Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Stability Boundaries Analysis of Non-Autonomous Systems with Resonant Solutions Based on Subharmonic Melnikov Functions

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Abstract— **This paper addresses stability boundaries in non-autonomous systems. An analytical criterion for stability boundaries in one degree of freedom (time-periodic) perturbed Hamiltonian systems was recently proposed. The criterion evaluates basin boundaries of non-resonant solutions. This paper discusses the stability boundaries with respect to the resonant solutions based on the above result and subharmonic Melnikov functions. At first one degree of freedom perturbed (time-independent) Hamiltonian systems for the resonant solutions is derived using coordinates transformations and second order averaging. Then an approximate expression for the basin boundaries of the resonant solutions is obtained based on the above analytical criterion. This paper also exhibits the effectiveness of the approximate expression through a simple example.**

I. INTRODUCTION

In many engineering fields it is much important to evaluate stability boundaries of dynamical systems precisely. The *stability boundaries* are basin boundaries of stable equilibrium points or periodic solutions which correspond to stable operating conditions of practical systems. The stability boundaries of autonomous dynamical systems are discussed based on several analytical methods: Lyapunov's direct methods [1], [2], dynamical systems theory [3], [4], passivity-based approach [5], [6] and so forth. However, for non-autonomous systems, any analytical criterion for the stability boundaries was not proposed; thus the evaluation depended on numerical simulation such as cell-to-cell mapping [7]. It was hence strongly required to derive an analytical criterion for the stability boundaries of the non-autonomous systems.

In [8] we proposed an analytical criterion for stability boundaries in one degree of freedom (abbreviated as ODF) (time-periodic) perturbed Hamiltonian systems based on a Melnikov's perturbation method [9], [10], [11]. Our proposed criterion has some advantages in its easy and quick evaluation and is applicable to various engineering systems. The criterion addresses the basin boundaries of non-resonant solutions, and is not therefore effective if the genesis of resonant solutions happens in the non-autonomous systems. Although the resonant solutions and associated basin structures were phenomenologically discussed in [12], [13], [14], [15], [16] and so forth, any analytical and effective approach to the basin boundaries of the resonant solutions has not been reported.

The present paper discusses basin boundaries of the resonant solutions based on subharmonic Melnikov functions and related theory [9], [10], [11], [17], [18], thereby obtaining an approximate expression for the basin boundaries. At first ODF (time-independent) perturbed Hamiltonian system for the resonant solutions is derived using coordinates transformations and second order averaging. Applying our proposed criterion to the perturbed Hamiltonian system, we obtain an approximate expression for the basin boundaries of the resonant solutions. The approximate expression can be derived with the original Hamiltonian system and enables for us to clarify the stability boundaries with respect to the resonant solutions analytically. Utilizing the expression we can also evaluate the stability boundaries of the non-autonomous systems with both non-resonant and resonant solutions.

II. SYSTEM MODEL AND PRELIMINARIES

This paper deals with ODF (time-periodic) perturbed Hamiltonian system as follows:

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = \mathrm{J}\mathrm{D}H(\boldsymbol{q}) + \varepsilon\boldsymbol{g}(\boldsymbol{q},t), \tag{1}$$

where H denotes the Hamiltonian function and $\varepsilon\,(\geq 0)$ the small parameter, and $\boldsymbol{q} \triangleq (x,y)^{\mathrm{T}}$ $(x, y \in \mathbf{R})$,

$$\begin{cases} \mathrm{J} & \triangleq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ \mathrm{D}H(\boldsymbol{q}) & \triangleq \left(\dfrac{\partial H}{\partial x}(x,y), \dfrac{\partial H}{\partial y}(x,y)\right)^{\mathrm{T}}, \\ \boldsymbol{g}(\boldsymbol{q},t) & \triangleq (g_1(x,y,t), g_2(x,y,t))^{\mathrm{T}}. \end{cases} \tag{2}$$

The symbol T denotes the transpose operation of vectors. $\boldsymbol{g}(\boldsymbol{q},t)$ has the periodicity of $T(=2\pi/\Omega)$ for t. The right-hand side of the system (1) is assumed to be tractable in the region we are interested in. Additionally the system (1) under $\varepsilon = 0$ holds the following assumptions [10], [11], [17], [18]: The assumed phase structure is schematically shown in Fig. 1.

Assumption 1 For $\varepsilon = 0$ the system (1) possesses a homoclinic orbit $\Gamma^0 \triangleq \{\boldsymbol{q}^0(t)\,|\,t \in \mathbf{R}\}$ to a saddle point $\boldsymbol{p}_0$.

This research is supported in part by the Ministry of Education, Culture, Sports, Sciences and Technology in Japan, The 21st Century COE Program No. 14213201, and Grant-in-Aid for Scientific Research (C) No. 14550264, 2003.

