)\| \geq \varepsilon\} \leq \frac{1}{\varepsilon_i} V^{i_0}(x_0)$$

or

$$P\{\sup_{t\geq 0} \|x(t)\| \geq \varepsilon\} \leq \max_i \{\frac{1}{\varepsilon_i} V^{i_0}(x_0)\}$$

The positive definiteness and continuity of each V^i imply that for each $\varepsilon_2 > 0$ there is a $\delta(\varepsilon_2) > 0$ such that $\max_i\{\frac{1}{\varepsilon_i} V^{i_0}(x_0)\} \leq \varepsilon_2$ is implied by $\|x_0\| \leq \delta$. Letting x_0 tend to zero we derive the desired result.$\diamondsuit$

We now present a restricted version of the above theorem.

Theorem 4: *Suppose that assumptions (a),(b),(c) hold for each subsystem. Let $\mathcal{S}$ denote the set of all switching sequences related to the system. If $\forall S \in \mathcal{S}$,$S = x_0;\ (i_0, t_0), (i_1, t_1), \ldots$ and $\forall i \in Q$ the following conditions are satisfied:*

$$LV^i(x) = \sum_{j=1}^{n} b_j^i(x)\frac{\partial V^i}{\partial x_j} + \\ \frac{1}{2}\sum_{j,k=1}^{n} \alpha_{jk}^i(x)\frac{\partial^2 V^i}{\partial x_j \partial x_k} \leq 0\ \forall t \in I(S|i) \quad (11)$$

$$EV^{i_{j+1}}(x(t_{j+1})) \leq EV^{i_j}(x(t_{j+1}^-)) \forall j \in \mathcal{N} \quad (12)$$

then the trivial solution $X(t) \equiv 0$ is stable in probability, according to definition (3).

Proof:From the proof of Theorem 2 we have that $EV^{i_j}(x(t)) \leq EV^{i_j}(x(t_j))\ \forall i, j \in \mathcal{N}, \forall t \in [t_j, t_{j+1}]$. Hence, equation (12) yields $EV^{i_j}(x(t)) \leq EV^{i_j}(x(t_j))$

$\leq EV^{i_{j-1}}(x(t_j^-)) \leq \ldots \leq EV^{i_0}(x(t_0)) = V^{i_0}(x_0)\forall i, j \in \mathcal{N}, \forall t$. Similar to the proof of theorem 3 we have

$$P\{\sup_{t \in I(S|i)} V^i(t) \geq \varepsilon_i\} \leq \frac{1}{\varepsilon_i} V^{i_0}(x_0)\,\forall i, \forall \varepsilon_i > 0, \forall j \in \mathcal{N}$$

and the rest of the proof is the same. $\diamondsuit$

C. Remarks on the Theorems

The following remarks are in order.

- Theorem 3 lacks in applicability compared to theorems 2,4, mainly because one has to know the global behavior of the trajectory, in order to check whether equations (9),(10) are satisfied or not. The stricter Theorem 4 is more applicable because it requires only local knowledge of the trajectory whenever switchings take place.
- The above theorems do not hold whenever the set of modes Q is infinite. The reason is that in this case a switch to a mode never before activated can always be made, so that conditions of the theorems still hold, although the system is unstable.
- If the Lyapunov functions in the above theorems are defined only in a sphere around the origin, then the results have only local validity.
- So far we haven't discussed asymptotic stability in probability, i.e. the case when the origin is stable in probability and the following relation holds for any $s \geq 0$ and $\varepsilon > 0$

$$\lim_{x \to 0} P\{\lim_{t \to \infty} X^{s,x}(t) = 0\} = 1 \tag{13}$$

  It is possible to strengthen the conditions of theorems 2,3,4 in order to achieve asymptotic stability. For example, if there are finite switches and the last subsystem activated is asymptotically stable in probability, then it is obvious that so is the whole system. Furthermore, if there are infinite switches and the energy strictly decreases at the switching times(strict inequality at equations (9),(12)) then the system can be proven to be asymptotically stable in probability. The proofs have the same structure of the corresponding theorems for one mode [9]. Since we deal with autonomous subsystems, we could also use another result of Hasminskii who has shown that if G in (5) satisfies $y^T G^T(x) G(x) y \geq m(x)\|y\|^2$ for all $x, y, \|x\| < r$,for some $r > 0$,where $m(x)$ is positive definite and bounded away from 0, then stochastic stability implies stochastic asymptotic stability.

IV. Examples

A. Example:Linear Subsystems

A variety of results regarding stability of deterministic switched linear systems can be found in literature. Sufficient conditions have been derived in the form of Linear Matrix Inequalities(LMI's) [12]. In this section we derive analogous results for the case where the subsystems of system (5) are linear, i.e. for switched stochastic systems of the form:

$$dX(t) = B^i X(t)dt + G^i X(t)d\xi(t) \tag{14}$$

where $B^i, G^i n \times n$ matrices and $\xi(t)$ a standard *1*-dimensional Wiener process. We shall make use of theorem 4. A common way to begin is to choose quadratic Lyapunov functions for each subsystem: $V^i(x) = x^T P^i x$ where P^i a positive definite real symmetric matrix. In the case when continuity in mean square is preserved at the switching instants it is easy to see that from equations (11),(12) we derive the following sufficient conditions for the system (13):

$$x^T \left(P^i B^i + \left(B^i\right)^T P^i\right) x + tr\left\{x^T \left(G^i\right)^T P^i G^i x\right\} \leq 0 \Rightarrow$$

$$\Rightarrow P^i B^i + (B^i)^T P^i + (G^i)^T P^i G^i \leq 0\ \forall i \tag{15}$$

$$P^{i_{j+1}} - P^{i_j} \leq 0 \forall j \in \mathcal{N} \tag{16}$$

where $tr(M)$ denotes the trace of the $n \times n$ matrix M and $M \leq 0$ denotes that the square matrix M is negative semi-definite. Equation (16) is stricter than the corresponding condition in [12]. This is because we haven't imposed conditions on subsets of the state space where switchings occur. If that were the case, then (16) could be rewritten in the form of a LMI.

