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A necessary and sufficient local controllability condition for bilinear discrete-time systems

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***Abstract*— This paper presents a local necessary and sufficient condition for the controllability of bilinear discrete-time systems. Based on an optimization approach of the problem of controllability, the method assumes a relaxed condition of inversibility of the state transition function, with respect to other similar approaches.**

I. INTRODUCTION

Controllability of bilinear discrete-time systems has been the subject of some relevant studies. This is for one part due to their analytical simplicity and has led to fairly developed theory. On the other hand, it appears that many important processes, not only in engineering, but also in biology, socio-economy, and ecology, can be modeled by bilinear systems, which accredits such studies.[1]

The homogeneous bilinear discrete-time system considered here can be classicaly modeled by using two matrices A and Q, where A is the matrix involved in the linear part of the state equation whereas Q is the matrix involved in the bilinear part.

The controllability of bilinear system, as studied in [4], raises two conditions for controllability: one for necessity and one for the sufficiency. Such approach is a local one and consists in decomposing the bilinear system model into a linear system and a multiplicative feedback. However, it requires that $rank(Q) = 1$ where Q must be factorized in two vectors ; in other terms this technic needs orthogonality property. The same problem as considered in [3] gives rise to a global necessary and sufficient condition. In addition to decomposing the system as in [4], the approach involves forward and backward composition of the transition function. It still ensues a condition of orthogonality on the matrix Q, more an inversibility condition on the matrix A. On a more general point of view, bilinear systems can be studied by specific tools used in nonlinear discrete-time system, like differential geometry. This is the case in[6] where local controllability of bilinear discrete-time system is investigated. The approach, which requires the inversibility of $(A+u(k)Q)$, where $u(k)$ denotes the discrete control vector at time k, issues in the determination of a feasible domain of control. This assumption is due to the definition of vector fields that are used to characterize the local controllability of the system.

The present paper rediscovers the same necessary and sufficient condition for local controllability condition of bilinear discrete-time system as in differential geometry. However due to a relaxed assumption on the system model property, it adresses to a larger feasible control domain.

The plan is organized as follows. Section II gives some preliminaries and definitions. Section III is aimed to the presentation of a local necessary and sufficient controllability condition for bilinear discrete-time systems. Section IV shows with some details an example of controllability analysis which do not fit the condition required by differential geometry approach, but which satisfies the relaxed condition for application proposed here.

II. PRELIMINARIES

This paper is devoted to the study of controllability of homogeneous discrete-time bilinear system, described by the following state equation in discrete-time :

$$x(k+1) = Ax(k) + \sum_{i=1}^{m} u_i(k) Q_i x(k) \qquad (1)$$

where $x(k)$ is the n-dimensioned state vector at time k, $u(k) = (u_i(k))$ is the m-dimensioned control vector at time k; A and $Q_1, \cdots, Q_m$ are square matrices of order n.

A. Controllability

We recall from [5] the definition of the controllability for discrete-time systems. A dynamic discrete-time system, described in state form, is said to be controllable on the interval $[k_0, k_1]$, if for any state x_0 and x_1, there exist an input $u(k)$ that drives the system to state $x(k_1) = x_1$ at time $k = k_1$, starting from the state $x(k_0) = x_0$ at time $k = k_0$.

B. A Necessary Controllability Condition

Let us consider first the more general nonlinear discrete-time system model:

$$x(k+1) = f(x(k), u(k)) \qquad (2)$$

where $x(k) \in \Re^n, u(k) \in \Re^m$ and $f(.)$ is a continous vector function.

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Provided that $\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}\right)$ has full rank, a necessary condition of controllability of this system, proposed in [2], is given by :

$$rank[P]=n$$

where P is the $n\times Nm$ matrix defined as :

$$\begin{aligned} P &= \Big[\tfrac{\partial f}{\partial u(N-1)} \quad \tfrac{\partial f}{\partial x(N-1)}\tfrac{\partial f}{\partial u(N-2)} \\ &\quad \cdots \quad \tfrac{\partial f}{\partial x(N-1)}\cdots\tfrac{\partial f}{\partial x(1)}\tfrac{\partial f}{\partial u(0)}\Big] \end{aligned}$$

III. A NECESSARY AND SUFFICIENT CONDITION OF CONTROLLABILITY

It results in what follows a necessary and sufficient condition for local controllability of bilinear discrete-time system. A proof is also given.