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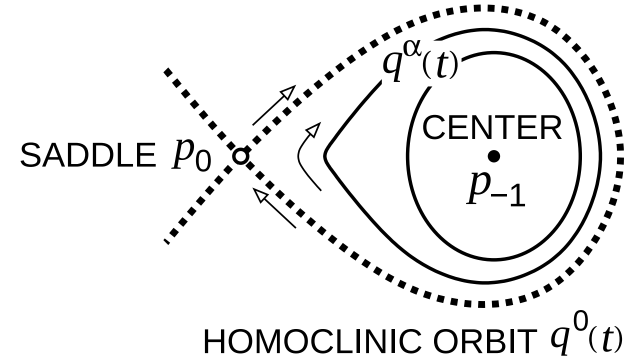

Fig. 1. Assumed phase structure of non-autonomous system (1) under $\varepsilon = 0$.

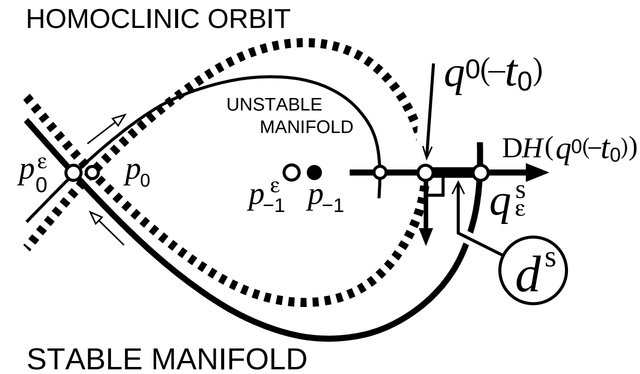

Fig. 2. Schematic phase structure of discrete dynamical system $P^{\varepsilon}_{\phi_0}$ for sufficiently small ε.

Assumption 2 The interior of $\Gamma^0 \cap \{\boldsymbol{p}_0\}$ is filled with a continuous family of periodic orbits $\boldsymbol{q}^{\alpha}(t), \alpha \in (-1, 0)$ with period T_α. Letting $d(\boldsymbol{x}, \Gamma^0) \triangleq \inf_{\boldsymbol{q} \in \Gamma^0} |\boldsymbol{x} - \boldsymbol{q}|$, we have $\lim_{\alpha \to 0} \sup_{t \in \mathbf{R}} d(\boldsymbol{q}^{\alpha}(t), \Gamma^0) = 0$ and $\lim_{\alpha \to 0} T_\alpha = +\infty$. In addition the system (1) under $\varepsilon = 0$ possesses a center $\boldsymbol{p}_{-1}$ surrounded by the continuous family of the periodic orbits.
Assumption 3 T_α is a differentiable function of the Hamiltonian value $h_\alpha \triangleq H(\boldsymbol{q}^{\alpha}(t))$ and $\mathrm{d}T_\alpha / \mathrm{d}h_\alpha > 0$ inside $\Gamma^0 \cap \{\boldsymbol{p}_0\}$.

A discrete dynamical system is introduced for the transformed system as follows:

$$\begin{cases} \dfrac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = \mathrm{J}\mathrm{D}H(\boldsymbol{q}) + \varepsilon \boldsymbol{g}(\boldsymbol{q}, \phi), \\ \dfrac{\mathrm{d}\phi}{\mathrm{d}t} = \Omega, \end{cases} \tag{3}$$

where ϕ has the periodicity of 2π, that is, $\phi \in S^1$. If we take a global section $\Sigma_{\phi_0} \triangleq \{(\boldsymbol{q}, \phi) \in \mathbf{R}^2 \times S^1 \mid \phi = \phi_0 \in S^1\}$, for some fixed phase ϕ_0 the autonomous system (3) is transformed into a discrete dynamical system:

$$P^{\varepsilon}_{\phi_0} : \Sigma_{\phi_0} \to \Sigma_{\phi_0}. \tag{4}$$

A periodic orbit with the period T in the system (1) is transformed into a fixed point of the same type stability in $P^{\varepsilon}_{\phi_0}$. $P^{\varepsilon}_{\phi_0}$ is often called Poincaré map.

III. AN ANALYTICAL CRITERION FOR BASIN BOUNDARIES OF NON-RESONANT SOLUTIONS BASED ON MELNIKOV'S METHOD

This section briefly introduces an analytical criterion for the basin boundary of the non-resonant solution in the discrete dynamical system $P^{\varepsilon}_{\phi_0}$ based on the Melnikov's method. The detail derivation and numerical examples are shown in [8].

A. Non-Resonant and Resonant Solutions

Before starting the discussion let us confirm non-resonant and resonant solutions in the non-autonomous system (1). A *non-resonant solution* $\boldsymbol{\gamma}^{\varepsilon}_0(t)$ in the system (1) is a periodic solution with the period T, which is represented as $\boldsymbol{\gamma}^{\varepsilon}_0(t) = \boldsymbol{p}_0 + \boldsymbol{\mathcal{O}}(\varepsilon)$, having the same stability as $\boldsymbol{p}_0$. For sufficiently small ε the non-resonant solution $\boldsymbol{\gamma}^{\varepsilon}_0(t)$ uniquely exists, and associated invariant manifolds are C^r-close to those of the unperturbed periodic orbit $\boldsymbol{p}_0 \times S^1$ [10], [11], [17], [18]; Here $\boldsymbol{\gamma}^{\varepsilon}_0(t)$ is transformed into a fixed point $\boldsymbol{p}^{\varepsilon}_0$ of $P^{\varepsilon}_{\phi_0}$. Furthermore we make the following assumption pertinent to the basin boundary:

Assumption 4 A non-resonant fixed point $\boldsymbol{p}^{\varepsilon}_{-1}$, associated with $\boldsymbol{p}_{-1}$, of $P^{\varepsilon}_{\phi_0}$ uniquely exists and is asymptotically stable.

