Nonlinear Stability Analysis for Non-polynomial Systems

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Abstract— **In this paper the stability analysis of nonlinear systems is studied through different approaches. The main idea of the paper is to map the original class of nonlinear systems into a smaller subclass of systems described by multivariate polynomial functions, for which the study of stability is available.**

I. INTRODUCTION

The objective of this work is to propose tools to study the stability of a large class of nonlinear systems using Lyapunov methods. From Lyapunov stability theory [8], we know that the stability of an autonomous system $\dot{x}(t) = f(x), x \in \mathbf{R}^n$ can be investigated by checking the sign definiteness of the function $\dot{V} = \frac{\delta V(x)}{\delta x} f(x)$ where $V(x) \geq 0$ is a Lyapunov function candidate. For the class of systems where $f(x)$ is a multivariate polynomial in the components of x, there are several available tools for stability analysis and design. Such tools include Quantifier eliminations (QE) [5], [16], Branch and Bound techniques [6], [10], probabilistic and statistical learning methods [1], and several positivity tests [4].

In [19], [20] approximation techniques are used to transform a nonlinear system that does not satisfy the involutivity conditions required for feedback linearization, into a feedback-linearizable system. Our work in contrast proposes different approximation techniques, the main objective being to transform the system into a polynomial form. Although in this work we are only concerned with stability analysis, all the techniques proposed can be extended to design by considering *Control Lyapunov Function* (CLF) [7]. The main advantage of multivariate polynomial approximations for $f(x)$ is that, if $V(x)$ is selected to be a multivariate polynomial function, then $\dot{V}$ is also a multivariate polynomial function, and Lyapunov stability tests are reduced to the study of sign-definiteness of multivariate polynomial functions.

II. DEFINITIONS AND NOTATION

We recall some standard definitions, and notations.

Definition 1: **Multivariate Monomials.** A multivariate real monomial of degree m, in n variables in $\mathbf{R}$ is a function defined as $M(m,n) := x_1^{m_1} x_2^{m_2} x_3^{m_3} ... x_n^{m_n}$, for $x = [x_1, \ldots, x_n] \in \mathbf{R}^n$ and the degree m of a monomial is defined as $m = \deg\{M(m,n)\} := \Sigma_{i=1}^{n} m_i$.

Definition 2: **Multivariate Polynomials.** A multivariate polynomial of degree m in n variables in $\mathbf{R}$ is a function defined as a finite sum of multivariate monomials $G_{m,n} := \Sigma_{i=1}^{N} M_i(m,n)$ where N is the number of monomials added in $G_{m,n}$. The degree of the polynomial is defined as $m = \max_i m_i$, where m_i are the degrees of the monomials.

Definition 3: **PSD Polynomials.** We define $P_{n,m}$ as the set of positive semi-definite (PSD) polynomials of degree m, with m even number, in n variables, i.e.

$$P_{n,m} := \{p \in \mathbf{R}[x] : p(x) \geq 0, \deg\{p\} = m, \forall x \in \mathbf{R}^n\} \quad (1)$$

Where $\mathbf{R}[x]$, is the set of polynomials with real coefficients in the variable $x = [x_1, \ldots, x_n]$.

Definition 4: **SOS Polynomials.** We define $\Sigma_{n,m}$ as the set of sum-of-squares polynomials in n variables, and degree $\frac{m}{2}$, where m is even, i.e.

$$\Sigma_{n,m} := \{p(x) = \Sigma_k h_k^2; \deg\{p\} = m, \deg\{h\} = \frac{m}{2};$$
$$p, h_k \in \mathbf{R}[x], x \in \mathbf{R}^n\} \quad (2)$$

We proceed by giving a formal definition of the class of functions we will be dealing with throughout this paper.

Definition 5: Consider the class S of a nonlinear multivariate function defined as follows

$$S = \quad \{f : f(x) = \sum_{i=1}^{l} p_i(x)g_i(x), p_i : \mathbf{R}^n \to \mathbf{R}, monomials$$
$$g_i : D_1^n \to D_2, nonlinear\ functions, D_1^n \subseteq \mathbf{R}^n,$$
$$D_2 \subseteq \mathbf{R}, l \in \mathbb{N}\} \quad (3)$$

In particular, the elements of S are sums of polynomial functions, non-polynomial functions, and product of both.

