An Approach to the Optimal Scanning Measurement Problem Using Optimum Experimental Design

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***Abstract*— We address the problem of activating discrete scanning sensors so as to maximize some quantitative observability measure for a given distributed parameter system. In contrast to the classical approach based on a direct application of non-linear programming algorithms, the key idea here is to operate on the density of sensors per unit area instead of the positions of individual sensors. Mathematically, this procedure involves searching for a family of "optimal" probability measures defined on subsets of the set of feasible measurement points. The method proposed for solving the problem so formulated, originates from an extremely efficient approach which is based on directly constrained design measures that are used in optimum experimental design theory. As a result, a fast iterative procedure is obtained whose each step reduces to replacing less informative sensor locations with points which furnish more information about the system state.**

I. INTRODUCTION

This paper seeks to study a problem which has relatively often been addressed in the control literature: *How should one choose the locations of measurement sensors for a given distributed parameter system (DPS) so as to increase its degree of observability quantified by a suitable observability measure?* Certainly the selection of these sensor positions may have such a dramatic effect on the performance possibilities as to far outweigh the optimal "tuning" of the control signals that takes place after sensor locations are selected. Specific features of this problem and past approaches are surveyed e.g. in [1], [2]. However, the results communicated by most authors are rather limited to the selection of stationary sensor positions. A generalization which imposes itself is to apply sensors which are capable of tracking points providing at a given time moment best information about the system state. In particular, it happens frequently that the observation system comprises multiple sensors whose positions are already specified and it is desired to activate only a subset of them during a given time interval while the other sensors remain dormant [3]. A reason for not using all the available sensors could be the reduction of the observation system complexity and the cost of operation and maintenance [1]. Such a scanning strategy of taking measurements can be also interpreted in terms of several sensors which are mobile. This line of research has drawn some attention of both scientists and engineers, but the existing methods are still of little use in practice.

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Ordinarily, the task is reduced to examining a given finite set of possible candidate locations. Then we seek the best subset of locations from among all the possible ones. Numerical algorithms for the construction of optimum sensor configurations by searching over a list of candidate locations customarily involve an iterative improvement of the initial sensor configuration. The combinatorial nature of the problem so formulated implies that with a long list of candidate points and the DPS discretization involving a high-dimensionality of the lumped representation, complicated search algorithms can readily consume appreciable computer time and space. In contrast to this approach, the key idea here is to operate on the density of sensors per unit area instead of the positions of individual sensors. Such conditions allow us to relax the discrete optimization problem in context and to replace it by its continuous approximation. Mathematically, this procedure involves looking for a family of 'optimal' probability measures defined on subsets of the set of feasible measurement points. In spite of its somewhat abstract assumptions, the resulting algorithm of exchange type is very easy to implement. The underlying concepts have already been applied in the context of sensor location for parameter estimation [4]–[6]. A main contribution of the present paper consists in a non-trivial generalization of those results to output selection in control system design, where to the best of the authors' knowledge the approach based on sensor densities has not been employed yet.

II. PROBLEM FORMULATION

A. Observability in a Quantitative Sense

Given a linear DPS described by a partial differential equation model, consider its finite-dimensional approximation (e.g. obtained via the finite-element method) in the form of the following system of linear ordinary differential equations:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = A(t)y(t), \quad t \in T = [t_0, t_f], \tag{1}$$

$$y(t_0) = y_0, \tag{2}$$

such that $y(t) \in \mathbb{R}^n$ and $A(t) \in \mathbb{R}^{n\times n}$, which is augmented by the sensor location parameterized counterpart of the output equation

$$z(t) = C(t;\zeta(t))y(t), \tag{3}$$

where $z(t) \in \mathbb{R}^N$, $C(t;\zeta(t)) \in \mathbb{R}^{N\times n}$, and the notation emphasizes the dependence of the output matrix C on the current spatial configuration of sensors $\zeta(t)$ (to be determined in what follows).

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As regards a quantitative measure for state observability, consider the observability Gramian

$$W(\zeta) = \int_{t_0}^{t_f} \Phi^{\mathrm{T}}(t,t_0) C^{\mathrm{T}}(t;\zeta(t)) C(t;\zeta(t)) \Phi(t,t_0)\,\mathrm{d}t, \tag{4}$$

where the fundamental (or transition) matrix $\Phi(t,t_0)$ obeys

$$\frac{\mathrm{d}\Phi(t,t_0)}{\mathrm{d}t} = A(t)\Phi(t,t_0), \quad \Phi(t_0,t_0) = I, \tag{5}$$

I being the identity matrix.

