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Input-output Analysis of Decentralized Relay Systems

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***Abstract*—A method of input-output analysis of decentralized relay systems is proposed in the paper. The proposed method furnishes a better accuracy than the describing function method and does not require the involvement of the filtering hypothesis. An example illustrating application of the method and assessment of its accuracy is given.**

I. INTRODUCTION

RELAY control systems are extensively used in various industries in many cases providing cheaper solutions and better performance compared to other types of control system. Some application examples are various on-off process control systems, DC motor, pneumatic and hydraulic relay servo actuators, etc. Relays used in those control systems normally have two-level ideal or hysteresis characteristic. Application of a decentralized relay may provide some advantages over a single relay control. Among those advantages the most important are: smaller amplitude of self-excited oscillations and higher tracking accuracy of servo control. A method of input-output analysis of dead-zone relay servo systems was given in [1], which was the extension of the methodology proposed in [2] and [3] for the hysteresis relay. The current paper extends mentioned above methods to the case of two relays. The obtained results well agree with known exact [4] and approximate [5] methods of analysis of periodic motions in relay systems as well as of periodic motions in decentralized relay systems [6].

II. PROBLEM FORMULATION AND MAIN RESULTS

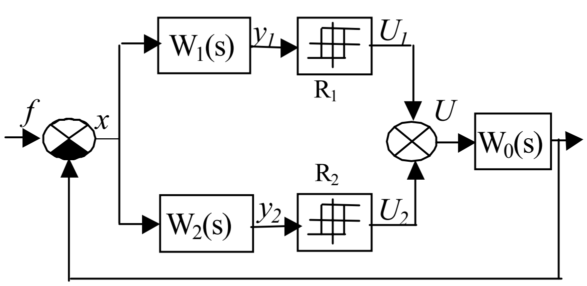

Fig.1. Decentralized relay system with two relays.

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The decentralized relay system, which is going to be analysed in the paper, can be represented by the block diagram (Fig. 1). It consists of a plant given by the transfer function $W_0(s)$, two identical hysteresis relays (R_1 and R_2) and two linear filters given by the transfer functions $W_1(s)$ and $W_2(s)$. The filters given by $W_1(s)$ and $W_2(s)$ transform signal x into signals y_1 and y_2 that have a phase shift in respect to each other. As a result, relays R_1 and R_2 switch at different times and control U, which is the sum of their outputs, is equivalent to the output of a certain dead-zone relay (Fig. 2).

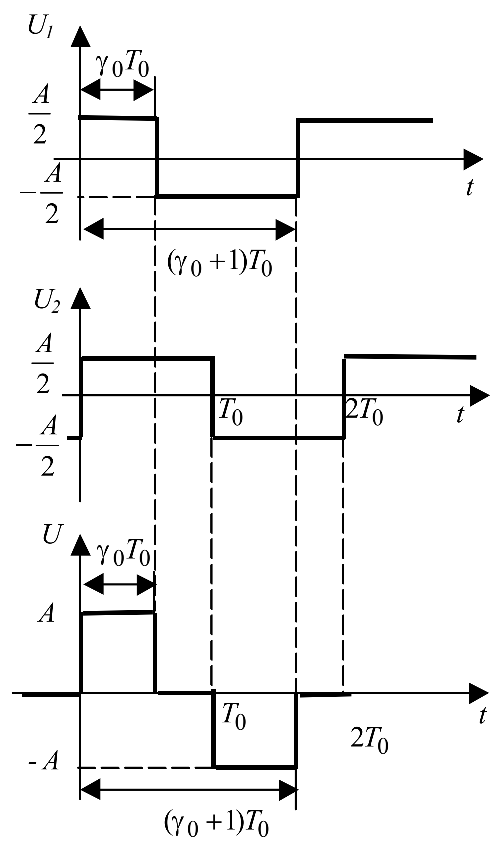

Fig. 2. Resultant control.

A. Analysis of the periodic motion

Let us denote the transfer functions: $W_1^*(s) = W_0(s)W_1(s)$; $W_2^*(s) = W_0(s)W_1(s)$; $\frac{A}{2}$ being the amplitude of the relay outputs (R_1 and R_2); and 2κ being the hysteresis of relays R_1 and R_2. We can assume without

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loss of generality that in periodic motion, the signal $y_2(t)$ has a phase lag with respect to signal the $y_1(t)$.The resultant control $U(t)$ is the sum of the two relay outputs as shown in Fig. 2. Let $\mathbf{x} = [x_1, x_2, ..., x_n]^T$ be the state vector of the system, *2T* be the period of self-excited oscillations and *γ* be the relative duration of the positive pulse (Fig. 2). Let us denote the value of **x** in a periodic motion at the time of the control *U* switch from zero to plus as $\mathbf{x}^1(T,\gamma)$. Let us denote the value of **x** in a periodic motion at the time of the control *U* switch from plus to zero as $\mathbf{x}^2(T,\gamma)$. Define the *phase locus of the relay system with two relays* as vectors $\mathbf{x}^1(T,\gamma)$ and $\mathbf{x}^2(T,\gamma)$. Let us call the components of the *phase locus R*-characteristics.

