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H_∞ Control of Differential Linear Repetitive Processes

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***Abstract*— Repetitive processes are a distinct class of two-dimensional systems (i.e. information propagation in two independent directions) of both systems theoretic and applications interest. They cannot be controlled by direct extension of existing techniques from either standard (termed 1D here) or two-dimensional (2D) systems theory. Here we give new results on the relatively open problem of the design of physically based control laws using an H_∞ setting. These results are for the sub-class of so-called differential linear repetitive processes which arise in applications areas such as iterative learning control.**

I. Introduction

Linear repetitive processes are a distinct class of 2D systems of both system theoretic and applications interest. The essential unique characteristic of such a process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, leads to the unique control problem for these processes in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction.

To introduce a formal definition, let $\alpha < +\infty$ denote the pass length (assumed constant). Then in a repetitive process the pass profile $y_k(t)$, $0 \leq t \leq \alpha$, generated on pass k acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile $y_{k+1}(t), 0 \leq t \leq \alpha, k \geq 0$.

Physical examples of repetitive processes include long-wall coal cutting and metal rolling operations (see, for example, [8]). Also in recent years applications have arisen where adopting a repetitive process setting for analysis has distinct advantages over alternatives. Examples of these so-called algorithmic applications include classes of iterative learning control (ILC) schemes [1] and iterative algorithms for solving nonlinear dynamic optimal control problems based on the maximum principle [7]. In the case of ILC for the linear dynamics case, the stability theory for so-called differential and discrete linear repetitive processes is the essential basis for a rigorous stability/convergence theory for a powerful class of such algorithms.

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Attempts to control these processes using standard (or 1D) systems theory/algorithms fail (except in a few very restrictive special cases) precisely because such an approach ignores their inherent 2D systems structure, i.e. information propagation occurs from pass-to-pass and along a given pass. In seeking a rigorous foundation on which to develop a control theory for these processes, it is natural to attempt to exploit structural links which exist between these processes and other classes of 2D linear systems.

The fact that the pass length is finite (and hence information in this direction only occurs over a finite duration) is the key difference with other classes of 2D linear systems. Hence there is a need to develop a systems theory for these processes for onward translation (where appropriate) into numerically reliable design algorithms.

A rigorous stability theory for linear repetitive processes has been developed. This theory [8] is based on an abstract model in a Banach space setting which includes all such processes as special cases. Also the results of applying this theory to a wide range of cases have been reported, including the so-called differential linear repetitive processes considered here. This has resulted in stability tests that can, if desired, be implemented by direct application of well known 1D linear systems tests.

One unique feature of repetitive processes is that it is possible to define physically meaningful control laws for them. For example, in the ILC application, one such family of control laws is composed of state feedback control action on the current pass combined with information 'feedforward' from the previous pass (or trial in the ILC context) which, of course, has already been generated and is therefore available for use. In the general case of repetitive processes it is clearly highly desirable to have an analysis setting where such control laws can be designed for stability and/or guaranteed performance. Also previous work has shown that an LMI re-formulation of the stability conditions for discrete linear repetitive processes leads naturally to design algorithms for control laws of this form — see, for example, [6].

The H_∞ setting for the control related analysis of 1D linear systems is now a very mature area and it is natural question to ask if such an approach can be extended to 2D linear systems/linear repetitive processes. In the case of 2D discrete linear systems, some work on an H_∞ approach to

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analysis has been reported — see, for example, [3]. In the case of differential linear repetitive processes, little or no work has yet been reported. The fact that it is possible to define physically meaningful control laws for these processes strongly suggests that an H_∞ based methodology should be very profitable with onward translation to, for example, the ILC area where the problem of what is meant by robustness of such schemes is still a largely open general question. Note also that since the dynamics along the pass is governed by a linear matrix differential equation, none of the results developed for the control of 2D discrete linear systems are applicable to the repetitive processes considered in this work.

In this paper, we first give new results on the control of differential linear repetitive processes which formulate and solve the fundamental problem of finding an admissible controller such that a transfer function (matrix) which defines closed-loop performance satisfies a scalar magnitude constraint. By optimizing the controller over the scalar magnitude constraint γ, we get as close as required to the minimal H_∞ norm. Also it is shown that the H_∞ control problem here can, in computational terms, be solved using linear matrix inequalities (LMIs) [2]. Finally, significant new results on the robust control of these processes are developed from this standpoint.