Assumption 4 is necessary for the evaluation of the basin boundary of the non-resonant solution $\boldsymbol{p}^{\varepsilon}_{-1}$. On the other hand a *resonant solution* in the system (1) is a periodic solution, which associated fixed or periodic point of $P^{\varepsilon}_{\phi_0}$ does not coincide with both $\boldsymbol{p}^{\varepsilon}_0$ and $\boldsymbol{p}^{\varepsilon}_{-1}$.

B. Derivation of An Analytical Criterion

Based on the above discussion Fig. 2 shows the schematic phase structure of the discrete dynamical system $P^{\varepsilon}_{\phi_0}$ for sufficiently small ε. In the figure $\boldsymbol{q}^0(-t_0)$ denotes a point on the homoclinic orbit Γ^0 as a parameter $t_0 \in \mathbf{R}$ and $\mathrm{D}H(\boldsymbol{q}^0(-t_0))$ the normal vector at the point $\boldsymbol{q}^0(-t_0)$. In addition, $\boldsymbol{q}^{\mathrm{s}}_{\varepsilon}$ represents the intersection point of the normal vector $\mathrm{D}H(\boldsymbol{q}^0(-t_0))$ and the stable manifold, which possibly coincides with the basin boundary of $\boldsymbol{p}^{\varepsilon}_{-1}$, of the saddle point $\boldsymbol{p}^{\varepsilon}_0$.

We now derive an analytical criterion for the basin boundary of $\boldsymbol{p}^{\varepsilon}_{-1}$ in $P^{\varepsilon}_{\phi_0}$ using the distance between $\boldsymbol{q}^0(-t_0)$ on Γ^0 and $\boldsymbol{q}^{\mathrm{s}}_{\varepsilon}$ on the stable manifold. The distance $d^{\,\mathrm{s}}(\boldsymbol{q}^0(-t_0), \phi_0, \varepsilon)$ is defined as follows:

$$\begin{aligned} d^{\,\mathrm{s}}\left(\boldsymbol{q}^0(-t_0), \phi_0, \varepsilon\right) &= \varepsilon \frac{\Delta^{\mathrm{s}}_1\left(\boldsymbol{q}^0(-t_0), \phi_0\right)}{|\mathrm{D}H(\boldsymbol{q}^0(-t_0))\,|} + \mathcal{O}(\varepsilon^2) \\ &\triangleq d^{\,\mathrm{s}}_1\left(\boldsymbol{q}^0(-t_0), \phi_0\right) + \mathcal{O}(\varepsilon^2) \end{aligned} \tag{5}$$

where $\Delta_1^{\rm s}\left(\boldsymbol{q}^0(-t_0),\phi_0\right)$ is given by

$$\begin{aligned}\Delta_1^{\rm s}\left(\boldsymbol{q}^0(-t_0),\phi_0\right) = -\int_{-t_0}^{+\infty} {\rm D}H(\boldsymbol{q}^0(t)) \\ \cdot\, \boldsymbol{g}\left(\boldsymbol{q}^0(t), \mathit{\Omega}(t+t_0)+\phi_0\right){\rm d}t. \quad (6)\end{aligned}$$

It is then expected that for sufficiently small ε the following modified point $\boldsymbol{q}^{0\prime}(-t_0,\phi_0)$ is close to the stable manifold:

$$\begin{aligned}\boldsymbol{q}^{0\prime}(-t_0,\phi_0) \triangleq \boldsymbol{q}^0(-t_0) \\ + \frac{d_1^{\rm s}(\boldsymbol{q}^0(-t_0),\phi_0)}{|{\rm D}H(\boldsymbol{q}^0(-t_0))|}{\rm D}H(\boldsymbol{q}^0(-t_0)). \quad (7)\end{aligned}$$

We hence propose the modified homoclinic orbit $\mathit{\Gamma}_{\phi_0}^{0\prime}$ as an analytical criterion for the basin boundary of $\boldsymbol{p}_{-1}^{\varepsilon}$ in $P_{\phi_0}^{\varepsilon}$:

$$\mathit{\Gamma}_{\phi_0}^{0\prime} \;\triangleq\; \{\boldsymbol{q}^{0\prime}(-t_0,\phi_0)\,|\, t_0\in\mathbf{R} \text{ and } \phi_0\in S^1\}. \quad (8)$$

Remark 1 Our proposed criterion has the following advantages:

- The criterion can be calculated with the information about the system (1) under $\varepsilon = 0$, that is, the integrable system.
- Since the criterion is based on the stable manifold which possibly coincides with the true stability boundary, the criterion is not conservative such as the Lyapunov's direct methods for the autonomous systems.