To determine the parameters of the self-excited oscillations (period 2*T* and relative pulse duration *γ*) the following set of equation should be solved:

$$y_1^2(T,\gamma) = \kappa \ , \widetilde{Z}_1^-(T,\gamma) > 0 \ ,$$

$$y_2^1(T,\gamma) = -\kappa \ , Z_2^-(T,\gamma) < 0 \ .$$

where $y_1^2(T,\gamma)$, $y_2^1(T,\gamma)$ are *R*-characteristics pertaining to the input signals of the relays R_1 and R_2 respectively; $\widetilde{Z}_1^-(T,\gamma)$ is the value of $\dot{y}_1(t)$ in periodic motion at the time of relay R_1 switch from plus to minus; $Z_2^-(T,\gamma)$ is the value of $\dot{y}_2(t)$ in periodic motion at the time of relay R_2 switch from mines to plus.

Consider the components of the phase locus. Let the linear part of system be given by:

$$\dot{\mathbf{x}} = C\mathbf{x} + DU \ , \qquad (1)$$

where $C \in R^{n\times n}, D \in R^{n\times 1}, K \in R^{1\times n}$, *U* is a scalar control. Find vector x value at the time of the control U switch from zero to minus (this time corresponds to half of the period):

$$\mathbf{x}(T) = V(T)\mathbf{x}(0) + V(T-\gamma T)C^{-1}(V(\gamma T) - E)DA, \quad (2)$$

where $V(t)$ is the solution of equation $\dot{\mathbf{x}} = C\mathbf{x}$; *E* is the identity matrix of respective dimension. Formula (2) is valid if vector **x** is continuous on $t\in[0,T]$. Since the solution is symmetric on period 2T, for periodic motion the following holds: $\mathbf{x}(0) = -\mathbf{x}(T)$ and the first component of the phase locus $\mathbf{x}^1(T,\gamma)$ is:

$$\mathbf{x}^1(T,\gamma) = -(E + V(T))^{-1}V(T-\gamma T)C^{-1}(V(\gamma T) - E)DA.$$

The second component $\mathbf{x}^2(T,\gamma)$ can be determined as a solution of (2) at t=γT subject to control is $U = A$ and initial state is $\mathbf{x}^1(T,\gamma)$:

$$\mathbf{x}^2(T,\gamma) == V(\gamma T)\mathbf{x}^1(T,\gamma) + C^{-1}(V(\gamma T) - E)DA.$$

B. Input-output analysis of type zero relay servo system

Suppose the transfer function doesn't have zero poles. Also suppose that in the autonomous mode there exists symmetric oscillations with parameters T_0 and γ_0.

In the case of non-symmetric control (fig. 3) (it can occur when a constant input signal is introduced) the parameters of the oscillations can be defined by four function-vectors

$$\mathbf{x}^1 = \mathbf{x}^1(\tau_1,\tau_2,\tau_3,T), \mathbf{x}^2 = \mathbf{x}^2(\tau_1,\tau_2,\tau_3,T),$$

$$\mathbf{x}^3 = \mathbf{x}^3(\tau_1,\tau_2,\tau_3,T), \ \mathbf{x}^4 = \mathbf{x}^4(\tau_1,\tau_2,\tau_3,T).$$

Here $\mathbf{x}^1$ corresponds to the control *U* switch from zero to plus, $\mathbf{x}^2$ - from plus to zero, $\mathbf{x}^3$ - from zero to minus, $\mathbf{x}^4$ - from minus to zero.

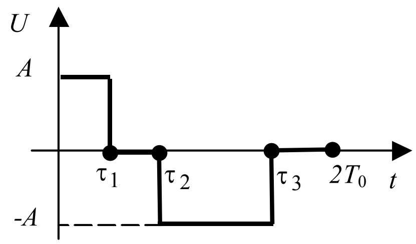

Fig. 3. Control signal.

Let the control *U* be as shown in the Fig. 3. Introduce a small constant input signal *f*. This will result in small deviations of the limit circle parameters $\Delta\tau_1$, $\Delta\tau_2$, $\Delta\tau_3$ and $\Delta\tau_4 = \Delta T$.