Throughout this paper, the null matrix and the identity matrix with the required dimensions are denoted by 0 and I, respectively. Moreover, $M > 0$ (< 0) denotes a real symmetric positive (negative) definite matrix. The L_2 norm of the $q \times 1$ vector $w_l(t)$ defined over $[0, \infty], [0, \infty]$ is given by

$$\|w\|_2 = \sqrt{\sum_{l=0}^{\infty} \int_0^{\infty} w_l(t)^T w_l(t)\ dt} \tag{1}$$

and w_l is said to be a member of $L_2^q\{[0, \infty], [0, \infty]\}$, or L_2^q for short, if $||w_l||_2 < \infty$. We use $(*)$ to denote the transpose of matrix blocks in some of the LMIs employed (which are required to be symmetric).

II. Background

The differential linear repetitive processes considered here are described by a state space model of the following form over $0 \leq t \leq \alpha$, $k \geq 0$

$$\begin{aligned} \dot{x}_{k+1}(t) &= Ax_{k+1}(t) + Bu_{k+1}(t) + B_0 y_k(t) + B_1 w_{k+1}(t) \\ y_{k+1}(t) &= Cx_{k+1}(t) + Du_{k+1}(t) + D_0 y_k(t) + D_1 w_{k+1}(t) \end{aligned} \tag{2}$$

Here on pass k, $x_k(t)$ is the $n \times 1$ state vector, $y_k(t)$ is the $m \times 1$ pass profile vector, $u_k(t)$ is the $l \times 1$ vector of control inputs and $w_k(t)$ is an $r \times 1$ disturbance input vector which belongs to L_2^r.

To complete the process description, it is necessary to specify the boundary conditions i.e. the state initial vector on each pass and the initial pass profile (i.e. on pass 0). The simplest possible choice for these is

$$\begin{aligned} x_{k+1}(0) &= d_{k+1},\ k \geq 0 \\ y_0(t) &= f(t) \end{aligned} \tag{3}$$

where the $n \times 1$ vector d_{k+1} has known constant entries and $f(t)$ is an $m \times 1$ vector whose entries are known functions of t over $[0, \alpha]$. (For ease of presentation, we will make no further explicit reference to the boundary conditions in this paper.)

The stability theory [8] for linear repetitive processes consists of two distinct concepts but here it is the stronger of these which is required. This is termed stability along the pass and several equivalent sets of necessary and sufficient conditions for processes described by (2) to have this property are known but here it is following which will be used.

Theorem 1: [8] A differential linear repetitive process described by (2) is stable along the pass if, and only if,

$$\rho(s, z) \neq 0,\ \forall\ (s, z) : \mathrm{Re}(s) \geq 0, |z| \leq 1$$

where

$$\rho(s, z) := \det \begin{bmatrix} sI_n - A & -B_0 \\ -zC & I_m - zD_0 \end{bmatrix}$$

Now, define the following matrices from the state space model (2)

$$\widehat{A}_1 = \begin{bmatrix} A & B_0 \\ 0 & 0 \end{bmatrix},\ \ \widehat{A}_2 = \begin{bmatrix} 0 & 0 \\ C & D_0 \end{bmatrix}$$

Then we have the following sufficient condition for stability along the pass in terms of an LMI [6].

Theorem 2: A differential repetitive process described by (2) is stable along the pass if there exist matrices $P_1 > 0$, $P_2 > 0$, and $P_3 > 0$ such that the following LMI is feasible

$$\begin{bmatrix} -S & S\widehat{A}_2 \\ \widehat{A}_2^T S & \widehat{A}_1^T P + P\widehat{A}_1 - R \end{bmatrix} < 0 \tag{4}$$

where $P = \mathrm{diag}\{P_1, 0\}$, $R = \mathrm{diag}\{0, P_2\}$, $S = \mathrm{diag}\{P_3, P_2\}$.