A. Proposition

The bilinear discrete-time system described by equation (1) is controllable on the interval $[0,N]$ if and only if :

$$rank[C]=n$$

where

$$\begin{aligned} C &= [Q_1A^{N-1}x(0) \quad AQ_1A^{N-2}x(0)\cdots A^{N-1}Q_1x(0) \\ &\quad Q_2A^{N-1}x(0) \quad AQ_2A^{N-2}x(0)\cdots A^{N-1}Q_2x(0) \\ &\quad \cdots \quad Q_mA^{N-1}x(0) \quad AQ_mA^{N-2}x(0)\cdots \\ &\quad A^{N-1}Q_mx(0)] \end{aligned} \tag{3}$$

with sequence of controls satisfying that matrix $\left(A+u_1Q_1+\cdots+u_mQ_m+Q_1x\frac{\partial u_1}{\partial x}+\cdots+Q_mx\frac{\partial u_m}{\partial x}\right)$ has full rank.

Proof For sake of simplicity, we will limit first to the single input case.

By expressing as follows the solution $x(N)$ of state equation (2) after function composition, one gets:

$$x(N)=f_{u(N-1)}\circ\cdots\circ f_{u(1)}\circ f_{u(0)}(x(0))$$

with $f(x(k),u(k))=f_{u(k)}(x(k))$

which will be also denoted by

$$x(N)=\Gamma_{N-1}(x(0),u)$$

where u means the control sequence $\{u(0),\cdots,u(N-1)\}$.

Using Taylor's development to expand the right-hand side of the previous equation yields:

$$\begin{aligned} &x(N)=f_{u(N-1)}\circ\cdots\circ f_{u(1)}\circ f_{u(0)}(x(0))|_{\overline{u}=0} \\ &+\left[\tfrac{\partial\Gamma_{N-1}(x(0),u)}{\partial u(N-1)} \quad \tfrac{\partial\Gamma_{N-1}(x(0),u)}{\partial u(N-2)} \quad \cdots \quad \tfrac{\partial\Gamma_{N-1}(x(0),u)}{\partial u(0)}\right]_{\overline{u}=0}\overline{u} \\ &+O(u^2) \end{aligned}$$

with $\overline{u}=[u(N-1) \quad \cdots \quad u(1) \quad u(0)]^T$, which can be rewritten as:

$$x(N)=A^Nx(0)+P|_{\overline{u}=0}\overline{u}+O(u^2)$$

Necessity

Necessary condition appears as obvious, since it merely consists in applying the necessary controllability condition, as depicted in section II, to the particular case of bilinear discrete-time system, namely:

$$\begin{aligned} x(k+1) &= f(x(k),u(k)) \\ &= Ax(k)+u(k)Qx(k) \end{aligned} \tag{4}$$

This gives :

$$\frac{\partial f(x(k),u(k))}{\partial x(k)}=A+u(k)Q, \quad \frac{\partial f(x(k),u(k))}{\partial u(k)}=Qx(k)$$

Then, it comes that $\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}\right)=(A+u(k)Q+Qx\frac{\partial u}{\partial x}$ and

$$\begin{aligned} P &= [Qx(N-1) \quad (A+u(N-1)Q)Qx(N-2) \\ &\quad \cdots \quad (A+u(N-1)Q)\cdots(A+u(1)Q)Qx(0)] \end{aligned}$$

Then, matrix C, as previously defined, results from the computation of P after function composition, when controls are assumed to be equal to zero.

In other words, it can also be noted that the solution of state equation (4) when neglecting higher order control terms write as:

$$x(N)=A^Nx(0)+\sum_{k=0}^{N-1}A^{N-1-k}QA^kx(0)u(k)+O(u^2) \tag{5}$$

which corresponds to:

$$x(N)=A^Nx(0)+C\overline{u} \tag{6}$$

It turns out that such a determination of the matrix C does confer to the controllability analysis, a local character, valid for small inputs.