Let us write the conditions of proper relays switching with an accuracy of up to the first order of small quantity:

$$f\,W_2(0) - \left(y_2^1 + \frac{\partial y_2^1}{\partial \tau_1}\Delta\tau_1 + \frac{\partial y_2^1}{\partial \tau_2}\Delta\tau_2 + \right.$$

$$\left. + \frac{\partial y_2^1}{\partial \tau_3}\Delta\tau_3 + \frac{\partial y_2^1}{\partial T}\Delta T \right) = \kappa$$

$$f\,W_1(0) - \left(y_1^2 + \frac{\partial y_1^2}{\partial \tau_1}\Delta\tau_1 + \frac{\partial y_1^2}{\partial \tau_2}\Delta\tau_2 + \right.$$

$$\left. + \frac{\partial y_1^2}{\partial \tau_3}\Delta\tau_3 + \frac{\partial y_1^2}{\partial T}\Delta T \right) = -\kappa$$

$$f\,W_2(0) - \left(y_2^3 + \frac{\partial y_2^3}{\partial \tau_1}\Delta\tau_1 + \frac{\partial y_2^3}{\partial \tau_2}\Delta\tau_2 + \right.$$

$$\left. + \frac{\partial y_2^3}{\partial \tau_3}\Delta\tau_3 + \frac{\partial y_2^3}{\partial T}\Delta T \right) = -\kappa$$

$$f\,W_1(0) - \left(y_1^4 + \frac{\partial y_1^4}{\partial \tau_1}\Delta\tau_1 + \frac{\partial y_1^4}{\partial \tau_2}\Delta\tau_2 + \right.$$

$$\left. + \frac{\partial y_1^4}{\partial \tau_3}\Delta\tau_3 + \frac{\partial y_1^4}{\partial T}\Delta T \right) = \kappa \ . \qquad (3)$$

Here

$$y_2^1 = y_2^1(\tau_1,\tau_2,\tau_3,T)\Big|_{\substack{\tau_1=\gamma_0 T_0,\tau_2=T_0,\\ \tau_3=(1+\gamma_0)T_0,T=T_0}} = -y_2^3,$$

$$y_1^2 = y_1^2(\tau_1,\tau_2,\tau_3,T)\Big|_{\substack{\tau_1=\gamma_0 T_0,\tau_2=T_0,\\ \tau_3=(1+\gamma_0)T_0,T=T_0}} = -y_1^4,$$

$$\frac{\partial y_i^k}{\partial \tau_j} = \frac{y_i^k(\tau_1,\tau_2,\tau_3,T)}{\partial \tau_j}\Bigg|_{\substack{\tau_1=\gamma_0 T_0,\tau_2=T_0,\\ \tau_3=(1+\gamma_0)T_0,T=T_0}}, k=\overline{1,4}.$$

Consider symmetric periodic functions $U^i(t)$ corresponding to *R*-characteristics where time $t=0$ stands for respective relay switching. For $U^i(t)$ $t=0$ stands for relay R_2 switch from minus to plus. Asymmetry of control *U* can be caused by a variation of the time of any switch of either of the relays. Denote asymmetric control as follows:

$\tilde{U}_j^i(t) = U^i(t) + \delta U_j^i(t)$,

where $U^i(t)$ is a symmetric periodic function; $\delta U_j^i(t)$ is a control increment, which too is a 2*T*-periodic function and caused by the increment τ_j. The typical control increment is shown in Fig. 4

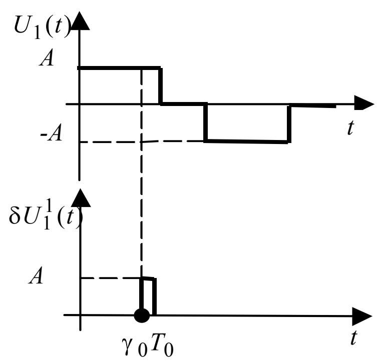

Fig. 4. Increment $\delta\, U_1^1(t)$.

Let us denote

$$\tilde{x}_j^1 = x^1(\tau_1,\ldots,\tau_j+\Delta\tau_j,\tau_{j+1},\ldots,T),$$

$$\tilde{x}_j^2 = x^2(\tau_1,\ldots,\tau_j+\Delta\tau_j,\tau_{j+1},\ldots,T),$$

$$\tilde{x}_j^3 = x^3(\tau_1,\ldots,\tau_j+\Delta\tau_j,\tau_{j+1},\ldots,T),$$

$$\tilde{x}_j^4 = x^4(\tau_1,\ldots,\tau_j+\Delta\tau_j,\tau_{j+1},\ldots,T).$$

By definition

$$\frac{\partial x^i}{\partial \tau_j} = \lim_{\Delta\tau_j\to 0}\frac{\tilde{x}_j^i - x^i}{\Delta\tau_j}. \qquad (4)$$