On the other hand we can indicate the disadvantages of our proposed criterion as follows:

- The criterion does not provides us with any sufficient condition for the basin boundary of the non-resonant solution.
- The criterion cannot necessarily grasp various stability boundaries which possibly appear in the system (1): for examples, (i) genesis of resonant solutions and associated basin boundaries, (ii) fractal growth in the basin boundary of the non-resonant solution.

In Section IV, to get rid of the disadvantage (i), we show an analytical approach to the basin boundaries of the resonant solutions.

Remark 2 If more than one saddle point with associated homoclinic orbit and family of periodic orbits exist in the system (1), our proposed criterion can be applied to each homoclinic orbit.

IV. AN ANALYTICAL APPROACH TO BASIN BOUNDARIES OF RESONANT SOLUTIONS BASED ON SUBHARMONIC MELNIKOV FUNCTIONS

This section discusses the basin boundaries of the resonant solutions based on the subharmonic Melnikov functions and related theory [9], [10], [11], [17], [18].

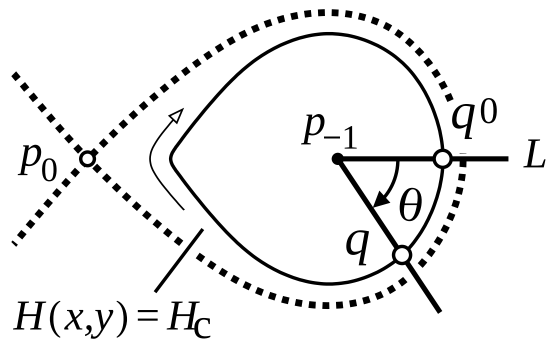

Fig. 3. Conceptual diagram of action angle coordinates transformation.

A. Action Angle Coordinates Transformation

At first, in the interior of $\mathit{\Gamma}^0 \cap \{\boldsymbol{p}_0\}$ the system (1) can be transformed into another system via an action angle coordinates transformation [9], [11]. Fig. 3 shows the conceptual diagram of the action angle transformation in the system (1) under $\varepsilon = 0$. The transformation can be found as follows:

$$\left\{\begin{array}{lclcl} I &=& T_I(x,y) &\triangleq& \dfrac{1}{2\pi}\displaystyle\oint_{H(x,y)=H_{\rm c}} y{\rm d}x, \\ \theta &=& T_\theta(x,y) &\triangleq& \dfrac{2\pi}{T(H_{\rm c})}s(x,y), \end{array}\right. \quad (9)$$

where $T(H_{\rm c})$ denotes the period of the periodic orbit satisfying $H(x,y) = H_{\rm c}(=\text{const.})$. $s(x,y)$ represents the time taken for the solution starting at a prefixed point on the periodic orbit $H(x,y) = H_{\rm c}$ to reach $\boldsymbol{q}$. The transformation (9) is one of canonical transformations, and its differentiable inverse exists: $x = T_x(I,\theta)$ and $y = T_y(I,\theta)$. As a result the system (1) is represented as follows:

$$\left\{\begin{array}{lcccl} \dfrac{{\rm d}I}{{\rm d}t} &=& & & \varepsilon\left(\dfrac{\partial T_I}{\partial x}g_1 + \dfrac{\partial T_I}{\partial y}g_2\right), \\ &\triangleq& & & \varepsilon F(I,\theta,t), \\ \dfrac{{\rm d}\theta}{{\rm d}t} &=& \tilde{\mathit{\Omega}}(I) &+& \varepsilon\left(\dfrac{\partial T_\theta}{\partial x}g_1 + \dfrac{\partial T_\theta}{\partial y}g_2\right), \\ &\triangleq& \tilde{\mathit{\Omega}}(I) &+& \varepsilon G(I,\theta,t), \end{array}\right. \quad (10)$$

where $\tilde{\mathit{\Omega}}(I)$ stands for the angular frequency of the periodic orbit satisfying $H(I) = H_{\rm c}(=\text{const.})$. The functions F and G apparently have the periodicity of T for t. A discrete dynamical system $\tilde{P}_{\phi_0}^{\varepsilon}$ can also be defined by the transformed system as follows:

$$\left\{\begin{array}{lcccl} \dfrac{{\rm d}I}{{\rm d}t} &=& & & \varepsilon F(I,\theta,\phi), \\ \dfrac{{\rm d}\theta}{{\rm d}t} &=& \tilde{\mathit{\Omega}}(I) &+& \varepsilon G(I,\theta,\phi), \\ \dfrac{{\rm d}\phi}{{\rm d}t} &=& \mathit{\Omega}. & & \end{array}\right. \quad (11)$$