Proof: Consider the state space model (2) with no control and disturbance inputs, then by introducing the change of variables

$$\begin{aligned} l &= k + 1 \\ v_l(t) &= y_{k+1}(t) \end{aligned} \tag{5}$$

it can be rewritten in the form

$$\begin{bmatrix} \dot{x}_l(t) \\ v_{l+1}(t) \end{bmatrix} = \widehat{A}_1 \xi_l(t) + \widehat{A}_2 \xi_l(t) \tag{6}$$

where

$$\xi_l(t) = \begin{bmatrix} x_l(t) \\ v_l(t) \end{bmatrix} \tag{7}$$

Now define the candidate Lyapunov function for this process as

$$\begin{aligned} V(l, t) &= V_1(l, t) + V_2(l, t) \\ &= x_l^T(t) P_1 x_l(t) + v_l^T(t) P_2 v_l(t) \end{aligned} \tag{8}$$

where $P_1 > 0$ and $P_2 > 0$. (This function is combination of two independent indeterminates due to the 2D nature of the repetitive processes considered here.) Since

$$\dot{V}_1(l, t) = \dot{x}_l^T(t) P_1 x_l(t) + x_l^T P_1 \dot{x}_l(t)$$

and

$$\Delta V_2(l,t) = v_{l+1}^T(t)P_2v_{l+1}(t) - v_l^T(t)P_2v_l(t)$$

the associated increment for (8) is

$$\begin{aligned}\Delta V(l,t) =& \dot{V}_1(l,t) + V_2(l+1,t) - V_2(l,t)\\ =& \dot{x}_l^T(t)P_1x_l(t) + x_l^TP_1\dot{x}_l(t)\\ &+ v_{l+1}^T(t)P_2v_{l+1}(t) - v_l^T(t)P_2v_l(t)\end{aligned} \tag{9}$$

Substitution of (6) and (7) into this last expression now yields

$$\Delta V(l,t)=\xi_l^T(t)\left(\widehat{A}_1^TP+P\widehat{A}_1+\widehat{A}_2^TR\widehat{A}_2-R\right)\xi_l(t) \tag{10}$$

where

$$P = \begin{bmatrix} P_1 & 0\\ 0 & 0\end{bmatrix},\ \ R = \begin{bmatrix} 0 & 0\\ 0 & P_2\end{bmatrix} \tag{11}$$

Hence stability along the pass holds if $\Delta V(l,t) < 0$ for $\xi_l(t) \neq 0$, and a sufficient condition for this is

$$\widehat{A}_1^TP+P\widehat{A}_1+\widehat{A}_2^TR\widehat{A}_2-R<0 \tag{12}$$

or, equivalently,

$$\widehat{A}_1^TP + P\widehat{A}_1 + \widehat{A}_2^TS\widehat{A}_2 - R < 0 \tag{13}$$

where $S = \mathrm{diag}\{P_3, P_2\}$, and $P_3 > 0$ is arbitrary. Finally, an obvious application of the Schur complement yields the equivalent condition of (4) and the proof is complete.

III. The H_∞ norm bound

Consider now the case of a differential linear repetitive process (2) with no control inputs but with external disturbance inputs and written in the form

$$\begin{bmatrix} \dot{x}_l(t)\\ v_{l+1}(t)\end{bmatrix} = \begin{bmatrix} A & B_0\\ C & D_0\end{bmatrix}\begin{bmatrix} x_l(t)\\ v_l(t)\end{bmatrix} + \begin{bmatrix} B_1\\ D_1\end{bmatrix}w_l(t) \tag{14}$$

and define the so-called measured output vector $z_l(t)$, which in this case is equal the pass profile vector, as

$$z_l(t) = \begin{bmatrix} 0 & I\end{bmatrix}\begin{bmatrix} x_l(t)\\ v_l(t)\end{bmatrix} \tag{15}$$

Also introduce

$$B_{11} = \begin{bmatrix} B_1\\ 0\end{bmatrix},\ D_{11} = \begin{bmatrix} 0\\ D_1\end{bmatrix},\ H = \begin{bmatrix} 0 & I\end{bmatrix} \tag{16}$$

then the major result in this section is Theorem 3 below which gives an H_∞ condition for stability along the pass and requires the following definition.