Definition 6: Recall again the class S of multivariate functions defined in (5) as the functions composed by sum of terms in which there are polynomial and non-polynomial elements. Consider a subset $S_1 \subset S$ in which a part of the variables only appear in the polynomial functions $p(x)$, i.e.

$$f(x) = \quad \sum_i p_i(x)g_i(x_g), x \in \mathbf{R}^n, x_g \in \mathbf{R}^{(n-k)} \quad (4)$$
$$x_g(j) = x(j), j = k+1, ..., n, k \leq n$$

where p_i are multivariate polynomial functions and g_i are multivariate non polynomial functions. Observe that the first k components of x only appear in the polynomial part and form a so-called polynomial vector $x_p(i) = x(i), i = 1, \ldots, k$ where $x_p \in \mathbf{R}^k$. We refer to these variables as "polynomial variables" and to the remaining variables in the state vector as "global variables", and they form a "global vector" $x_g(j) = x(j), j = k+1, \ldots, n$ and $x_g \in \mathbf{R}^{(n-k)}$, we have $x = [x_p \ x_g]^T$. The meaning of such notation will be made clearer next.

Definition 7: Consider the class of systems that produce a derivative of the quadratic Lyapunov function along the trajectory that belongs to the class S_1 defined in (6), we refer to this class of system as "decoupled state systems", in which the state vector can be split into two parts, the first part of the state vector only contribute to the dynamic of the system through polynomial functions, i.e.

$$\dot{x} = P(x)G(x_g), \; x \in \mathbf{R}^n, x_g \in \mathbf{R}^{(n-k)} \tag{5}$$
$$x_g(j) = x(j) \;\; j = k+1,...,n, \; k \leq n$$

Where P and G are respectively a vector polynomial function and a vector non-polynomial function. Consider the state vector x and split it into two parts $x = [\xi \; \varphi]^T$, in which ξ is the vector of polynomial variables and φ is the vector of global variables as defined in (6), we can then rewrite (5) as follows

$$\begin{bmatrix} \dot{\xi} \\ \dot{\varphi} \end{bmatrix} = P(\xi, \varphi)G(\varphi) \tag{6}$$
$$\xi(i) = x_i \;\; i = 1,...,k \tag{7}$$
$$\varphi(j) = x_j \;\; j = k+1,...n \;\; k \leq n. \tag{8}$$

Next we present conditions for the stability of a nonlinear system based on the stability of its polynomial approximation.

III. POLYNOMIAL APPROXIMATIONS

In this section, we study the local stability of a nonlinear system through its approximation with a polynomial system. This analysis is valid as long as our system function is within an error ε from the polynomial approximation. Our results, of course, hinge on the fact that we are able to get a 'good' approximation. Several references on approximation with multivariate polynomials can be found in [11], [12]. The main idea of approximating with multivariate polynomials is to sample (deterministically or probabilistically) the original function, the to interpolate the samples using polynomials. Since almost all approximation schemes are valid in a region, our stability results will naturally be local For our purposes, we assume that a polynomial that approximates the system function on the region of interest is available.

A. Approximated by Polynomial Functions

In most Lyapunov tests, we have little knowledge on how to verify the sign-definiteness of the resulting complicated multivariate functions. Since many tools are however available for determining the sign-definiteness of polynomial functions, a potential solution to the original problem is to consider the approximation of a generic nonlinear function by a polynomial, and to study how the local stability of the original system may be deduced to the local stability of the approximated system. This is the case of the next theorem.

Theorem 1: Consider the nonlinear system

$$\dot{x} = f(x), \; f(0) = 0, \; x \in \mathbf{R}^n \tag{9}$$

where f is a vector function $f : D_1^n \rightarrow D_2^n$ where $D_1^n, D_2^n \subseteq \mathbf{R}^n$, and $f_i, i = 1,...,n$ are continuous multivariate functions. Also consider the polynomial approximation p of f, with $p : \mathbf{R}^n \rightarrow \mathbf{R}^n$ on the intervals $[a_i, b_i], i = 1,...,n$, with a bound on the approximation error given by $\varepsilon = [\varepsilon_1 \, \varepsilon_2 \, ... , \varepsilon_n]^T$, i.e.