Sufficiency

Let us suppose that C has full rank. We will proof in what follows, that one can find a sequence of controls tansferring any initial state $x(0)$ to any final state $x(N)$. So, by considering state equation solution (6), it results that one can determine a sequence of controls which tranfers initial state $x(0)$ to final state $x(N)$, in other words, that there exist a solution for the equation (6). Moreover, it can be easily shown that the sequence of controls $\overline{u}$ that minimizes the particular cost function :

$$\sum_{k=0}^{N-1}r(x(k),u(k))$$

$$\text{with} \quad r(x(k),u(k))=u^2(k)$$

corresponds to the generalized right-inverse of the controllability matrix, namely :

$$\overline{u}=C^T[CC^T]^{-1}[x(N)-A^Nx(0)]$$

Note that for single input system, this matrix takes the following simplified form :

$$\begin{bmatrix} QA^{N-1}x(0) & AQA^{N-2}x(0) & \cdots & A^{N-1}Qx(0) \end{bmatrix}$$

Remark

Generalization of the proof to multi-input system do not raise any particular difficulty.

Let's consider for this purpose the following multivariable system:

$$\begin{array}{rcl} x(k+1) & = & Ax(k) + \sum_{i=1}^{m} u_i(k)Q_i x(k) \\ & = & f(x(k), u(k)) \end{array}$$

It results that:

$$\begin{array}{rcl} \frac{\partial f(x(k),u(k))}{\partial x(k)} & = & A + \sum_{i=1}^{m} u_i(k)Q_i \\ \frac{\partial f(x(k),u(k))}{\partial ui(k)} & = & Q_i x(k) \end{array}$$

and by substituting $x(k) = A^k x(0)$ and by assuming the controls to be equal to zero in the calculation of matrix P, we obtain matrix C as presented in (3).

The calculation of the terminal state $x(N)$ by using the composition function, gives:

$$x(N) = A^N x(0) + \sum_{k=0}^{N-1} \sum_{i=1}^{m} A^{N-1-k} Q_i A^k x(0) u_i(k) + O(u^2)$$

and by neglecting higher order control terms $O(u^2)$, we get the same form as in (6).

IV. AN EXAMPLE

Let us consider the following bilinear discrete-time system.

$$\begin{array}{rcl} x_1(k+1) & = & x_2(k) \\ x_2(k+1) & = & x_2(k) + u(k)x_2(k) \end{array}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

It can be shown that a linear state feedback control law of the form :

$$u(k) = \alpha^T x(k)$$

assures the controllability of the system in two steps, whenever $x(2) \in \Re^2$ and $x(0) \in \{\Re^2/(0,0)\}$.

The domain of control admissibility, defined in [6] as the set of control sequences that makes that $(A + u(k)Q)$ has full rank, can be determined from the analysis of the rank of the following matrix :

$$A + \alpha^T xQ = \begin{bmatrix} 0 & 1 \\ 0 & 1 + \alpha_1 x_1(k) + \alpha_2 x_2(k) \end{bmatrix}$$

It is obvious that $(A + \alpha^T xQ)$ has not full rank, so we cannot settle that the control law $\alpha^T x$ belongs to the domain of admissibility. It results that one can't conclude about the controllability of this system.

Now, considering the relaxed condition proposed in this paper leads to analyze the matrix :

$$A + \alpha^T xQ + Qx\alpha^T = \begin{bmatrix} 0 & 1 \\ \alpha_1 x_2(k) & 1 + \alpha_1 x_2(k) + 2\alpha_2 x_2(k) \end{bmatrix}$$

which appears to have full rank, provided that α_1 doesn't equal to zero. Controllability matrix is then defined as:

$$C = [AQx(0) \quad QAx(0)] = \begin{bmatrix} x_2(0) & 0 \\ x_2(0) & x_2(0) \end{bmatrix}$$

which shows that the system is controllable, for any $x_2(0)$ different from zero.

V. CONCLUSION

We presented a necessary and sufficient condition for local controllability of bilinear discrete-time systems, valid for small inputs. It is mentioned that the same criterion has been obtained yet by using differential geometry. However, the approach was based on a condition of local inversibility of the state tansition matrix which is relaxed into a less restrictive condition in the present analysis issued from optimization theory. A numerical example application satisfying only the latter condition is shown.

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