Definition 1: A differential linear repetitive process which can be written in the form (14) is said to have H_∞ noise attenuation (or norm bound) γ if it is stable along the pass and its induced norm is bounded by γ i.e.

$$\sup_{0\neq w\in L_2^r}\frac{\|z\|_2}{\|w\|_2} < \gamma \tag{17}$$

Theorem 3: A differential linear repetitive process which can be written in the form (14) and (15) is stable along the pass and has H_∞ norm bound $\gamma > 0$ if there exists matrices $P_1 > 0$, $P_2 > 0$, and $P_3 > 0$, such that the following LMI holds

$$\begin{bmatrix} \widehat{A}_1^TP+P\widehat{A}_1+\widehat{A}_2^TR\widehat{A}_2 + H^TH - R & (\star)\\ B_{11}^TP + D_{11}^TR\widehat{A}_2 & D_{11}^TRD_{11} - \gamma^2I\end{bmatrix} < 0 \tag{18}$$

where S, P, R are defined in (11) and (13).

Proof: In order to ensure the H_∞ noise attenuation γ holds, it is required that the associated Hamiltonian defined by

$$H(l,t) = \Delta V(l,t)+v_l^T(t)v_l(t) - \gamma^2w_l^T(t)w_l(t) \tag{19}$$

satisfies

$$H(l,t) < 0 \tag{20}$$

The remainder of the proof now involves extensive but routine manipulations. In summary, these involve the construction of the increment $\Delta V(l,t)$ and then appropriate substitutions to yield

$$\begin{aligned}H(l,t) =& \Delta V(l,t)+z_l^T(t)H^THz_l(t)-\gamma^2w_l^T(t)w_l(t)\\ =& \begin{bmatrix} z_l^T(t) & w_l^T(t)\end{bmatrix}\\ &\times\begin{bmatrix} \widehat{A}_1^TP+P\widehat{A}_1+\widehat{A}_2^TR\widehat{A}_2+H^TH-R & (\star)\\ B_{11}^TP + D_{11}^TR\widehat{A}_2 & D_{11}^TRD_{11} - \gamma^2I\end{bmatrix}\begin{bmatrix} z_l(t)\\ w_l(t)\end{bmatrix}\end{aligned} \tag{21}$$

Finally (18) guarantees that (20) holds for any $z_l(t), w_l(t) \neq 0$ and the proof is complete.

IV. H_∞ Control via state feedback

The design of control laws for 2D discrete linear systems described by the Roesser and Fornasini Marchesini state space models (see, for example, the references cited in [8]) has received considerable attention in the literature over the years. A valid criticism of such work, however, is that the structure of the control algorithms are not well founded physically due to the fact that, for example, the concept of a state for these systems is not uniquely defined. For example, it is possible to define a state feedback law based on the local or global state vectors. Also in the absence of generalizations of well defined and understood 1D concepts, e. g. the pole assignment problem and error actuated output feedback control action, it has not been really possible to formulate a control design problem beyond that of obtaining conditions for stabilization under the control action. Similar comments also hold for 2D continuous-discrete systems but with the extra remark that much less work has been reported for these systems.

The first difficulty above does not arise with differential linear repetitive processes. For example, it is physically meaningful to define the current pass error as the difference, at each point along the pass, between a specified reference trajectory for that pass, which in most cases will be the same on each pass, and the actual pass profile produced.

Then it is possible to define a so-called current pass error actuated controller which uses the generated error vector to construct the current pass control input vector. In which context, preliminary work, see, for example, [8], has shown that, except in a few very restrictive special cases, the controller used must be actuated by a combination of current pass information and 'feedforward' information from the previous pass to guarantee even stability along the pass closed-loop. Note also here that in the ILC application area the previous trial output vector is an obvious signal to use as feedforward action.

One control law with this structure is

$$u_l(t) = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} x_l(t) \\ v_l(t) \end{bmatrix} \tag{22}$$

where K_1 and K_2 are appropriately dimensioned matrices to be designed. In effect, this control law uses feedback of the current state vector (which is assumed to be available for use) and 'feedforward' of the previous pass profile vector. Note that in repetitive processes the term 'feedforward' is used to describe the case where state or pass profile information from the previous pass (or passes) is used as (part of) the input to a control law applied on the current pass, i.e. to information which is propagated in the pass-to-pass (k) direction.