$$|f_i(x) - p_i(x)| < \varepsilon_i, \forall x_i \in [a_i, b_i], i = 1,...,n. \tag{10}$$

From now on we will use the notation $x \in [a, b]$, meaning $x_i \in [a_i, b_i], i = 1,...,n$, also $|A|_n$ will denote the absolute value of all the elements of the $n \times 1$ vector A. Assume we can find a quadratic Lyapunov function for the polynomial system $V_p = x^T Q x$ where $Q = Q^T$ is a positive definite, $n \times n$ matrix. Then, the original system is stable in $[a_i, b_i], i = 1,...,n$ if and only if

$$\dot{V}_p = \frac{\partial V_p}{\partial x} p(x) < -(|x^T Q|_n \varepsilon + \varepsilon^T |Qx|_n), \forall x \in [a, b] \tag{11}$$

Proof: We will denote with V_f the Lyapunov function of the original system, and with V_p that of the approximated system. We choose $V_f = V_p = x^T Q x$. Then

$$\dot{V}_f = x^T Q f(x) + f^T(x) Q x \tag{12}$$
$$\dot{V}_p = x^T Q p(x) + p^T(x) Q x \tag{13}$$

Let $\Delta_v = \dot{V}_f - \dot{V}_p$. Then two cases might arise:

1) $\Delta_v > 0$: we get

$$\dot{V}_f - \dot{V}_p = x^T Q(f(x) - p(x)) + (f(x) - p(x))^T Q x$$
$$\leq |x^T Q(f(x) - p(x)) + (f(x) - p(x))^T Q x|$$
$$\leq |x^T Q|_n(\varepsilon) + (\varepsilon)^T |Qx|_n \tag{14}$$

from which we can conclude that a necessary and sufficient condition for $\dot{V}_f$ to be negative is that

$$\dot{V}_p \leq -(|x^T Q|_n \varepsilon + \varepsilon^T |Qx|_n), \forall x \in [a, b] \tag{15}$$

2) $\Delta_v < 0$: In this case we observe that from the condition $\Delta_v < 0$, it follows that $\dot{V}_p - \dot{V}_f > 0$ from which we guarantee the negativity of $\dot{V}_f$ directly from the negativity of $\dot{V}_p$, i.e. $\dot{V}_p \leq 0$. Since condition (15) is stronger than this last condition, we can summarize the result and state that the local stability of the original system for $x \in [a, b]$ is guaranteed for $\dot{V}_p \leq -(|x^T Q|_n \varepsilon + \varepsilon^T |Qx|_n), \forall x \in [a, b]$. ∎

B. Stability of Perturbed Systems

Using stability results of perturbed systems [8], we can state sufficient conditions for a nominally stable system to remain stable after it is subjected to a perturbation depending on the size of the perturbation. We consider the polynomial approximation of the original system as the nominal system, and the original system as the perturbed system. In particular consider the nonlinear system

$$\dot{x} = f(t, x). \tag{16}$$

Where $f : [0, \infty) \times D \rightarrow \mathbf{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $[0, \infty) \times D$, $D \subset \mathbf{R}^n$ and $x = 0 \in D$.

Also consider the approximation of $f(t,x)$ in the interval $x \in [a,b] \subset D$ by a polynomial function $p(t,x)$ with error of approximation $e(t,x)$ such that $e : [0,\infty) \times [a,b] \to \mathbf{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $[0,\infty) \times [a,b]$, $[a,b] \subset D \subset \mathbf{R}^n$ and $x = 0 \in [a,b]$. Also assume we have an upper bound ε on the error of approximation $e(t,x)$ such that $|f(t,x) - p(t,x)| = |e(t,x)| \leq \varepsilon$. With this assumption, we have the nominal and perturbed systems

$$\dot{x} = p(t,x) \qquad (17)$$
$$\dot{x} = p(t,x) + e(t,x) \qquad (18)$$

Knowing that the polynomial system $p(t,x)$ has a uniformly asymptotically stable equilibrium point at the origin, we want to determine the size of approximation for which the original system remains stable. We can now apply the stability results of perturbed systems.