An optimal sensor configuration strategy $\zeta^\star$ can be found by minimizing some convex function Ψ defined on $W(\zeta)$ [1]. Common choices include the following:

1) $\Psi(W) = -\ln\det W$,
2) $\Psi(W) = \operatorname{trace} W^{-1}$,
3) $\Psi(W) = -\operatorname{trace} W$,
4) $\Psi(W) = \lambda_{\max}(W^{-1})$,

where $\lambda_{\max}(\,\cdot\,)$ stands for the largest eigenvalue of its matrix argument. Since the last criterion is non-differentiable when there are repeated eigenvalues, its use will not be considered here.

B. Scanning Problem for Optimal Observability

Let us form an arbitrary partition of the time interval $T = [t_0, t_f]$ by choosing points $t_0 < t_1 < \cdots < t_L = t_f$ defining subintervals $T_\ell = [t_{\ell-1}, t_\ell)$, $\ell = 1,\ldots,L$. We then consider N scanning sensors which will possibly be changing their locations at the beginning of every time subinterval, but will be remaining stationary for the duration of each of the subintervals. Thus the sensor configuration ζ can be viewed as follows:

$$\zeta(t) = (x_\ell^1,\ldots,x_\ell^N) \quad \text{for } t \in T_\ell, \quad \ell = 1,\ldots,L, \tag{6}$$

where $x_\ell^j \in X \subset \mathbb{R}^d$ stands for the location of the j-th sensor on the subinterval T_ℓ, X being the part of the spatial domain where the measurements can be taken.

Assume that the consecutive rows of the matrix $C(t;\zeta(t))$ in (3) correspond to contributions from different sensors, i.e.

$$C(t,\zeta(t)) = \begin{bmatrix} \gamma^{\mathrm{T}}(x_\ell^1,t) \\ \vdots \\ \gamma^{\mathrm{T}}(x_\ell^N,t) \end{bmatrix} \quad \text{for } t \in T_\ell, \quad \ell = 1,\ldots,L, \tag{7}$$

where $\gamma : \mathbb{R}^{d+1} \to \mathbb{R}$ is a given function. Then we can decompose the Gramian as follows:

$$W(\zeta) = \sum_{\ell=1}^{L} \sum_{j=1}^{N} \Upsilon_\ell(x_\ell^j), \tag{8}$$

where

$$\Upsilon_\ell(x) = \int_{t_{\ell-1}}^{t_\ell} g(x,t) g^{\mathrm{T}}(x,t)\,\mathrm{d}t, \tag{9}$$

$$g(x,t) = \Phi^{\mathrm{T}}(t,t_0)\gamma(x,t). \tag{10}$$

We have thus arrived at the crucial point for the presented approach, as the proposed algorithm of finding best sensor locations may only be employed on condition that the Gramian constitutes the sum of some matrices, each of them being completely defined by the position of only one scanning sensor on one subinterval T_ℓ, cf. (8).

C. Conversion to Finding Optimal Sensor Densities

When the number of sensors N is large, which is rather a common situation in applications such as air pollution monitoring networks or control architectures for smart material systems, the optimal sensor location problem becomes extremely difficult from a computational point of view. Consequently, we propose to *operate on the spatial density of sensors*, rather than *on the sensor locations*. This is proved reasonable for a sufficiently large N and potential solutions would be satisfactory for many technical processes.

Performing such a conversion does not eliminate the discrete nature of the original formulation, and therefore the resultant computational problem is still not amenable to solution. Thus we relax the definition of the set of admissible solutions by observing that the density of sensors over the subinterval T_ℓ can be approximately described by a probability measure $\xi_\ell(\mathrm{d}x)$ on the space $(X,\mathcal{B})$, where $\mathcal{B}$ is the σ-algebra of all Borel subsets of X. As regards the practical interpretation of the so produced solutions, one possibility is to partition X into non-overlapping subdomains ΔX_i of relatively small areas and then, on the subinterval T_ℓ, to allocate to each of them the number

$$N_\ell(\Delta X_i) = \left\lceil N \int_{\Delta X_i} \xi_\ell(\mathrm{d}x) \right\rceil \tag{11}$$

of sensors ($\lceil\rho\rceil$ is the smallest integer $\geq \rho$).

Thus our aim is to find probability measures ξ_ℓ, $\ell = 1,\ldots,L$ over X. For notational convenience, in what follows we shall briefly write $\xi = (\xi_1,\ldots,\xi_L)$ and call ξ a *design measure* (or a *design* for short).