Since the superposition principle is valid for the linear part of the system, the following can be set forth for R-characteristics $\tilde{x}_j^i$

$$\tilde{x}_j^i = x^i + \delta x_j^i,$$

where δx_j^i - *R*-characteristic corresponding to periodic signal $\delta U_j^i(t)$. According to (4):

$$\frac{\partial x^i}{\partial \tau_j} = \lim_{\Delta\tau_j\to 0}\frac{\delta x_j^i}{\Delta\tau_j}. \qquad (5)$$

The analysis of increment $\delta U_j^i(t)$ proves the following formulas to be valid if the plant is linear:

$$\frac{\partial x^1}{\partial \tau_1} = -\frac{\partial x^3}{\partial \tau_3},\ \frac{\partial x^1}{\partial \tau_2} = -\frac{\partial x^2}{\partial \tau_3} = \frac{\partial x^4}{\partial \tau_1},\ \frac{\partial x^1}{\partial \tau_3} = -\frac{\partial x^3}{\partial \tau_1},$$

$$\frac{\partial x^2}{\partial \tau_1} = \frac{\partial x^1}{\partial \tau_2} + \frac{\partial x^4}{\partial \tau_2} - \frac{\partial x^2}{\partial \tau_2},\ \frac{\partial x^3}{\partial \tau_1} = \frac{\partial x^1}{\partial \tau_1} + \frac{\partial x^1}{\partial \tau_2} + \frac{\partial x^1}{\partial \tau_3},$$

$$\frac{\partial x^4}{\partial \tau_3} = \frac{\partial x^2}{\partial \tau_2} - \frac{\partial x^1}{\partial \tau_2} - \frac{\partial x^4}{\partial \tau_2},\ \frac{\partial x^1}{\partial T} = -\frac{\partial x^3}{\partial T},\ \frac{\partial x^2}{\partial T} = -\frac{\partial x^4}{\partial T}. \qquad (6)$$

Derive from (3) the following equalities with the use of equations (6):

$$\Delta T = 0,\ \Delta\tau_1 = \Delta\tau_2 - \Delta\tau_3, \qquad (7)$$

If the input signal is not introduced the following equalities are valid:

$$-y_2^1 = \kappa,\quad -y_1^2 = -\kappa,\ -y_2^3 = -\kappa,\quad -y_1^4 = \kappa.$$

Taking into consideration (7), rewrite equations (3) as:

$$\left(\frac{\partial y_2^1}{\partial \tau_1} - \frac{\partial y_2^1}{\partial \tau_3}\right)\Delta\tau_1 + \left(\frac{\partial y_2^1}{\partial \tau_2} + \frac{\partial y_2^1}{\partial \tau_3}\right)\Delta\tau_2 = f\,W_2(0),$$

$$\left(\frac{\partial y_1^2}{\partial \tau_1} - \frac{\partial y_1^2}{\partial \tau_3}\right)\Delta\tau_1 + \left(\frac{\partial y_1^2}{\partial \tau_2} + \frac{\partial y_1^2}{\partial \tau_3}\right)\Delta\tau_2 = f\,W_1(0). \qquad (8)$$

The equivalent gains of the relays can be found as quotients of averaged over the period control component *U* and averaged input signals of the relays y_1 and y_2.

$$K_1 = \frac{\overline{U_1}}{\overline{y_1}},\qquad K_2 = \frac{\overline{U_2}}{\overline{y_2}}. \qquad (9)$$

For $\overline{U_1}, \overline{U_2}, \overline{y}_1, \overline{y}_2$ the following formulas are obviously true:

$$\overline{U_1} = \frac{A(\Delta\tau_1 - \Delta\tau_3)}{2T_0},\ \overline{U_2} = \frac{A\Delta\tau_2}{2T_0}, \qquad (10)$$

$$\overline{y}_1 = f\,W_1(0) - \overline{U}\,W_1(0)\,W(0),$$

$$\overline{y}_2 = f\,W_2(0) - \overline{U}\,W_2(0)\,W(0). \qquad (11)$$

where $\overline{U} = \dfrac{A\Delta\tau_1}{T_0}$.