Lemma 1: [8] Let $x = 0$ be an exponentially stable equilibrium point of the approximated system (17). Consider a Lyapunov function associated with the approximated system $V_p(t,x)$, and assume that it satisfy the following

$$c_1 ||x||^2 \leq V_p(t,x) \leq c_2 ||x||^2 \qquad (19)$$
$$\frac{\partial V_p}{\partial t} + \frac{\partial V_p}{\partial t} p(t,x) \leq -c_3 ||x||^2 \qquad (20)$$
$$\left\| \frac{\partial V_p}{\partial x} \right\| \leq c_4 \leq ||x||, \forall (t,x) \in [0,\infty) \times [a,b], \quad (21)$$
$$[a,b] = \{ x \in \mathbf{R}^n : ||x|| < r \}$$

Where c_1, c_2, c_3, c_4 are positive constants. Also assume that the error of approximation $e(t,x)$ satisfies

$$||e(t,x)|| \leq \varepsilon < \frac{c_3}{c_4} \sqrt{\frac{c_1}{c_2}} \theta r \qquad (22)$$
$$\forall t \geq 0, \forall x \in [a,b], 0 < \theta < 1. \qquad (23)$$

Where ε is an upper bound on the approximation error i.e. $|f(t,x) - p(t,x)| = |e(t,x)| \leq \varepsilon$ Then $\forall ||x(t_0)|| < \sqrt{\frac{c_1}{c_2}} r$, the solution $x(t)$ of the original system (18) satisfy

$$||x(t)|| \leq k \exp[-\gamma(t - t_0)] ||x(t_0)||, t_0 \leq t < t_0 + T \quad (24)$$

and

$$||x(t)|| \leq b, t \geq t_0 + T \qquad (25)$$

for some finite time T, and

$$k = \sqrt{\frac{c_2}{c_1}}; \ \gamma = \frac{(1-\theta)c_3}{2c_2}; \ b = \frac{c_4}{c_3} \sqrt{\frac{c_2}{c_1}} \frac{\varepsilon}{\theta} \qquad (26)$$

C. Krasovskii's Method

In studying the stability of nonlinear systems, the only unanswered question is usually the choice of a Lyapunov function candidate. Krasovskii showed that choosing the Lyapunov function candidate as a quadratic form of the system function, i.e. $V(x) = f(x)^T Q f(x), Q = Q^T > 0$, the asymptotic stability conditions can be reformulated as follows.

Theorem 2: Given the autonomous system

$$\dot{x} = f(x), f(0) = 0 \qquad (27)$$

with $f(x)$ continuously differentiable, a sufficient condition for the asymptotic stability of the system is that the Jacobian of $f(x)$, $\left[\frac{\partial f}{\partial x}(x) \right]$ satisfies

$$Q \left[\frac{\partial f}{\partial x}(x) \right] + \left[\frac{\partial f}{\partial x}(x) \right]^T Q \leq -I, \forall x \in D \subseteq \mathbf{R}^n \qquad (28)$$

or equivalently

$$x^T Q f(x) + f^T(x) Q x \leq -x^T x, \forall x \in D \subseteq \mathbf{R}^n \qquad (29)$$

If we apply Theorem 2 to the system rewritten in terms of its approximated version, we obtain the following stability conditions in terms of $p(x)$ and ε

$$x^T Q p(x) + p^T(x) Q x \leq -x^T x - (x^T Q \varepsilon + \varepsilon^T Q x) \forall x \in [a,b]$$

which is obviously a positivity condition on a multivariate polynomial function.

IV. S-PROCEDURE APPROACH

There are several available tools to study the sign definiteness of polynomial functions. Our goal in this section is to simplify the structure of non-polynomial functions through a transformation that allows us to rewrite the function as a multivariate polynomial whose variables are subject to some inequality constraints. The positivity of the original function can then be investigated, by studying the new set of inequalities, of the transformed function and the constraints. In [18] a technique to test the polynomial nonnegativity over a finite set described by polynomial equalities and inequalities is proposed. We propose here an alternative approach. The S-procedure will allow us to obtain sufficient conditions for the positivity of the system of inequalities. We start by stating the problem of determining the positivity of a multivariate nonlinear function over $\mathbf{R}^n$.