Such an extension of the concept of the sensor configuration allows us to replace (8) by

$$W(\xi) = \sum_{\ell=1}^{L} \int_X \Upsilon_\ell(x)\,\xi_\ell(\mathrm{d}x). \tag{12}$$

A rather natural additional assumption is that the density of sensors $N_\ell(\Delta X_i)/N$ in a given part ΔX_i must not exceed some prescribed level. In terms of the probability measures, this amounts to imposing the conditions

$$\xi_\ell(\mathrm{d}x) \leq \omega(\mathrm{d}x), \quad \ell = 1,\ldots,L, \tag{13}$$

where $\omega(\mathrm{d}x)$ is a given measure satisfying $\int_X \omega(\mathrm{d}x) \geq 1$.

Defining $J(\xi) = \Psi[W(\xi)]$, we may phrase the scanning sensor location problem as the selection of

$$\xi^\star = \arg\min_{\xi\in\Xi(X)} J(\xi), \tag{14}$$

where $\Xi(X)$ denotes the set of all competing designs whose components satisfy (13) (note that $\Xi(X)$ is non-empty and convex). We call $\xi^\star$ the (Ψ,ω)-*optimal solution*.

The idea of working with sensor densities in lieu of sensor positions was proposed by Fedorov [7] who sought spatially distributed observations maximizing the accuracy of parameter estimates of a given static system (the problem pertains to the general setting of optimum experimental design theory [8], [9]). Fedorov's ideas were then generalized to the context of optimum stationary [6] and scanning [5] sensor locations for parameter estimation in dynamic DPS's.

In the sequel, we will need the following assumptions:

(A1) X is compact,

(A2) $\Upsilon_\ell \in C(X;\mathbb{R}^{N\times N})$,

(A3) Ψ is convex,

(A4) If $W_1 \preceq W_2$, then $\Psi(W_1) \geq \Psi(W_2)$,

(A5) $\omega(\mathrm{d}x)$ is atomless, i.e. for any $\Delta X \subset X$ there exists a $\Delta X' \subset \Delta X$ such that

$$\int_{\Delta X'} \omega(\mathrm{d}x) < \int_{\Delta X} \omega(\mathrm{d}x), \tag{15}$$

(A6) There exists a finite real q such that

$$\big\{\xi = (\xi_1,\ldots,\xi_L) : J(\xi) \leq q < \infty,\ \xi_\ell(\mathrm{d}x) \leq \omega(\mathrm{d}x), \ \ell = 1,\ldots,L\big\} = \widetilde{\Xi}(X) \neq \emptyset,$$

(A7) For any $\xi \in \widetilde{\Xi}(X)$ and $\bar{\xi} \in \Xi(X)$, we have

$$\mathrm{d}J(\xi;\bar{\xi}-\xi) = \sum_{\ell=1}^{L} \int_X \psi_\ell(x,\xi)\,\bar{\xi}_\ell(\mathrm{d}x), \tag{16}$$

where the left-hand side stands for the one-sided directional derivative of J at ξ in the direction $\bar{\xi}-\xi$,

$$\begin{aligned}\mathrm{d}J(\xi;\bar{\xi}-\xi) &= \lim_{\lambda\downarrow 0} \frac{J(\xi+\lambda(\bar{\xi}-\xi)) - J(\xi)}{\lambda} \\ &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\Psi[W(\xi)+\lambda(W(\bar{\xi})-W(\xi))]\Big|_{\lambda=0^+}\end{aligned} \tag{17}$$

and $\psi_\ell(\,\cdot\,,\xi)$, $\ell = 1,\ldots,L$ are $C(X)$ functions.

Assumption (A4) characterizes Ψ as a linear ordering of $\Xi(X)$ ($W_1 \preceq W_2$ iff $W_2 - W_1$ is non-negative definite). In turn, Assumption (A7) means that the directional derivative of J must be somewhat specific. Note, however, that requiring Ψ to be differentiable with respect to individual elements of its matrix argument, we obtain

$$\psi_\ell(x,\xi) = c(\xi) - \phi_\ell(x,\xi), \tag{18}$$

the functions c and φ being respectively defined as

$$c(\xi) = -\frac{1}{L}\,\mathrm{trace}\Big[\overset{\circ}{\Psi}(\xi)W(\xi)\Big], \tag{19}$$

$$\phi_\ell(x,\xi) = -\,\mathrm{trace}\Big[\overset{\circ}{\Psi}(\xi)\Upsilon_\ell(x)\Big], \tag{20}$$

where

$$\overset{\circ}{\Psi}(\xi) = \frac{\partial\Psi(W)}{\partial W}\bigg|_{W=W(\xi)}.$$

Table I lists specific forms of the so introduced functions for the most popular design criteria.