In the case of the second difficulty, the following result shows that the LMI setting extends to allow the design of a control law of the form (22) for stability along the pass closed loop with a prescribed H_∞ bound.

Theorem 4: Suppose that a differential linear repetitive process described by (2) is subject to a control law defined by (22). Then the resulting closed loop process is stable along the pass and has prescribed H_∞ norm bound $\gamma > 0$ if there exist matrices $W_1 > 0$, $W_2 > 0$, $W_3 > 0$ and N_1, N_2 such that the following LMI holds

$$\begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \end{bmatrix} < 0 \tag{23}$$

where

$$W_{11} = \begin{bmatrix} -W_3 & 0 \\ 0 & -W_2 \\ 0 & W_1C^T + N_1^TD^T \end{bmatrix}$$

$$W_{12} = \begin{bmatrix} 0 \\ CW_1 + DN_1 \\ W_1A^T + N_1B^T + AW_1 + BN_1 \end{bmatrix}$$

$$W_{13} = \begin{bmatrix} 0 & 0 & 0 \\ D_0W_2 + DN_2 & D_1 & 0 \\ B_0W_2 + BN_2 & B_1 & 0 \end{bmatrix}$$

$$W_{21} = \begin{bmatrix} 0 & W_2D_0^T + N_2^TD^T \\ 0 & D_1^T \\ 0 & 0 \end{bmatrix}$$

$$W_{22} = \begin{bmatrix} W_2B_0^T + N_2^TB^T \\ B_1^T \\ 0 \end{bmatrix}$$

$$W_{23} = \begin{bmatrix} -W_2 & 0 & I \\ 0 & -\gamma^2 I & 0 \\ I & 0 & -I \end{bmatrix}$$

In this condition holds, the controller matrices K_1 and K_2 are given by $N_1W_1^{-1}$ and $N_2W_2^{-1}$ respectively.

Proof: Application Theorem 3 gives that the closed loop process in this case is stable along the pass with prescribed H_∞ norm bound $\gamma > 0$ if

$$\begin{bmatrix} -S & S\overline{A}_2 & SD_{11} \\ \overline{A}_2^TS & \overline{A}_1^TP + P\overline{A}_1 + H^TH - R & PB_{11} \\ D_{11}S & B_{11}^TP & -\gamma^2 I \end{bmatrix} < 0 \tag{24}$$

where

$$\overline{A}_1 = \begin{bmatrix} A + BK_1 & B_0 + BK_2 \\ 0 & 0 \end{bmatrix}$$

$$\overline{A}_2 = \begin{bmatrix} 0 & 0 \\ C + DK_1 & D_0 + DK_2 \end{bmatrix}$$

Note that this last condition is not linear in S, P, R, K_1 and K_1. To proceed, first apply the Schur complement to yield

$$\begin{bmatrix} -S & S\overline{A}_2 & SD_{11} & 0 \\ \overline{A}_2^TS & \overline{A}_1^TP + P\overline{A}_1 - R & PB_{11} & H \\ D_{11}^TS & B_{11}^TP & -\gamma^2 I & 0 \\ 0 & H & 0 & -I \end{bmatrix} < 0 \tag{25}$$

Then substituting for $\overline{A}_1$ and $\overline{A}_2$ in this last expression, pre and post-multiplying the result by

$$\text{diag}\left\{P_3^{-1}, P_2^{-1}, P_1^{-1}, P_2^{-1}, I, I\right\}$$

and finally setting $W_1 = P_1^{-1}$, $W_2 = P_2^{-1}$, $W_3 = P_3^{-1}$, $N_1 = K_1P_1^{-1}$, $N_2 = K_2P_2^{-1}$ results in (23) and the proof is complete.

Remark 1: In many cases of practical interest it is desirable to compute the minimum H_∞ norm bound γ. This minimum can be obtained by solving a linear objective minimization problem [2] (the EVP problem) of the following form

$$\begin{aligned} &\text{minimize } \varepsilon \text{ subject to} \\ &W_1 > 0, W_2 > 0, W_3 > 0, N_1, N_2, \varepsilon > 0 \\ &\quad \text{and } (23) \end{aligned} \tag{26}$$

where $\gamma = \sqrt{\varepsilon}$.