Problem 1: Consider a multivariate nonlinear function composed of sum of an arbitrary number of nonlinear functions, $f = \sum_1^l f_i, f_i : D_1^n \to D_2$ where $D_1^n \subseteq \mathbf{R}^n$ and $D_2 \subseteq \mathbf{R}$. Our objective is to determine if $f(x)$ is non-negative for all $x \in D_1^n$.

Next we consider the problem of deciding positivity of a multivariate polynomial function, whose variables are subject to inequality constraints.

Problem 2: Consider a multivariate polynomial function $p : \mathbf{R}^n \to \mathbf{R}$ where D is a n-dimensional domain. We aim to determine if $p(x)$ is non-negative for all $x \in \mathbf{R}^n$ subject to inequality constraints i.e. $p(x) \geq 0, \forall x \in \mathbf{R}^n, \underline{x}_i < x_i < \bar{x}_i, i = 1, \ldots, n$.

Next we show how using a special transformation, Problem (1) can be reformulated as Problem (2). Then the S-procedure [3] can be used to solve problem (2).

A. S-procedure for quadratic functions

Let $F_0 \ldots, F_k$ be quadratic functions of the variables $z \in \mathbf{R}^n$:

$$F_i(z) = z^T T_i z + 2u_i^T z + v_i, \, i = 0, \ldots, k \qquad (30)$$

where $T_i = T_i^T$, are $n \times n$, u_i are $n \times 1$ vectors and v_i are scalars. Then a sufficient condition for the following statement

$$\forall z \text{ such that } F_i(z) \geq 0, i = 1, \ldots k \Rightarrow F_0 \geq 0 \qquad (31)$$

is that there exists $\tau_1, \ldots, \tau_k \geq 0$ such that

$$F_0 \geq \tau_1 F_1 + \cdots + \tau_k F_k \qquad (32)$$

B. Function Transformation

In order to simplify the structure of the problem we refine the class of functions described in Problem (1). Starting with the class S defined in Definition 5, we will, through a two-step transformation, rewrite a function from the class S as a quadratic function. As a first step, we observe that a function in class S with n variables and $n_1 \leq l$ non-polynomial elements g_i, can be viewed as a polynomial function in $n + n_1$ variables, where the vector z of variables is an extension of the original vector x, if we rewrite the non-polynomial functions as new variables defined over the range of the function as follows:

$$f(x) = \tilde{f}(z) = \sum_{i=1}^{l} p_i(z) h_i(z), z \in \mathbf{R}^{(n+n_1)} \qquad (33)$$

$$z_i = x_i, i = 1, \ldots, n$$
$$z_j = g_j(x), z_j \in D_2 = \{[\underline{z}_{j1}, \bar{z}_{j1}] \bigcup [\underline{z}_{j2}, \bar{z}_{j2}] \bigcup \ldots\},$$
$$j = n+1, \ldots, n+n_1$$

The second step of the transformation is to rewrite the function $\tilde{f}$, and the variable constraints in a quadratic form as follows:

$$\tilde{f}(z) = z^T T z + 2u^T z + v \qquad (34)$$

Where T is a $(n+n_1) \times (n+n_1)$ symmetric matrix, u is a $(n+n_1) \times (1)$ vector and v is a scalar. Also consider intervals defining the domain D_2

$$D_2 = \{\underline{z}_{ji} \leq z_{ji} \leq \bar{z}_{ji}, , j = n+1, \ldots, n+n_1,$$
$$i = 1, \ldots m\}. \qquad (35)$$

Note that the domain D_2 is composed of all the domains of definition of the functions g_j, $j = n+1, \ldots, n+n_1$, each of those domains may be composed as the union of many intervals $[\underline{z}_i, \bar{z}_i]$, $i = 1, \ldots m$ m is the number of intervals for each domain. Consider now the set of intervals in the domain D_2 expressed as in (35), in order to express this constraint as inequality of quadratic forms as follows. Let us focus first on the lower bounds

$$\underline{F}_j(z) = z_j - \underline{z}_j \geq 0, j = n+1, \ldots, n+n_1 \qquad (36)$$
$$\bar{F}_j(z) = \bar{z}_j - z_j \geq 0.$$