TABLE I

FUNCTIONS DEFINING THE DIRECTIONAL DERIVATIVES.

$\Psi[W(\xi)]$	$\phi_\ell(x,\xi)$	$c(\xi)$
$-\ln\det W(\xi)$	$\mathrm{trace}\big[W^{-1}(\xi)\Upsilon_\ell(x)\big]$	N
$\mathrm{trace}\,W^{-1}(\xi)$	$\mathrm{trace}\big[W^{-2}(\xi)\Upsilon_\ell(x)\big]$	$\mathrm{trace}\,W^{-1}(\xi)$
$-\,\mathrm{trace}\,W(\xi)$	$\mathrm{trace}\,\Upsilon_\ell(x)$	$\mathrm{trace}\,W(\xi)$

In what follows, we write $\bar{\Xi}(X)$ for the collection of all the designs ξ whose components satisfy the requirement[1]

$$\xi_\ell(\Delta X) = \begin{cases}\omega(\Delta X) & \text{for } \Delta X \subset \mathrm{supp}\,\xi_\ell,\\ 0 & \text{for } \Delta X \subset X\setminus \mathrm{supp}\,\xi_\ell.\end{cases} \tag{21}$$

The point of this definition is that the designs from $\bar{\Xi}(X)$ turn out to be vital while formulating optimality conditions. The main feature of a design $\xi \in \bar{\Xi}(X)$ is that for each of its components ξ_ℓ the design domain X can be split into two subsets for which ξ_ℓ coincides either with the zero-measure of with the upper bound ω.

III. CHARACTERIZATION OF OPTIMAL DESIGNS

We begin with a fundamental result regarding the form of (Ψ,ω)-optimal designs.

Theorem 1: Under Assumptions (A1)–(A7), a (Ψ,ω)-optimal design exists in $\bar{\Xi}(X)$.

Consequently, we can focus our attention on designs from the set $\bar{\Xi}(X)$. Our goal now is to develop a method for checking whether or not a given design $\xi \in \bar{\Xi}(X)$ is (Ψ,ω)-optimal. The test stated below in Theorem 2 is based on the following notion of the separability of two sets:.

Definition 1: Given a design ξ, we will say that the function $\psi_\ell(\,\cdot\,,\xi)$ defined by (18) *separates* sets X_1 and X_2 with respect to $\omega(\mathrm{d}x)$ if for any two sets $\Delta X_1 \subset X_1$ and $\Delta X_2 \subset X_2$ satisfying

$$\int_{\Delta X_1} \omega(\mathrm{d}x) = \int_{\Delta X_2} \omega(\mathrm{d}x) \tag{22}$$

we have

$$\int_{\Delta X_1} \psi_\ell(x,\xi)\,\omega(\mathrm{d}x) \leq \int_{\Delta X_2} \psi_\ell(x,\xi)\,\omega(\mathrm{d}x). \tag{23}$$

Theorem 2: A necessary and sufficient condition for $\xi^\star = (\xi_1^\star,\ldots,\xi_L^\star) \in \bar{\Xi}(X)$ to be (Ψ,ω)-optimal is that the functions $\psi_\ell(\,\cdot\,,\xi^\star)$ separate $X_\ell^\star = \mathrm{supp}\,\xi_\ell^\star$ and $X\setminus X_\ell^\star$ for $\ell = 1,\ldots,L$.

As a companion to the above result, we next consider the special case where $\omega(\mathrm{d}x)$ has a continuous density ϱ.

Corollary 1: Let $\xi^\star \in \bar{\Xi}(X)$ and $X_\ell^\star = \mathrm{supp}\,\xi_\ell^\star$, $\ell = 1,\ldots,L$. If $\omega(\mathrm{d}x) = \varrho(x)\,\mathrm{d}x$, where $\varrho(x)$ is a positive

[1]The support of a measure ξ_ℓ is defined as the closed set $\mathrm{supp}\,\xi_\ell = X\setminus\bigcup\{G : \xi_\ell(G) = 0,\ G - \text{open}\}$, cf. [10, p.80].

continuous function, then $\xi^\star$ is (Ψ,ω)-optimal iff

$$\sup_{x\in X_\ell^\star}\psi_\ell(x,\xi^\star)\le\inf_{x\in X\setminus X_\ell^\star}\psi_\ell(x,\xi^\star),\qquad \ell=1,\ldots,L. \tag{24}$$