Considering (7), (9) – (11) the following formulas of the equivalent gains of the relays can be derived:

$$K_1 = \frac{A(2\Delta\tau_1 - \Delta\tau_2)}{f\,W_1(0)\,2T_0 - 2A\Delta\tau_1\,W_1(0)\,W(0)},$$

$$K_2 = \frac{A\Delta\tau_2}{f\,W_2(0)\,2T_0 - 2A\Delta\tau_1\,W_2(0)\,W(0)}. \qquad (12)$$

Variables $\Delta\tau_1$ and $\Delta\tau_2$ can be obtained from (8). As a result, the following final formulas of the equivalent gains can be obtained:

$$K_1 = \frac{A(2d_1 - d_2)}{d\, W_1(0)\, 2T_0 - 2Ad_1\, W_1(0)\; W(0)},$$

$$K_2 = \frac{Ad_2}{d\, W_2(0)\, 2T_0 - 2Ad_1\, W_2(0)\; W(0)}. \qquad (13)$$

where:

$$d = \left[\frac{\partial y_2^1}{\partial\tau_1} - \frac{\partial y_2^1}{\partial\tau_3}\right]\left[\frac{\partial y_1^2}{\partial\tau_2} + \frac{\partial y_1^2}{\partial\tau_3}\right] - \left[\frac{\partial y_1^2}{\partial\tau_1} - \frac{\partial y_1^2}{\partial\tau_3}\right]\left[\frac{\partial y_2^1}{\partial\tau_2} + \frac{\partial y_2^1}{\partial\tau_3}\right],$$

$$d_1 = W_2(0)\left[\frac{\partial y_1^2}{\partial\tau_2} + \frac{\partial y_1^2}{\partial\tau_3}\right] - W_1(0)\left[\frac{\partial y_2^1}{\partial\tau_2} + \frac{\partial y_2^1}{\partial\tau_3}\right],$$

$$d_2 = W_1(0)\left[\frac{\partial y_2^1}{\partial\tau_1} - \frac{\partial y_2^1}{\partial\tau_3}\right] - W_2(0)\left[\frac{\partial y_1^2}{\partial\tau_1} - \frac{\partial y_1^2}{\partial\tau_3}\right].$$

C. *Input-output analysis of type one relay servo system*

Let us consider input-output analysis of the system if the plant contains one zero pole. Let the system be as depicted in Fig. 5

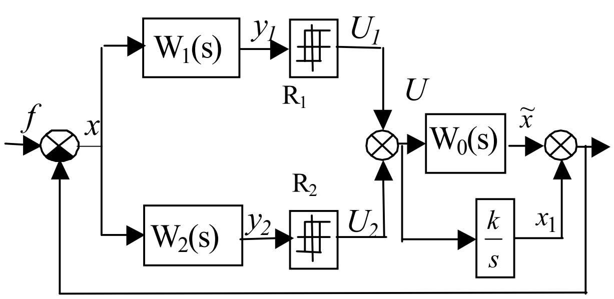

Fig. 5. Type one control system.

Let the signal U be

$$U(t) = U^0(t) + \overline{U},$$

where $U^0(t)$ is a symmetric $2T$-periodic function (Fig. 2).

If control U is asymmetric the following formula is valid:

$$x_1(t) = k\overline{U}\,t + \nu + x_1^0(t), \qquad (14)$$

where ν is a constant value determined by the initial conditions, $x_1^0(t)$ is a $2T$-periodic function with zero average over the period value. Asymmetric periodic motion may exist in the system if the input signal $f(t) = g_1 t + g_0$ is introduced. Let in the steady oscillatory process the ramp input applied to the relays be:

$$L^{-1}(f(s)W_1(s)) = a_1 t + a_0, \quad L^{-1}(f(s)W_2(s)) = b_1 t + b_0.$$

Write the conditions of relay switches with an accuracy of up to the first order of small quantity – with the use of equations (7):

$$\left(\frac{\partial y_2^1}{\partial\tau_1} - \frac{\partial y_2^1}{\partial\tau_3}\right)\Delta\tau_1 + \left(\frac{\partial y_2^1}{\partial\tau_2} + \frac{\partial y_2^1}{\partial\tau_3}\right)\Delta\tau_2 = b_0 - \nu_2, \qquad (15)$$

Consider propagation of signal f through the filter $W_1(s)$ with the constant and ramp components being of interest, with the use of Laplace transform.

$$L^{-1}\left(\left(\frac{g_1}{s^2} + \frac{g_0}{s}\right)W_1(s)\right) = L^{-1}\left(\frac{g_1}{s^2}W_1(0) + \frac{g_1}{s}C_1 + \frac{g_0}{s}W_1(0) + Z(s)\right)$$

Here, $Z(s)$ is the Laplace transform of the motion component vanishing with time. It will not be taken into account below. Factor C_1 can be determined through the representation of $\frac{g_1}{s^2}W_1(s)$ as partial fractions and taking the factor at $\frac{g_1}{s}$. Obviously:

$$L^{-1}\left(\left(\frac{g_1}{s^2} + \frac{g_0}{s}\right)W_1(s)\right) = W_1(0)g_1 t + C_1 g_1 + W_1(0)g_0 + Z(t). \qquad (16)$$