B. One Degree of Freedom Perturbed Hamiltonian Systems for Resonant Solutions

We consider small perturbation in the neighborhood of the following resonance relation:

$$m\tilde{\Omega}(I^{m/n}) \;=\; n\Omega \;=\; n\frac{2\pi}{T}, \tag{12}$$

where m and n are relatively prime integers and $I^{m/n}$ an action value satisfying the above relation; A region near $I = I^{m/n}$ in $\tilde{P}^{\varepsilon}_{\phi_0}$ at $\phi_0 = 0$ (abbreviated as $\tilde{P}^{\varepsilon}_0$) is called a resonance band. The small perturbation is now introduced as follows:

$$I = I^{m/n} + \sqrt{\varepsilon}h, \quad \theta = \tilde{\Omega}(I^{m/n})t + \sigma. \tag{13}$$

The perturbation is regarded as a kind of van der Pol transformations [10], [11], [19]. A variational system is then obtained as follows:

$$\begin{cases} \dfrac{\mathrm{d}h}{\mathrm{d}t} &= \sqrt{\varepsilon}F(I^{m/n},\theta,t) \\ &+ \varepsilon\dfrac{\partial F}{\partial I}(I^{m/n},\theta,t)h + \mathcal{O}(\varepsilon^{3/2}), \\ \dfrac{\mathrm{d}\sigma}{\mathrm{d}t} &= \sqrt{\varepsilon}\dfrac{\partial\tilde{\Omega}}{\partial I}(I^{m/n})h \\ &+ \varepsilon\left\{\dfrac{1}{2}\dfrac{\partial^2\tilde{\Omega}}{\partial I^2}(I^{m/n})h^2 + G(I^{m/n},\theta,t)\right\} \\ &+ \mathcal{O}(\varepsilon^{3/2}). \end{cases} \tag{14}$$

We are in a position to derive ODF perturbed Hamiltonian system for the resonant solutions by applying the second order averaging [10], [11], [19], [20] to the variational system (14). A second order averaged system is obtained as follows:

$$\begin{cases} \dfrac{\mathrm{d}\overline{h}}{\mathrm{d}t} &= \mu\dfrac{1}{2\pi n}\bar{M}_1^{m/n}\left(\dfrac{\overline{\sigma}}{\tilde{\Omega}(I^{m/n})}\right) \\ &+ \varepsilon\overline{\dfrac{\partial F}{\partial I}}(\overline{\sigma})\overline{h}, \\ \dfrac{\mathrm{d}\overline{\sigma}}{\mathrm{d}t} &= \mu\dfrac{\partial\tilde{\Omega}}{\partial I}(I^{m/n})\overline{h} \\ &+ \varepsilon\left\{\dfrac{1}{2}\dfrac{\partial^2\tilde{\Omega}}{\partial I^2}(I^{m/n})\overline{h}^2 + \overline{G}(\overline{\sigma})\right\}, \end{cases} \tag{15}$$

where $\mu \triangleq \sqrt{\varepsilon}$ is treated as a dependent parameter, and

$$\begin{cases} \bar{M}_1^{m/n}\left(\dfrac{\overline{\sigma}}{\tilde{\Omega}(I^{m/n})}\right) \\ \triangleq \tilde{\Omega}(I^{m/n})\displaystyle\int_0^{mT} F(I^{m/n},\tilde{\Omega}(I^{m/n})\tau + \overline{\sigma},\tau)\mathrm{d}\tau, \\ \overline{\dfrac{\partial F}{\partial I}}(\overline{\sigma}) \\ \triangleq \dfrac{1}{mT}\displaystyle\int_0^{mT}\dfrac{\partial F}{\partial I}(I^{m/n},\tilde{\Omega}(I^{m/n})\tau + \overline{\sigma},\tau)\mathrm{d}\tau, \\ \overline{G}(\overline{\sigma}) \\ \triangleq \dfrac{1}{mT}\displaystyle\int_0^{mT} G(I^{m/n},\tilde{\Omega}(I^{m/n})\tau + \overline{\sigma},\tau)\mathrm{d}\tau. \end{cases} \tag{16}$$

$\bar{M}_1^{m/n}(\overline{\sigma}/\tilde{\Omega}(I^{m/n}))$ is well-known as a subharmonic Melnikov function [10], [11], [17], [18]. The averaged system (15) is ODF (time-independent) perturbed Hamiltonian system with the Hamiltonian $K(\overline{\sigma},\overline{h})$:

$$\frac{\mathrm{d}\overline{\boldsymbol{q}}}{\mathrm{d}t} \;=\; \mathrm{J}\mathrm{D}K(\overline{\boldsymbol{q}}) + \varepsilon\overline{\boldsymbol{g}}(\overline{\boldsymbol{q}}), \tag{17}$$

where $\overline{\boldsymbol{q}} \triangleq (\overline{\sigma},\overline{h})^{\mathrm{T}}$ and