Corollary 2: Under the assumptions of Corollary 1, if moreover $\partial\Psi(W)/\partial W\big|_{W=W(\xi^\star)}$ exists and is bounded, then $\xi^\star$ is (Ψ,ω)-optimal iff

$$\inf_{x\in X_\ell^\star}\phi_\ell(x,\xi^\star)\ge\sup_{x\in X\setminus X_\ell^\star}\phi_\ell(x,\xi^\star),\qquad \ell=1,\ldots,L. \tag{25}$$

According to the above result, the functions ϕ_ℓ play a leading role in indicating spatial points which provide the most valuable information in terms of the adopted optimality criterion Ψ. They constitute a good starting point for constructing numerical procedures of determining best sensor configurations in practice.

IV. NUMERICAL PROCEDURE OF EXCHANGE TYPE

A. General Algorithm

Corollary 2 forms a basis for an efficient numerical algorithm of determining (Ψ,ω)-optimal designs. Its main idea is to move some measure from an area of lower values of $\phi_\ell(\cdot,\xi^{(k)})$ to those with higher values, as we expect that such a procedure will improve the current design $\xi^{(k)}$. Details regarding this scheme are summarized as follows:

Algorithm 1: General scanning strategy algorithm:

1) Guess an initial design $\xi^{(0)}\in\bar\Xi(X)$. Select a tolerance $0<\eta\ll 1$. Set $k=0$.
2) For $\ell=1,\ldots,L$ separately set $X_{1\ell}^{(k)}=\operatorname{supp}\xi_\ell^{(k)}$ and $X_{2\ell}^{(k)}=\overline{X\setminus X_{1\ell}^{(k)}}$ (the bar over the symbol denoting a set stands for its closure), and determine
$$\begin{aligned}x_{1\ell}^{(k)}&=\arg\min_{x\in X_{1\ell}^{(k)}}\phi_\ell(x,\xi^{(k)}),\\ x_{2\ell}^{(k)}&=\arg\max_{x\in X_{2\ell}^{(k)}}\phi_\ell(x,\xi^{(k)}).\end{aligned} \tag{26}$$
If $\phi_\ell(x_{1\ell}^{(k)},\xi^{(k)})>\phi_\ell(x_{2\ell}^{(k)},\xi^{(k)})-\eta$ for all $\ell=1,\ldots,L$, then STOP.
3) For $\ell=1,\ldots,L$ proceed as follows: If $\phi_\ell(x_{1\ell}^{(k)},\xi^{(k)})>\phi_\ell(x_{2\ell}^{(k)},\xi^{(k)})-\eta$, then fix $S_{1\ell}^{(k)}(\alpha)=S_{2\ell}^{(k)}=\emptyset$. Otherwise define $S_{1\ell}^{(k)}(\alpha)$ as the intersection of $X_{1\ell}^{(k)}$ and the ball centered at $x_{1\ell}^{(k)}$ with a radius adjusted so that $\int_{S_{1\ell}^{(k)}(\alpha)}\rho(x)\,\mathrm{d}x=\alpha$, and similarly, let $S_{2\ell}^{(k)}(\alpha)$ be the intersection of $X_{2\ell}^{(k)}$ and the ball centered at $x_{2\ell}^{(k)}$ with a radius adjusted so as to satisfy $\int_{S_{2\ell}^{(k)}(\alpha)}\rho(x)\,\mathrm{d}x=\alpha$. Then construct $\xi^{(k+1)}=(\xi_1^{(k)}(\alpha^{(k)}),\ldots,\xi_L^{(k)}(\alpha^{(k)}))\in\bar\Xi(X)$ by choosing $\alpha^{(k)}$ so that
$$\begin{aligned}&\Psi[W(\xi_1^{(k)}(\alpha^{(k)}),\ldots,\xi_L^{(k)}(\alpha^{(k)}))]\\ &\quad=\min_{\alpha\in(0,\bar\alpha]}\Psi[W(\xi_1^{(k)}(\alpha),\ldots,\xi_L^{(k)}(\alpha))],\end{aligned} \tag{27}$$
where
$$\operatorname{supp}\xi_\ell^{(k)}(\alpha)=\overline{(X_{1\ell}^{(k)}\setminus S_{1\ell}^{(k)}(\alpha))\cup S_{2\ell}^{(k)}(\alpha)}, \tag{28}$$
and $\bar\alpha=\min\{1,\int_X\varrho(x)\,\mathrm{d}x-1\}$. Increment k and go to Step 2.