Reasoning along similar lines, consider the input signal $f(t)$ propagation through the filter $W_2(s)$. Therefore, the following equations are valid with (15) and (16) taken into account:

$$b_0 = W_2(0)g_0 + C_2 g_1,$$
$$a_0 = W_1(0)g_0 + C_1 g_1. \qquad (17)$$

Here, C_2 is the factor at $\frac{g_1}{s}$ in the partial fraction representation (similar to C_1). In accordance with (14) and (16),

$$W_1(0)g_1 = k_1\overline{U},$$
$$W_2(0)g_1 = k_2\overline{U}, \qquad (18)$$

where k_1 and k_2 are the gains of the integrators in partial fraction representations of $(W_0(s) + \frac{k}{s})W_1(s)$ and $(W_0(s) + \frac{k}{s})W_2(s)$ respectively.

Since $\overline{U} = \frac{A\Delta\tau_1}{T_0}$, then

$$g_1 = \frac{k_1 A\Delta\tau_1}{2T_0 W_1(0)},$$

$$g_1 = \frac{k_2 A\Delta\tau_1}{2T_0 W_2(0)}. \qquad (19)$$

Considering (17), (19), and the following equations:

$\nu_1 = W_1(0)\nu$, $\nu_2 = W_2(0)\nu$,

the set of equations (15) can be rewritten as:

$$\left[\frac{\partial y_2^1}{\partial \tau_1}-\frac{\partial y_2^1}{\partial \tau_3}\right]\Delta\tau_1+\left[\frac{\partial y_2^1}{\partial \tau_2}+\frac{\partial y_2^1}{\partial \tau_3}\right]\Delta\tau_2-$$
$$-\frac{C_2 k_2 A\Delta\tau_1}{2T_0 W_2(0)}=W_2(0)(g_0-\nu),$$
$$\left[\frac{\partial y_1^2}{\partial \tau_1}-\frac{\partial y_1^2}{\partial \tau_3}\right]\Delta\tau_1+\left[\frac{\partial y_1^2}{\partial \tau_2}+\frac{\partial y_1^2}{\partial \tau_3}\right]\Delta\tau_2-$$
$$-\frac{C_1 k_1 A\Delta\tau_1}{2T_0 W_1(0)}=W_1(0)(g_0-\nu). \quad (20)$$

Let us denote $q=g_0-\nu$. The equivalent gains of the relays are determined as above (formulas (9)) but y_1 and y_2 are now equal to:

$$y_1=qW_1(0)-\widetilde{W}_1^*(0)\overline{U}+C_1 g_1,$$
$$y_2=qW_2(0)-\widetilde{W}_2^*(0)\overline{U}+C_2 g_1.$$

Here, $\widetilde{W}_1^*(s)$ and $\widetilde{W}_2^*(s)$ are the partial fraction representation of $(W_0(s)+\frac{k}{s})W_1(s)$ and $(W_0(s)+\frac{k}{s})W_2(s)$ respectively with the integrators being not considered. Now we can write the formulas of the equivalent gains of the relays:

$$K_1=\frac{A(2d_1-d_2)W_1(0)}{2(d\,(W_1(0))^2 T_0-Ad_1\,\widetilde{W}_1^*(0)W_1(0)+C_1 k_1 A d_1)},$$
$$K_2=\frac{Ad_2 W_2(0)}{2W_2(0)^2\left(d\,T_0-\frac{Ad_1\,\widetilde{W}_2^*(0)}{(W_2(0))^2}+\frac{C_2 k_2 A d_1}{(W_2(0))^2}\right)}. \quad (21)$$

In formulas (21):

$$d=\left[\frac{\partial y_2^1}{\partial \tau_1}-\frac{\partial y_2^1}{\partial \tau_3}-\frac{C_2 k_2 A}{2T_0 W_2(0)}\right]\left[\frac{\partial y_1^2}{\partial \tau_2}+\frac{\partial y_1^2}{\partial \tau_3}\right]-$$
$$-\left[\frac{\partial y_1^2}{\partial \tau_1}-\frac{\partial y_1^2}{\partial \tau_3}-\frac{C_1 k_1 A}{2T_0 W_1(0)}\right]\left[\frac{\partial y_2^1}{\partial \tau_2}+\frac{\partial y_2^1}{\partial \tau_3}\right],$$
$$d_1=W_2(0)\left[\frac{\partial y_1^2}{\partial \tau_2}+\frac{\partial y_1^2}{\partial \tau_3}\right]-W_1(0)\left[\frac{\partial y_2^1}{\partial \tau_2}+\frac{\partial y_2^1}{\partial \tau_3}\right],$$
$$d_2=W_1(0)\left[\frac{\partial y_2^1}{\partial \tau_1}-\frac{\partial y_2^1}{\partial \tau_3}-\frac{C_2 k_2 A}{2T_0 W_2(0)}\right]-$$
$$-W_2(0)\left[\frac{\partial y_1^2}{\partial \tau_1}-\frac{\partial y_1^2}{\partial \tau_3}-\frac{C_1 k_1 A}{2T_0 W_1(0)}\right].$$