V. Robust H_∞ control

Here we extend the analysis of the previous section to the case when there is uncertainty in the process state space model. Space limitations preclude detailed analysis of all possible uncertainty models and here we only consider the case when the uncertainty is norm bounded in both the state and pass profile updating equations. In such a case we can write the process state space model in the form

$$\begin{bmatrix} \dot{x}_l(t) \\ v_{l+1}(t) \end{bmatrix} = \left(\begin{bmatrix} A & B_0 \\ C & D_0 \end{bmatrix} + \begin{bmatrix} \Delta A & \Delta B_0 \\ \Delta C & \Delta D_0 \end{bmatrix} \right) \begin{bmatrix} x_l(t) \\ v_l(t) \end{bmatrix} + \left(\begin{bmatrix} B \\ D \end{bmatrix} + \begin{bmatrix} \Delta B \\ \Delta D \end{bmatrix} \right) u_l(t) + \left(\begin{bmatrix} B_1 \\ D_1 \end{bmatrix} + \begin{bmatrix} \Delta B_1 \\ \Delta D_1 \end{bmatrix} \right) w_l(t) \tag{27}$$

where

$$\begin{bmatrix} \Delta A & \Delta B_0 & \Delta B & \Delta B_1 \\ \Delta C & \Delta D_0 & \Delta D & \Delta D_1 \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \mathcal{F} \begin{bmatrix} E_1 & E_2 & E_3 & E_3 \end{bmatrix} \tag{28}$$

H_1, H_2, E_1, E_2, E_3, E_4 are known matrices with constant entries, and $\mathcal{F}$ is an unknown matrix with constant entries which satisfies

$$\mathcal{F}^T \mathcal{F} \leq I \tag{29}$$

In this case, we have the following result.

Theorem 5: Suppose that a linear differential repetitive process of the form described by (27), with uncertainty structure modelled by (28) and (29) is subject to a control law of the form (22). Then the resulting closed-loop process is stable along the pass for all admissible uncertainties and has prescribed H_∞ norm bound $\gamma > 0$ if there exist matrices $W_1 > 0$, $W_2 > 0$, $W_3 > 0$, $W_3 > 0$, N_1, N_2 and a scalar $\epsilon > 0$ such that

$$\begin{bmatrix} -W_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -W_2+3\epsilon H_2 H_2^T & (\star) & (\star) & (\star) & 0 & 0 & 0 & 0 \\ 0 & W_1 C^T + N_1^T D^T + 3\epsilon H_1 H_2^T & \Psi_{33} & (\star) & (\star) & 0 & (\star) & 0 & 0 \\ 0 & W_2 D_0^T + N_2^T D^T & W_2 B_0^T + N_2^T B^T & -W_2 & 0 & I & 0 & (\star) & 0 \\ 0 & D_1^T & B_1^T & 0 & -\gamma^2 I & 0 & 0 & 0 & (\star) \\ 0 & 0 & 0 & I & 0 & -I & 0 & 0 & 0 \\ 0 & 0 & E_1 W_1 + E_3 N_1 & 0 & 0 & 0 & -\epsilon I & 0 & 0 \\ 0 & 0 & 0 & E_2 W_2 + E_3 N_2 & 0 & 0 & 0 & -\epsilon I & 0 \\ 0 & 0 & 0 & 0 & E_4 & 0 & 0 & 0 & -\epsilon I \end{bmatrix} < 0 \tag{30}$$

where

$$\Psi_{33} = W_1 A^T + N_1^T B^T + A W_1 + B N_1 + 3\epsilon H_1 H_1^T$$

If (30) holds then the stabilizing matrices K_1 and K_2 in the control law are given by $N_1 W_1^{-1}$ and $N_2 W_2^{-1}$ respectively.