The properties of this feasible-direction-like algorithm can be considered in some detail, but in practice the scheme outlined in what follows is preferred as it is much easier to implement.

B. Implementation Issues

Within the framework of the sensor placement, we usually deal with a constant allowable sensor density $\varrho(x)=\text{const}$. Moreover, while implementing Algorithm 1 on a computer, all integrals are most often replaced by sums over some finite grid elements (the grid produced by the finite-element method can be employed for that purpose). Analogously, the sets X, $X_{1\ell}^{(k)}$, $X_{2\ell}^{(k)}$, $S_{1\ell}^{(k)}$ and $S_{2\ell}^{(k)}$ then simply consist of grid nodes. Thus Algorithm 1 can be interpreted as an exchange-type algorithm (in each iteration some points are deleted from the current design and replaced by the same number of vacant points). In practice, $\alpha^{(k)}$ is usually fixed and, what is more, one-point exchanges are most often adopted, i.e. $S_{1\ell}^{(k)}=\{x_{1\ell}^{(k)}\}$ and $S_{2\ell}^{(k)}=\{x_{2\ell}^{(k)}\}$, which substantially simplifies the implementation.

Taking account of the above remarks, the following computational scheme can be developed:

Algorithm 2: Practical exchange-type scanning strategy algorithm:

1) Construct *a priori* a sufficiently dense set of possible sensor locations $\tilde X=\{x^j\}_{j=1}^{N'}$ covering the domain X, where $N'>N$. For each node of this grid, determine and store the matrices $\Upsilon_\ell(x^j)$, $\ell=1,\ldots,L$. Select N-element sets $\tilde X_{1\ell}^{(0)}\subset\tilde X$, $\ell=1,\ldots,L$ which constitute initial guesses regarding best sites for locating sensors over the consecutive subintervals T_ℓ. They will possibly be improved in what follows. Set $k=0$.
2) Assemble the Gramian
$$W^{(k)}=\sum_{\ell=1}^{L}\sum_{x\in\tilde X_{1\ell}^{(k)}}\Upsilon_\ell(x) \tag{29}$$
and compute
$$G^{(k)}=-\frac{\partial\Psi(W)}{\partial W}\bigg|_{W=W^{(k)}}. \tag{30}$$
For $\ell=1,\ldots,L$ separately, determine
$$\begin{aligned}x_{1\ell}^{(k)}&=\arg\min_{x\in\tilde X_{1\ell}^{(k)}}\operatorname{trace}\{G^{(k)}\Upsilon_\ell(x)\},\\ x_{2\ell}^{(k)}&=\arg\max_{x\in\tilde X\setminus\tilde X_{1\ell}^{(k)}}\operatorname{trace}\{G^{(k)}\Upsilon_\ell(x)\}.\end{aligned} \tag{31}$$

If $\mathrm{trace}\{G^{(k)}\Upsilon_\ell(x_{1\ell}^{(k)})\} \le \mathrm{trace}\{G^{(k)}\Upsilon_\ell(x_{2\ell}^{(k)})\} - \eta$, where $0 < \eta \ll 1$, then set $S_{1\ell}^{(k)} = \{x_{1\ell}^{(k)}\}$ and $S_{2\ell}^{(k)} = \{x_{2\ell}^{(k)}\}$, otherwise fix $S_{1\ell}^{(k)} = S_{2\ell}^{(k)} = \emptyset$.

3) If $\mathrm{trace}\{G^{(k)}\Upsilon_\ell(x_{1\ell}^{(k)})\} > \mathrm{trace}\{G^{(k)}\Upsilon_\ell(x_{2\ell}^{(k)})\} - \eta$ for all $\ell = 1, \ldots, L$, then STOP. Otherwise, set

$$\tilde{X}_{1\ell}^{(k+1)} = (\tilde{X}_{1\ell}^{(k)} \setminus S_{1\ell}^{(k)}) \cup S_{2\ell}^{(k)}. \tag{32}$$

Increment k and go to Step 2.

The integration required for determining the matrices $\Upsilon_\ell(x^j)$ can be performed using common quadratures.

Algorithm 2 performs well and turns out to be extremely fast despite the high dimensionality of the original problem. Switching from the formulation in terms of seeking the best sensor locations to that in terms of determining best sensor densities makes it possible to avoid the complications caused by the inherent combinatorial nature of the sensor location problem.