$$\begin{cases} K(\overline{\boldsymbol{q}}) \\ \triangleq \mu\left\{\dfrac{\partial\tilde{\Omega}}{\partial I}(I^{m/n})\dfrac{\overline{h}^2}{2} - \displaystyle\int\bar{M}_1^{m/n}\left(\dfrac{\overline{\sigma}}{\tilde{\Omega}(I^{m/n})}\right)\dfrac{\mathrm{d}\overline{\sigma}}{2\pi n}\right\}, \\ \mathrm{D}K(\overline{\boldsymbol{q}}) \\ \triangleq \left(\dfrac{\partial K}{\partial\overline{\sigma}}(\overline{\boldsymbol{q}}), \dfrac{\partial K}{\partial\overline{h}}(\overline{\boldsymbol{q}})\right)^{\mathrm{T}}, \\ \overline{\boldsymbol{g}}(\overline{\boldsymbol{q}}) \\ \triangleq \left(\dfrac{1}{2}\dfrac{\partial^2\tilde{\Omega}}{\partial I^2}(I^{m/n})\overline{h}^2 + \overline{G}(\overline{\sigma}), \overline{\dfrac{\partial F}{\partial I}}(\overline{\sigma})\overline{h}\right)^{\mathrm{T}}. \end{cases} \tag{18}$$

Remark 3 The subharmonic Melnikov function $\bar{M}_1^{m/n}(\overline{\sigma}/\tilde{\Omega}(I^{m/n}))$ provides us with existence and stability conditions of the resonant fixed or periodic points in the discrete dynamical system $\tilde{P}^{\varepsilon}_0$. In [11] $\bar{M}_1^{m/n}(\overline{\sigma}/\tilde{\Omega}(I^{m/n}))$ is discussed in terms of $\tilde{P}^{\varepsilon}_{\phi_0}$ and its first derivation, and is derived as the existence condition for m resonant periodic points of $\tilde{P}^{\varepsilon}_0$. Roughly speaking, the condition of $\tilde{P}^{\varepsilon}_0$ is identical to the existence condition for equilibrium points in the averaged system (17) under $\varepsilon = 0$, that is, $\bar{M}_1^{m/n}(\overline{\sigma}/\tilde{\Omega}(I^{m/n})) = 0$ and $\overline{h} = 0$. In addition, if $\bar{M}_1^{m/n}(\overline{\sigma}/\tilde{\Omega}(I^{m/n}))$ possesses some zero points, then we have $2m$ zero points; Associated m equilibrium points have the saddle-type stability. This property directly leads to the assumptions which the averaged system (17) will hold in the next subsection.

Remark 4 The averaged system (17) possibly has the sufficient property to clarify the phase structure of the discrete dynamical system $\tilde{P}^{\varepsilon}_0$ qualitatively. As discussed in [10], [11], [17], [18], provided that $\partial\Omega(I^{m/n})/\partial I$ is bounded and sufficiently small ε, the second order averaging is generally relevant to determine the stability for all equilibrium points of the averaged system (17). In addition, through the phase structure of the averaged system (17), we can grasp the phase structure in the resonance band (12) of $\tilde{P}^{\varepsilon}_0$ qualitatively.

Remark 5 The phase structure of the averaged system (17) can be analytically examined based on the original system (1) under $\varepsilon = 0$. In addition we can understand the phase structure of the averaged system (17) under $\varepsilon = 0$ analytically because of its integrable property.

C. Main Result: An Approximate Expression for Basin Boundaries of Resonant Solutions

This subsection states the main result obtained in this paper: an approximate expression for the basin boundaries of the resonant solutions based on the averaged system (17) and our proposed criterion in Section III. In order to apply our proposed criterion to the averaged system (17), we introduce the following assumptions which are identical to Assumptions 1 and 2:

Assumption 5 For $\varepsilon = 0$ the averaged system (17) possesses a homoclinic orbit $\overline{\Gamma}^0_{(i)} \triangleq \{\overline{\boldsymbol{q}}^0_{(i)}(t) \,|\, t \in \mathbf{R}\}$ to each saddle point $\overline{\boldsymbol{p}}_{0(i)}$ for $i = 1, \cdots, m$.

Assumption 6 Each interior of $\overline{\Gamma}^0_{(i)} \cap \{\overline{\boldsymbol{p}}_{0(i)}\}$ is filled with a continuous family of periodic orbits $\overline{\boldsymbol{q}}^\alpha_{(i)}(t)$, $\alpha \in (-1, 0)$ with period $\overline{T}_{\alpha(i)}$. We assume $\lim\limits_{\alpha\to 0}\sup\limits_{t\in\mathbf{R}} d(\overline{\boldsymbol{q}}^\alpha_{(i)}(t), \overline{\Gamma}^0_{(i)}) = 0$ and $\lim\limits_{\alpha\to 0}\overline{T}_{\alpha(i)} = +\infty$. In addition the averaged system (17) under $\varepsilon = 0$ possesses a center $\overline{\boldsymbol{p}}_{-1(i)}$ surrounded by each continuous family of the periodic orbits.