B. Application to Decentralized Air Traffic Management

The motivation for this work has its origins in the domain of decentralized motion planning. In previous work [5], we derived control laws for decentralized navigation of multiple agents in a closed-loop fashion. Each agent was assumed to have perfect knowledge of the positions of the other agents and the overall system model was purely deterministic. However this is not the case when one has to deal with distributed air traffic management systems. Each aircraft can only have knowledge of an estimate of the current positions of the other aircraft in a neighborhood of its center of a certain radius, namely its *protected zone*(Fig.1). The primary reason of uncertainty is the wind. Hence the dynamics of such a system include both stochastic and discrete dynamics(whenever a new aircraft enters the protected zone of another).

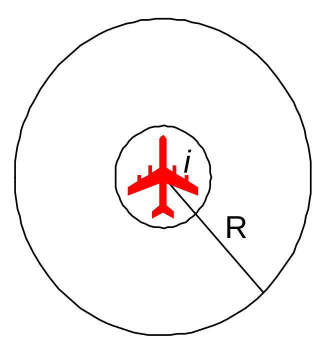

Fig. 1. Aircraft i and its protected zone of radius R.

Our approach will be based on [6]. Whenever the protected zone of aircraft i is empty the dynamics of the aircraft are purely deterministic, namely a potential function driving i towards its destination. Whenever an aircraft enters the protected zone of aircraft i, the control law is switched in order to meet both specifications:destination convergence(DC) and collision avoidance(CA). Hence the switching control strategy is given by:

$$\dot{q_i} = \begin{cases} DC & \text{if } N(i) = 0 \\ DC \wedge CA & \text{if } N(i) \neq 0 \end{cases} \tag{17}$$

where q_i the configuration of i, $N(i) \triangleq \{\#j : \|q_i - q_j\| \leq d_C\}$ is the number of aircraft in the protected zone of i and d_C is the separation minimum between two aircraft which corresponds to the radius of the protected zone. In this equation DC denotes the control imposed on i in order to meet the destination convergence goal whereas $DC \wedge CA$ denotes the control imposed on i in order to meet the destination convergence and collision avoidance goals simultaneously whenever there are intruding aircraft in i's neighborhood. The switching control strategy is shown in Figure 2.

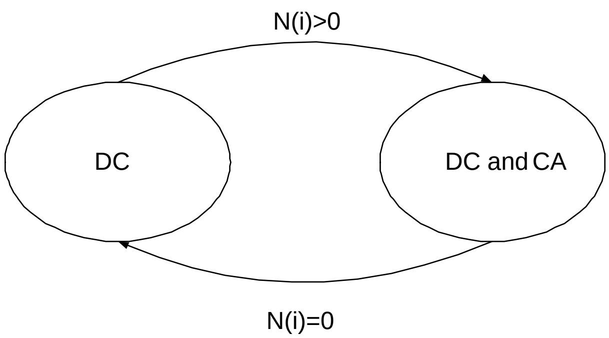

Fig. 2. Switching control strategy according to eq.(17).

It is our current goal to produce closed-loop decentralized control laws for each mode of (17) in the spirit of [5]. Each agent treats the movement of the other agents as a stochastic differential equation. For example let the dynamics of aircraft i be given by $dq_i(t) = b_i(q(t))dt$ and the dynamics of an intruding aircraft j be given by: $dq_j(t) = b_j(q(t))dt + G_j(q(t))d\xi(t)$, where $q(t) = [q_i(t), q_j(t)]^T$. Then the relative position of aircraft i with respect to j is given by: $dq_{ij}(t) = (b_i(q(t)) - b_j(q(t)))dt - G_j(q(t))d\xi(t)$. A possible interpretation of the two performance objectives (*DC* and *CA*) could then have the form:

DC:Design $b_i(q)$ so that

$$P\{\sup_{t\geq 0} \|q_i - q_{di}\| > \varepsilon\} \leq N(\varepsilon) > 0, \forall \varepsilon > 0$$

CA:Design $b_i(q)$ so that

$$P\{\inf_{t\geq 0} \|q_{ij}\| \leq d_C\} \leq M, M \text{ suff. small}$$

where q_{di} denotes the desired destination of i.

It is obvious that the system (17) is of the form (5). Hence the satisfaction of one of the theorems 2,3 or 4 is crucial for system (17) in order to achieve (asymptotic) stability in probability to the desired destination point.

V. Conclusions

In this paper we proved three theorems on stability of switched stochastic systems. These theorems are extensions of existing results of the past decade on stability of deterministic switched systems. The motivation comes from the field of distributed air traffic management where the dynamics encountered include discrete and stochastic components. The theorems proved in this work provide sufficient conditions for the stability of the stochastic hybrid system.

Current research aims at improving the results presented in an application-wise fashion as well as producing a satisfactory statement and solution of the stochastic decentralized problem discussed in section IV.B.

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