Apart from the decided advantages of the approach, two issues should be addressed as potential shortcomings. First of all, note that one-point exchanges in Algorithm 2 being a simplified version of Algorithm 1 correspond to the situation in which all $\alpha^{(k)}$'s are the same, while the convergence of the proposed scheme is guaranteed only for a sequence of properly selected $\alpha^{(k)}$'s, cf. (27). As a result, some minor oscillations of the quantity $\Psi[W(\xi^{(k)})]$ may be observed after the initial stage of a monotonic decrease in its values. In practice, however, if the grid $\tilde{X}$ is sufficiently dense, the reduction in the value of the performance index is so significant that we may hope that the obtained designs do not deviate too much from the optimal ones.

Another delicate question concerns the memory management, as the storage of large dense matrices $\Upsilon_\ell(x^j)$ for all points x^j, $j = 1, \ldots, N'$ and all time subintervals T_ℓ, $\ell = 1, \ldots, L$ requires $(N')^3 \times L$ words. Consequently, sufficiently large random access memory should be available for the execution of the relevant program. For reasonably dense grids and two spatial dimensions, the Matlab environment is applicable in this respect, the more so that the powerful and flexible Partial Differential Equation Toolbox [11] can then be employed to generate an unstructured two-dimensional spatial mesh and the related grid of nodes (they can be treated as potential points at which sensors may be located) and to form the matrices $A(t)$ in (1) via the finite element method (it is then used to produce an approximation to the transition matrix through solving (5) using e.g. the backward-difference method).

V. NUMERICAL EXAMPLE

The following numerical example serves as a vehicle for the display of the practical performance of the solution technique presented in Section IV. Consider the two-dimensional heat equation

$$\frac{\partial y}{\partial t} - \mu \Delta y = 0 \quad \text{in } \Omega \times T, \tag{33}$$

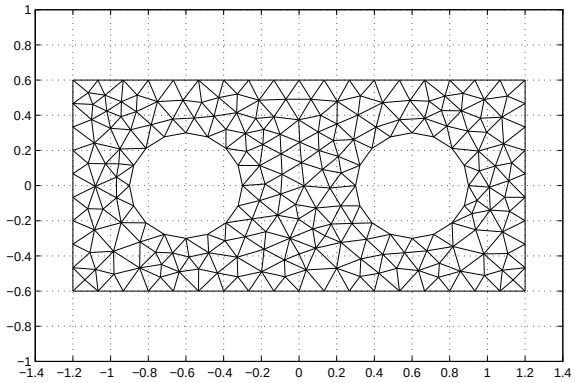

Fig. 1. Spatial domain Ω and an unstructured mesh on it.

which describes the diffusion of heat over the time interval $T = [0, 1]$ in a body represented geometrically by the spatial domain Ω, where $\mu = 0.1$ is the diffusion coefficient. The form of Ω, which can represent e.g. a metal block with two circular cracks or cavities, is given in Fig. 1.

Equation (33) is supplemented with the zero Dirichlet boundary conditions on the outer boundaries of Ω and the zero Neumann conditions on both the inner circular boundaries. Using the PDE Toolbox which provides a powerful and flexible environment for the study and solution of partial differential equations in MATLAB, the triangular mesh of 243 nodes shown in Fig. 1 was built on the domain Ω using the graphical user interface implemented in the routine `pdetool`. The mesh nodes which do not lie on the outer boundary (there were 189 such nodes) were treated as candidates for locating $N = 90$ pointwise sensors, i.e. they formed the set $\tilde{X}$ in Algorithm 2. The observation horizon T was partitioned into four subintervals

$$T_\ell = \Big[\frac{\ell - 1}{4}, \frac{\ell}{4}\Big), \quad \ell = 1, \ldots, 4. \tag{34}$$

The matrices $\Upsilon_\ell(x^j)$ for $\ell = 1, \ldots, 4$ and $j = 1, \ldots, 189$ were then computed in accordance with the developments of Section IV-B. In particular, approximation of the integrals (9) was performed by dividing the interval T_ℓ into seven equal subintervals and then using the trapezoidal rule. The ODE (5) was integrated by fixing the time step $\Delta t = 1/28$ and employing the backward difference method.