We can now express $\underline{F}_j$ and $\bar{F}_j$ as quadratic forms

$$\underline{F}_j(z) = z^T T_{j1} z + 2u_{j1}^T z - \underline{z}_j \geq 0 \qquad (37)$$
$$\bar{F}_j(z) = z^T T_{j2} z + 2u_{j2}^T z - \bar{z}_j \geq 0 \qquad (38)$$

in which T_{j1}, T_{j2} are zeros $(n+n_1) \times (n+n_1)$ matrices, u_{j1}, u_{j2} are $(n+n_1) \times (1)$ vectors with all the components null except those in the j position that are respectively $u_{j1}(j) = \frac{1}{2}$, $u_{j2}(j) = -\frac{1}{2}$. Applying the S-procedure we obtain that a sufficient condition for $f(x)$ to be positive under the constraints (37) and (38), is that there exist $\tau_i \in R$, $\tau_i \geq 0$ such that $\tilde{f}(z) - \sum_{i=1}^{2} \tau_i F_i(z) \geq 0$. Next we show an example on the applicability of this technique.

Example 1: Consider the nonlinear system

$$\dot{x}_1 = -x_1^3 + x_2^2 + x_1^2 x_2 \qquad (39)$$
$$\dot{x}_2 = x_2 sin(x_1) - 2x_2 - x_1^2 x_2 \qquad (40)$$

and the Lyapunov function candidate $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$. We want to analyze the stability of the system around the origin, which is an equilibrium point for the system. Since $V(x)$ is quadratic, in order to check the stability we need to test that $\dot{V}(x) < 0$. Then,

$$\dot{V}(x) = -x_1^4 + x_1 x_2^2 + x_1^3 x_2 + x_2^2 sin(x_1) - 2x_2^2 - x_1^2 x_2^2 \quad (41)$$

We aim to determine whether or not $-\dot{V} \geq 0$, $\forall x \in \mathbf{R}^2$. First we reformulate the problem as a multivariate polynomial problem with interval constraints i.e.

$$z_1 = x_1^2; \, z_2 = x_2; \, z_3 = x_1 x_2; \, z_4 = x_2^2; \, z_5 = sin(x_1)$$

we obtain

$$\tilde{f}(z) = -\dot{V}(z) = z_1^2 - z_3 z_2 - z_3 z_1 - z_4 z_5 + 2z_2^2 + z_3^2$$
$$-1 \leq z_5 \leq 1; \, z_1 \geq 0; \, z_4 \geq 0$$

Consider the quadratic form associated with $f(z) = z^T Q z$. Following the usual procedure we obtain the following decomposition for $-\dot{V}(z)$

$$\tilde{f}(z) = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 2 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix}$$

Also consider the quadratic forms associated with the constraints

$$sin(x_1) \geq -1 \rightarrow z_5 + 1 \geq 0; \, sin(x_1) \leq 1 \rightarrow 1 - z_5 \geq 0 \quad (42)$$

observe that since $1 - z_5$ and z_4 are both nonnegative, so is their product $(1 - z_5)z_4$. Rewriting the constraints in a quadratic form $F_1(z)$ and $F_1(z)$ and applying the S-procedure with $\tau_1 = 0$, $\tau_2 = 1$ we obtain

$$\tilde{f} - \sum_{i=1}^{2} \tau_i F_i = \tilde{f} - x_2^2(1 + sin(x_1))$$
$$= x_1^4 - x_1 x_2^2 - x_1^3 x_2 + x_2^2 + x_1^2 x_2^2$$

Applying the SOS procedure, which will be explained in the next section, we get

$$\tilde{f} - \sum_{i=1}^{2} \tau_i F_i = \begin{bmatrix} x_1^2 \\ x_2 \\ x_1 x_2 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2 \\ x_1 x_2 \end{bmatrix}$$

from the positivity of the matrix it follows the positivity of $\tilde{f} - \sum_{i=1}^{2} \tau_i F_i$ and consequently positivity of $-\dot{V}, \forall x \in \mathbf{R}^2$, and the stability of the system.

V. GENERALIZED SUM OF SQUARES

In [14], it was shown how SOS programming can be applied to analyze the stability of nonlinear systems described by polynomial functions. The tool has also been extended to several applications other than stability analysis [17], [15]. We aim in this section to extend this approach to systems that are not characterized by polynomial functions. The main advantage of the proposed approach is the computational tractability of the SOS decomposition for multivariate polynomials.