D. Derivatives of R-characteristics

To calculate the equivalent gains of the relays it is necessary to know the derivatives $\partial x^i/\partial\tau_j$. Suppose that transfer functions $W_1^*(s)$ and $W_2^*(s)$ are represented as partial fractions. Obviously the following statements are true: $\frac{\partial y_1^i}{\partial\tau_j}=\sum_{l=1}^{n}\frac{\partial x_l^i}{\partial\tau_j}$, $\frac{\partial y_2^i}{\partial\tau_j}=\sum_{l=1}^{m}\frac{\partial x_l^i}{\partial\tau_j}$, where $\frac{\partial x_l^i}{\partial\tau_j}$ is the derivative corresponding to a particular fraction; n, m the number of partial fractions in transfer functions $W_1^*(s)$ and $W_2^*(s)$ respectively. Derivatives $\frac{\partial x_l^i}{\partial\tau_j}$ can be obtained from a variational equation pertaining to a particular partial fraction.

Let us obtain the formula of $\partial x^1/\partial\tau_1$ for transfer function $W(s)=\frac{k}{s+a}$. Variational equation for this element is:

$$\frac{d\,\delta x_1^1}{d\,t}+a\,\delta x_1^1=k\,\delta U_1^1 \quad (22)$$

Increment $\delta\,U_1^1(t)$ is depicted in Fig. 4. Control variation $\delta\,U_1^1(t)$ is approximately equal to δ-function: $\delta U_1^1(t)=A\Delta\tau_1\delta(t-\tau_1)$. Equation (22) has the following solution:

$$\delta x(t)=\delta x(0)\,e^{-at}+kAe^{-a(t-\tau_1)}\Delta\tau_1,\ 0\le t\le 2T. \quad (23)$$

In a periodic motion $\delta x(2T)=\delta x(0)$. Then according to (23), the following formula of derivative of *R*-characteristic can be obtained:

$$\frac{\partial x^1}{\partial\tau_1}=\frac{kAe^{T^0 a\gamma^0}}{e^{2T^0 a}-1}.$$

To obtain the formula of the derivative of *R*-characteristic for the element with transfer function $W(s)=\frac{Cs+D}{s^2+2\alpha s+\alpha^2+\beta^2}$, this transfer function can be represented as a sum of two transfer functions with complex conjugate coefficients.

Formulas of the derivatives *R*-characteristics for various elements are presented in Table 1. Note that the table provides not all the derivatives but only those, which are necessary for calculation of the equivalent gains of the relays.

Table 1 has the following notation

$$L_1=e^{T\alpha(2+\gamma)}\sin T\beta(2-\gamma)+e^{T\alpha\gamma}\sin T\beta\gamma,$$
$$L_2=e^{T\alpha(2+\gamma)}\cos T\beta(2-\gamma)-e^{T\alpha\gamma}\cos T\beta\gamma,$$
$$M_1=e^{T\alpha(3+\gamma)}\sin T\beta(1-\gamma)+e^{T\alpha(1+\gamma)}\sin T\beta(1+\gamma),$$
$$M_2=e^{T\alpha(3+\gamma)}\cos T\beta(1-\gamma)-e^{T\alpha(1+\gamma)}\cos T\beta(1+\gamma),$$
$$N_1=e^{T\alpha(3-\gamma)}\sin T\beta(1+\gamma)+e^{T\alpha(1-\gamma)}\sin T\beta(1-\gamma),$$
$$N_2=e^{T\alpha(3-\gamma)}\cos T\beta(1+\gamma)-e^{T\alpha(1-\gamma)}\cos T\beta(1-\gamma),$$

$$P_1 = e^{T\alpha(4-\gamma)} \sin T\beta\gamma + e^{T\alpha(2-\gamma)} \sin T\beta(2-\gamma),$$
$$P_2 = e^{T\alpha(4-\gamma)} \cos T\beta\gamma - e^{T\alpha(2-\gamma)} \cos T\beta(2-\gamma).$$

TABLE 1.