As mentioned in Section III, for sufficiently small ε, each non-resonant equilibrium point $\overline{\boldsymbol{p}}^\varepsilon_{0(i)}$ related to $\overline{\boldsymbol{p}}_{0(i)}$ in the averaged system (17) uniquely exists and becomes a saddle point. In addition we make the following assumption:

Assumption 7 Each non-resonant equilibrium point $\overline{\boldsymbol{p}}^\varepsilon_{-1(i)}$ associated with $\overline{\boldsymbol{p}}_{-1(i)}$ for $i = 1, \cdots, m$ in the averaged system (17) uniquely exists and is asymptotically stable.

Thus, for sufficiently small ε, the averaged system (17) has the same phase structure as Fig. 2 qualitatively.

We now derive an approximate expression for the basin boundaries of the resonant solutions as same as the process in Section III. From the above discussion each modified homoclinic orbit $\overline{\Gamma}^{0\prime}_{(i)}$ for $i = 1, \cdots, m$ is obtained as follows:

$$\overline{\Gamma}^{0\prime}_{(i)} \triangleq \{\overline{\boldsymbol{q}}^{0\prime}_{(i)}(-\bar{t}_0) \mid \bar{t}_0 \in \mathbf{R}\}, \tag{19}$$

where $\bar{t}_0$ parameterizes each point on $\overline{\Gamma}^0_{(i)}$ and

$$\left\{\begin{aligned} &\overline{\boldsymbol{q}}^{0\prime}_{(i)}(-\bar{t}_0) \triangleq \overline{\boldsymbol{q}}^0_{(i)}(-\bar{t}_0) \\ &\qquad + \frac{\overline{d}^{\,\mathrm{s}}_{1(i)}(\overline{\boldsymbol{q}}^0_{(i)}(-\bar{t}_0))}{|\mathrm{D}K(\overline{\boldsymbol{q}}^0_{(i)}(-\bar{t}_0))|}\mathrm{D}K(\overline{\boldsymbol{q}}^0_{(i)}(-\bar{t}_0)), \\ &\overline{d}^{\,\mathrm{s}}_{1(i)}(\overline{\boldsymbol{q}}^0_{(i)}(-\bar{t}_0)) \triangleq \varepsilon\frac{1}{|\mathrm{D}K(\overline{\boldsymbol{q}}^0_{(i)}(-\bar{t}_0))|} \\ &\qquad \cdot\left(-\int_{-\bar{t}_0}^{+\infty}\mathrm{D}K(\overline{\boldsymbol{q}}^0_{(i)}(t))\cdot\overline{\boldsymbol{g}}(\overline{\boldsymbol{q}}^0_{(i)}(t))\mathrm{d}t.\right). \end{aligned}\right. \tag{20}$$

As shown in Section III each modified homoclinic orbit $\overline{\Gamma}^{0\prime}_{(i)}$ is close to the stable manifold, which coincides with the basin boundary of $\overline{\boldsymbol{p}}^\varepsilon_{-1(i)}$, of the saddle point $\overline{\boldsymbol{p}}^\varepsilon_{0(i)}$. We hence propose $\bigcup\limits_{i=1}^{m}\overline{\Gamma}^{0\prime}_{(i)}$ as an approximate expression for the basin boundaries of the resonant solutions.

Remark 6 The expression can be directly derived with the original Hamiltonian system (1) which is the integrable system.

Remark 7 To describe the expression $\bigcup\limits_{i=1}^{m}\overline{\Gamma}^{0\prime}_{(i)}$ in the original $x - y$ plane, it is necessary that each point on $\bigcup\limits_{i=1}^{m}\overline{\Gamma}^{0\prime}_{(i)}$ in $\bar{\sigma} - \bar{h}$ plane is transformed into a point in $x - y$ plane. The transformation is performed based on the following formula:

$$I = I^{m/n} + \sqrt{\varepsilon}\,\bar{h}, \quad \theta = \bar{\sigma}, \tag{21}$$

and

$$x = T_x(I, \theta), \quad y = T_y(I, \theta). \tag{22}$$

Remark 8 The expression approximately represents the basin boundaries of the resonant solutions. We obtained $\bigcup\limits_{i=1}^{m}\overline{\Gamma}^{0\prime}_{(i)}$ through the averaged system (17) which is derived by truncating the variational system (14) until $\mathcal{O}(\varepsilon)$ terms. Although the phase structure of the averaged system (17) is identical to that of $\tilde{P}^\varepsilon_0$ in the resonance band (12) qualitatively, it should be noted that the expression provides us with second order information about the basin boundaries of the resonant solutions.

D. Example

Our present approach is applied to the following concrete non-autonomous system [10], [18]:

$$\left\{\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y\{\Omega - (x^2 + y^2)\} \\ &+ \varepsilon\{\delta x - x(x^2 + y^2) + \gamma x\cos t\}, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -x\{\Omega - (x^2 + y^2)\} \\ &+ \varepsilon\{\delta y - y(x^2 + y^2)\}. \end{aligned}\right. \tag{23}$$

For $\varepsilon = 0$ the system (23) is ODF Hamiltonian system with the Hamiltonian $H(x, y)$:

$$H(x, y) = \Omega\frac{x^2 + y^2}{2} - \left(\frac{x^2 + y^2}{2}\right)^2. \tag{24}$$

The system (23) under $\varepsilon = 0$ does not have any hyperbolic saddle point and associated homoclinic orbit. However, in the interior of the circle $\{(x, y) \,|\, x^2 + y^2 = \Omega\}$ the Hamiltonian system possesses a center at the origin and a family of periodic solutions satisfying Assumption 3; We can therefore analyze the system (23). In addition, for $\varepsilon \neq 0$, $\gamma = 0$ and $0 < \delta < \Omega$, the system (23) has a non-resonant unstable focus at the origin and a stable limit cycle with the period $2\pi/(\Omega - \delta)$.