A. Global nonnegativity

As stated repeatedly in this paper, many problems in nonlinear systems can be reduced to the basic problem of checking the global nonnegativity of a function of several variables [4]. The problem is to give equivalent conditions or procedure for checking the validity of the proposition

$$F(x_1, \dots, x_n) \geq 0, \forall x_1, \dots, x_n \in \mathbf{R} \tag{43}$$

If we limit our study to polynomial functions, $F(x) = p(x)$, then a sufficient condition for $p(x) \geq 0$, is that $p(x)$ be a sum of squares. The general problem of testing global positivity can then be reformulated as a condition for the existence of SOS decomposition.

Observe first that a necessary condition for a multivariate polynomial function (43) to satisfy global nonnegativity is that the degree of the polynomial be even. In 1900 (First congress of Mathematicians) Hilbert presented the following conjecture, that will be referred as Hilbert 17th problem [9]: Consider $p \in P_{n,m}$, then there exist polynomials q_i and r_i such that

$$\sum_i (r_i(x))^2 p(x) = \sum_i (q_i(x))^2 \tag{44}$$

In other words any PSD polynomial $p(x)$ can be expressed as a sum of squares of ratios of polynomials. Hilbert proved the conjecture for ternary forms $(n = 3)$, and Artin proved the conjecture for any n in 1927. In [9], a theorem is presented, and a step-by-step algorithm is given allowing us to obtain the Hilbert decomposition of a polynomial form. From Hilbert's conjecture it is proven that any PSD form can be written as a sum of squares of ratios of polynomials. This is no longer true if instead of a ratio of polynomials we limit ourselves to polynomial functions. In general, having an SOS form is a sufficient but non-necessary condition for PSD. As Hilbert proved however, there are three cases for

which the two classes (SOS and PSD) are equivalent [14]: polynomials in two variables, polynomials of degree two and polynomials in three variables of degree four.

B. Sum of square decomposition

First we will show that using semi-definite programming (SDP) it is possible to test if a given polynomial admits an SOS decomposition [14].

Theorem 3: Given a multivariate polynomial $p : x \in \mathbf{R}^n \to \mathbf{R}^{2m}$ of degree $2m$, a sufficient condition for the existence of SOS representation $p(x) = p(z) = z^T Q z$ is $Q \geq 0$ where z is a vector of monomials in x of degree m.

So the test for *SOS* of a polynomial function has been reduced to a linear matrix inequality (LMI) [3]. Then for a symmetric matrix Q we obtain the following eigenvalue factorization [2] $Q = L^T T L$, from which follows the decomposition $p(x) = \sum_i (Lz)_i^2$. In general we have that the SOS representation might not be unique, depending on the choice of the components of the z vector. In particular, different choices of the vector z correspond to different matrices Q that satisfy the SOS representation. It could be that only some of those matrices are PSD, so the existence of SOS decomposition for a polynomial may depend on the representation. If at least one of the matrices of the linear subspace is positive semidefinite (i.e. the intersection of the linear subspace of matrices satisfying the SOS representation with the positive semidefinite matrix cone is non empty), then $p(x)$ is SOS and therefore PSD. In general we will choose the components of z to be linearly independent, and we will say that the corresponding representation is minimal.

C. SOS Generalization: A Partial State Vector Approach

We will show how, under certain assumptions, it is possible to apply the SOS procedure to a nonlinear, non-polynomial function. The main idea is based on the use of SOS procedure, considering the generic nonlinear function as a polynomial function, in which the non-polynomial parts are treated as coefficients of the function. Rewriting the function as a quadratic form we get $f(x) = z(x_p)^T Q(x_g) z(x_p)$ where z is a vector of monomial of $x_1, \dots, x_k \in x_p$, and Q is a matrix of appropriate dimension, which depend on the variable $x_{k+1}, \dots, x_n \in x_g$ trough the non-polynomial functions g_i. From SOS theory, a sufficient condition for $f(x) > 0$ is that $Q(x_g)$ to be positive definite. In order to apply the SOS procedure to a generic nonlinear non-polynomial function, we need to restrict the class of system we deal with, in particular we will consider the class of systems defined in (7). The state vector x is divided into two parts, φ and ξ. In fact, choosing a quadratic Lyapunov function $V = x^T x$ and applying the SOS procedure for determining the sign of $-\dot{V}$ we want to find conditions on φ that guarantee V is decreasing along the trajectory of the system for all ξ i.e.

$$-\dot{V} = z(\xi)^T Q(\varphi) z(\xi) > 0 \quad \forall \xi \in \mathbf{R}^k \tag{45}$$

Next, we present an example to illustrate these results.