$W(s)$	$\frac{k}{s}$	$\frac{k}{s+a}$
$\frac{\partial x^1}{\partial \tau_1}$	$\frac{kA(\gamma-1)}{2}$	$\frac{kAe^{Ta\gamma}}{e^{2Ta}-1}$
$\frac{\partial x^1}{\partial \tau_2}$	0	$\frac{kAe^{Ta}}{e^{2Ta}-1}$
$\frac{\partial x^1}{\partial \tau_3}$	$-\frac{kA\gamma}{2}$	$\frac{kAe^{Ta(1+\gamma)}}{e^{2Ta}-1}$
$\frac{\partial x^2}{\partial \tau_1}$	$\frac{kA}{2}$	$\frac{kA[e^{Ta}+e^{Ta(2-\gamma)}-e^{Ta(1-\gamma)}]}{e^{2Ta}-1}$
$\frac{\partial x^2}{\partial \tau_2}$	$-\frac{kA\gamma}{2}$	$\frac{kAe^{Ta(1-\gamma)}}{e^{2Ta}-1}$
$\frac{\partial x^2}{\partial \tau_3}$	0	$-\frac{kAe^{Ta}}{e^{2Ta}-1}$
$W(s)$	$\frac{Cs+D}{s^2+2\alpha s+\alpha^2+\beta^2}$	
$\frac{\partial x^1}{\partial \tau_1}$	$\frac{A[(D-c\alpha)L_1+c\beta L_2]}{\beta(e^{4T\alpha}-2e^{2T\alpha}\cos 2T\beta+1)}$	
$\frac{\partial x^1}{\partial \tau_2}$	$\frac{A[(D-c\alpha)(e^{3T\alpha}+e^{T\alpha})\sin T\beta}{\beta(e^{4T\alpha}-2e^{2T\alpha}\cos 2T\beta+1)}+\frac{+c\beta(e^{3T\alpha}-e^{T\alpha})\cos T\beta)]}{\beta(e^{4T\alpha}-2e^{2T\alpha}\cos 2T\beta+1)}$	
$\frac{\partial x^1}{\partial \tau_3}$	$\frac{A[(D-C\alpha)M_1+c\beta M_2]}{\beta(e^{4T\alpha}-2e^{2T\alpha}\cos 2T\beta+1)}$	
$\frac{\partial x^2}{\partial \tau_1}$	$\frac{\partial x^1}{\partial \tau_2}+\frac{\partial x^4}{\partial \tau_2}-\frac{\partial x^2}{\partial \tau_2}$	
$\frac{\partial x^2}{\partial \tau_2}$	$\frac{A[(D-c\alpha)N_1+C\beta N_2]}{\beta(e^{4T\alpha}-2e^{2T\alpha}\cos 2T\beta+1)}$	
$\frac{\partial x^2}{\partial \tau_3}$	$-\frac{\partial x^1}{\partial \tau_2}$	
$\frac{\partial x^4}{\partial \tau_2}$	$\frac{A[(D-c\alpha)P_1+c\beta P_2]}{\beta(e^{4T\alpha}-2e^{2T\alpha}\cos 2T\beta+1)}$	

III. EXAMPLE OF INPUT-OUTPUT ANALYSIS

Let the transfer functions of system be:

$$W_0(s) = \frac{2}{(0.5s+1)(0.04s^2+0.2s+1)},\ W_1(s)=1,$$

$$W_2(s) = \frac{2}{0.8s+1},\ A=2,\ \kappa=0.02.$$

In the autonomous mode, stable self-excited oscillations of period $T_0 = 0.7$ and parameter $\gamma_0 = 0.58$ exist in the system. The equivalent gains of the relays calculated with the use of the above formulas are:

$$K_1 = 0.24,\ K_2 = 0.636$$

Simulation of the system via integration of the original differential equation (DE) with successive Fourier analysis was done for the proposed method accuracy assessment. Both frequency responses: calculated on the basis of the developed approach and through DE solution are presented in Table 2.

TABLE 2

Frequency, s^{-1}	Proposed approach		DE solution	
	ϕ,°	A	ϕ,°	A
0.1	-1.9	0.751	-1.879	0.752
0.2	-3.92	0.75357	-3.85	0.7538
0.3	-5.891	0.75631	-5.803	0.7562
0.4	-7.718	0.76015	-7.879	0.7594

The proposed method does not require the filter hypothesis to be valid, unlike the describing functions method. Moreover, the actual shape of the oscillations is taken into account when deriving the formulas of the equivalent gains.

IV. CONCLUSION

A method of input-output analysis of decentralized relay systems is proposed above. Although the proposed approach results in more complex final formulas and the analysis procedure compared to one relay case, it still remains reasonably simple and provides many advantages over the describing function method.

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