We now consider 1/2–harmonic entrainment and associated basin boundary in system (23). The entrainment is mathematically represented by the resonance relation (12) at $m = 2$ and $n = 1$. Using the following transformations:

$$x = \sqrt{2I}\sin\theta, \quad y = \sqrt{2I}\cos\theta, \tag{25}$$

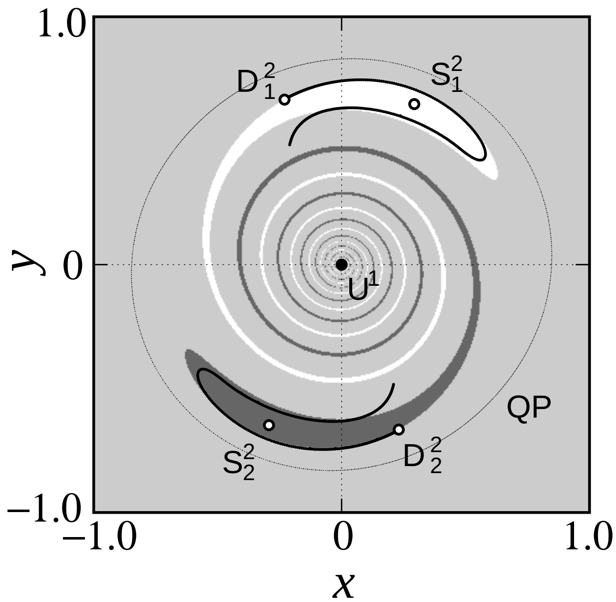

Fig. 4. Basin structure and approximate expression for discrete dynamical system P_0^ε associated with non-autonomous system (23). The fixed point U^1 is completely unstable, that is, source, and $\mathsf{S}(\text{or}\,\mathsf{D})_i^2$ $(i = 1, 2)$ stand for the completely stable (or directly unstable) 2-periodic points.

and

$$\theta = \frac{t}{2} + \sigma, \quad I = \frac{\omega}{2} + \sqrt{\varepsilon} h, \tag{26}$$

where $\omega \stackrel{\triangle}{=} \Omega - 1/2$, and averaging $\mathcal{O}(\varepsilon)$ terms we obtain the second order averaged system as follows:

$$\left\{\begin{array}{rcl} \dfrac{\mathrm{d}\sigma}{\mathrm{d}t} & = & \mu(-2h) \\ & + & \varepsilon \dfrac{\gamma}{4} \sin 2\sigma, \\ \dfrac{\mathrm{d}h}{\mathrm{d}t} & = & \mu\omega \left(\delta - \omega - \dfrac{\gamma}{4} \cos 2\sigma\right) \\ & + & \varepsilon \left\{2(\delta - 2\omega) - \dfrac{\gamma}{2} \cos 2\sigma\right\} h, \end{array}\right. \tag{27}$$

where $\mu \stackrel{\triangle}{=} \sqrt{\varepsilon}$ and all single bars are dropped. For $\varepsilon = 0$ the averaged system (27) is ODF Hamiltonian system with the Hamiltonian $K(\sigma, h)$:

$$K(\sigma, h) = \mu \left[-h^2 - \omega \left\{(\delta - \omega)\sigma - \frac{\gamma}{8} \sin 2\sigma\right\}\right]. \tag{28}$$

By calculating the approximate expression for the averaged system (27) we can discuss the basin boundary in the non-autonomous system (23).

Figure 4 shows the basin structure and the approximate expression in the discrete dynamical system P_0^ε associated with the system (23) at $\Omega = 1.0$, $\varepsilon = 0.05$, $\delta = 0.7$ and $\gamma = 1.1$. In the figure the *black* point, plotted at the origin, denotes the source U^1 and QP the quasi-periodic solution. The *white* and *dark-gray* regions in Fig. 4 represent the basins of 2-periodic points S_i^2 $(i = 1, 2)$ and the *light-gray* region the basin of QP. The approximate expression is described with two *black* solid lines, and precisely grasps a part of the basin boundaries of S_i^2. This example shows the effectiveness of the approximate expression although it can not clarify the detail of the basin boundaries of S_i^2.

ACKNOWLEDGMENTS

The authors would like to express their sincere gratitude to Professor Philip J. Holmes, Princeton University for his suggestive advice. The first author greatly appreciates stimulating discussions with Professor Yoshisuke Ueda and his research group, Future University–Hakodate.

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