Example 2: Consider a nonlinear system characterized by a decoupled vector space as described

$$\dot{x}_1 = -x_1 \sin(x_3) \tag{46}$$

$$\dot{x}_2 = -x_2^3 + x_1 \cos(x_3) \tag{47}$$

$$\dot{x}_3 = -x_2^2 x_3 log(x_3) \tag{48}$$

The state vector $x = [x_1 \, x_2 \, x_3]^T$, satisfies the condition for the system to be *decoupled state*, i.e. x_1, x_2 only appear in a polynomial form. Considering the quadratic Lyapunov function $V = \frac{1}{2}x^2$, sufficient condition for the stability of the system is the positivity of the following function

$$-\dot{V} = x_1^2 sin(x_3) + x_2^4 - x_1 x_2 cos(x_3) + x_2^2 x_3^2 log(x_3) > 0, \tag{49}$$

$\forall x_1, x_2 \in \mathbf{R}^2$. The corresponding quadratic form is

$$-\dot{V} = \begin{bmatrix} x_1 \\ x_2 \\ x_2^2 \end{bmatrix}^T \begin{bmatrix} sin(x_3) & -\frac{1}{2}cos(x_3) & 0 \\ -\frac{1}{2}cos(x_3) & x_3^2 log(x_3) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_2^2 \end{bmatrix}$$

the expression above result to be positive if all the minors of the matrix are positive, i.e.

$$sin(x_3) > 0; \ \& \ x_3^2 log(x_3) sin(x_3) - \frac{1}{4}cos^2(x_3) > 0$$

from which we get the condition on x_3 to guarantee the stability of the system. The system is stable in the domain

$$D = \{x_1 \in \mathbf{R}, x_2 \in \mathbf{R}, x_3 > 2.7183,$$
$$2\pi K < x_3 < \pi(2K+1), K = 0, 1, 2, ...\}$$

In order to verify the stability of the system in the domain D we evaluate $-\dot{V}(x)$ in a set of sampled value taken from D and outside of D. More precisely consider the domains

$$Q_1 = \{x_1 = 100x_3, x_2 = 1, x_3 = [0, 2\pi]\} \tag{50}$$

$$Q_2 = \{x_1 = 10, x_2 = \frac{x_3}{100}, x_3 = [0, 2\pi]\} \tag{51}$$
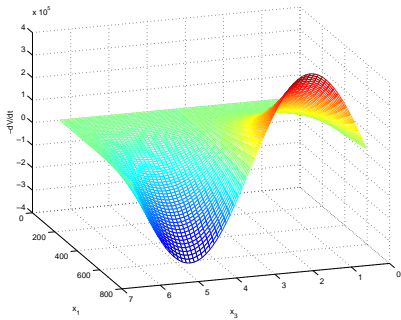

Fig. 1.   Values of $-\dot{V}$ in Q_1.

In Figure (1) $-\dot{V}(x)$ is plotted for $x \in Q_1$, we can notice how the function is positive, and then the system is stable for values of $x_3 \in (0, \pi)$, and is negative for values of $x_3 \in (\pi, 2\pi)$.

The result is however conservative, in fact even for values of $x_3 < 2.7$ the function $-\dot{V}$ is still positive as we can see in the Figure (1).

VI. Conclusion

In this paper, we presented various methods to analyze the stability of nonlinear systems. The main idea was to reformulate the nonlinear stability problem into a polynomial setting, and then utilize results pertaining to the stability of polynomial systems, or extend existing results. We presented a new analysis relating a class of nonlinear systems to approximated polynomials systems. Also, we utilized the S-procedure in a novel setting , and extended the SOS procedure to that end. All these reformulations present a new way to analyze nonlinear systems, through analyzing their polynomial counterparts.

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