At this step, the stiffness and mass matrices resulting from applying the method of lines semidiscretization were needed. They were therefore assembled using the procedure `assempde`. Computation of the $\Upsilon_\ell(x^j)$'s took approximately 90 s on a Pentium IV 2.40 GHz computer equipped with 524 MB RAM and running Windows 2000. The initial selection of sensor configurations was performed at random several times so as to avoid getting stuck in a local minimum. Algorithm 2 was then run for each such starting configuration for the performance indices $\Psi_1[W] = -\ln\det(W)$ and $\Psi_2[W] = -\mathrm{trace}(W)$. For comparison, optimal locations of stationary sensors were also determined using the same technique (i.e. by setting

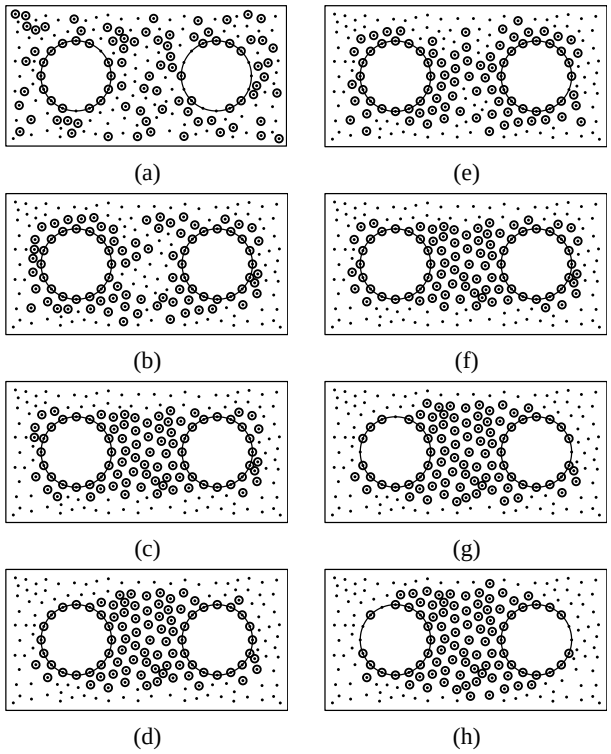

Fig. 2. Optimal selection of consecutive sensor configurations for the criteria $\Psi_1[W] = -\ln\det(W)$ (panels (a)–(d)) and $\Psi_2[W] = -\operatorname{trace}(W)$ (panels (e)–(h)).

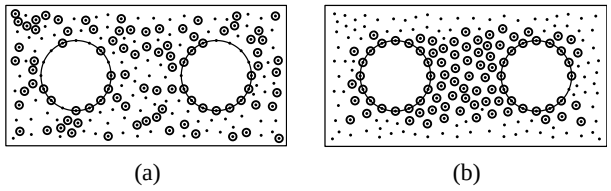

Fig. 3. Optimal selection of stationary sensor locations for the criteria $\Psi_1[W] = -\ln\det(W)$ (panel (a)) and $\Psi_2[W] = -\operatorname{trace}(W)$ (panel (b)).

$L = 1$). Convergence of the algorithm is rapid, as in the most time-consuming case of the criterion Ψ_1 one run took no more than three minutes. The results are shown in Figs. 2 and 3.

As was expected, major improvements are observed when scanning sensors are applied, since the ratios of the best absolute values for the scanning case to those for the stationary case equal 1.242 and 1.116 for the criteria $\Psi_1[W]$ and $\Psi_2[W]$, respectively. In applications, $\Psi_2[W]$ is often preferred, as its linearity with respect to the contributions from different sensors leads to extremely simple computations. Note, however, that this apparent advantage is illusory, since the resulting Gramian is close to singularity, which may cause some problems with the system observability. But this phenomenon is rather obvious, since a larger trace of a matrix does not necessarily imply that the matrix is non-singular. This drawback is eliminated through the use of the determinant of the Gramian as the performance index.

As for the interpretation of the produced solution, the scanning sensors are to occupy positions which give best information about the initial state. As time elapses, the candidate points close to the outer boundary provide less relevant information since the state at them is mostly determined by the Dirichlet boundary conditions. Thus, intuitively, the sensors should tend toward the center of Ω, Such a behavior is exhibited by the solution for the criterion $\Psi_1[W]$, which constitutes an additional argument for its superiority over $\Psi_2[W]$.

VI. CONCLUSIONS

The paper presents a new approach to the problem of scanning sensor location for linear distributed parameter dynamic systems based on various criteria defined on the observability Gramian. A close connection was established between this problem and modern optimum experimental design theory. The main idea is to operate on the density of sensors per unit area instead of the positions of individual sensors. It was shown that the optimal solutions obey certain minimax properties that lead to a rapidly convergent algorithm. The technique can be extended to Kalman filtering and robust control. A version suitable for on-line control architectures is currently developed.

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