Proof: Firstly, we need the following standard matrix result

Lemma 1: Let Σ_1, Σ_2 be real matrices of compatible dimensions then for any matrix $\mathcal{F}$ satisfying (29) and a scalar $\epsilon > 0$ the following inequality holds

$$\Sigma_1 \mathcal{F} \Sigma_2 + \Sigma_2^T \mathcal{F} \Sigma_1^T \leq \epsilon^{-1} \Sigma_1 \Sigma_1^T + \epsilon \Sigma_2^T \Sigma_2 \tag{31}$$

Next, incorporate the norm-bounded uncertainties into (23) to obtain sum of a matrix which has uncertainty terms and one whose entries are completely known. The former of these is the one of interest here and it can be written in the form

$$\tilde{H} = \overline{H \mathcal{F} E} + \overline{E}^T \overline{\mathcal{F}}^T \overline{H}^T \tag{32}$$

where

$$\overline{H} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & H_2 & H_2 & H_2 & 0 \\ 0 & 0 & H_1 & H_1 & H_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overline{\mathcal{F}} = \text{diag}\{\mathcal{F}, \mathcal{F}, \mathcal{F}, \mathcal{F}, \mathcal{F}, \mathcal{F}\}$$

$$\overline{E} = \text{diag}\{0, 0, E_1 W_1 + E_3 N_1, E_2 W_2 + E_3 N_2, E_4, 0\}$$

An obvious application of (31) now yields

$$\tilde{H} \leq \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Lambda_{22} & \Lambda_{23} & 0 & 0 & 0 \\ 0 & \Lambda_{32} & \Lambda_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Lambda_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \Lambda_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{33}$$

where the sub-matrices on the right-hand side of this last expression are given by

$$\begin{aligned} \Lambda_{22} &= 3\epsilon H_2 H_2^T, \ \Lambda_{23} = 3\epsilon H_2 H_1^T, \ \Lambda_{32} = 3\epsilon H_1 H_2^T \\ \Lambda_{33} &= 3\epsilon H_1 H_1^T + \epsilon^{-1}(E_1 W + E_3 N_1)^T (E_1 W + E_3 N_1) \\ \Lambda_{44} &= \epsilon^{-1}(E_2 W + E_3 N_2)^T (E_2 W + E_3 N_2), \\ \Lambda_{55} &= \epsilon^{-1} E_4^T E_4 \end{aligned}$$

Finally, the result follows by an obvious application of the Schur complement and the proof is complete.

Remark 2: The controller which ensures a minimum H_∞ norm bound γ can be computed by converting the LMI (30) into a linear objective minimization problem of the form

$$\begin{aligned} &\text{minimize } \varepsilon \text{ subject to} \\ &W_1 > 0, W_2 > 0, W_3 > 0, N_1, N_2, \epsilon > 0, \varepsilon > 0 \\ &\quad \text{and } (30) \end{aligned} \tag{34}$$

where $\gamma = \sqrt{\varepsilon}$.

A. Numerical example

As an example, the metal rolling process is considered. This process is an extremely common industrial process where, in essence, deformation of the workpiece takes place between two rolls with parallel axes revolving in opposite directions.

Appropriate algebraic manipulations [5] show that the metal rolling process can modelled by a differential repetitive process model of the form considered in this paper. In the design studies considered here the matrices in the process model (2) which approximate the dynamics are

$$\begin{aligned} B &= \begin{bmatrix} -\zeta\omega_n & (\zeta^2\omega_n^2 - \omega_n^2 - \frac{c_1 k_a k_c}{M}) \\ 1 & -\zeta\omega_n \end{bmatrix} \ B_1 = \begin{bmatrix} 0 \\ \hat{a} \end{bmatrix}, \\ D &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \ D_1 = c_3 \end{aligned} \qquad (35)$$

where

$$\hat{a} = c_3(c_2 - c_1 k_a k_c)/(\omega_n^2 M + c_1 k_a k_c - \zeta^2 \omega_n^2 M)$$

and $\omega_n = \sqrt{\frac{k_a + \lambda}{M}}$ and $\zeta = \frac{k_b}{2\omega_n M}$ are the (angular) natural frequency and damping ratio of the local servomechanism loop respectively, M is the lumped mass of the roll-gap adjusting mechanism, λ_1 is the stiffness of the adjustment mechanism spring, λ_2 is the hardness of the metal strip, and $\lambda = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}$ is the composite stiffness of the metal strip and the roll mechanism. Furthermore,

$$c_1 = \frac{\lambda}{\lambda_2}, \ c_2 = \lambda c_1, \ c_3 = \frac{\lambda}{\lambda_1} \qquad (36)$$

and

$$C = \begin{bmatrix} 0.4057 \\ 0.9355 \end{bmatrix}, \ D = 0.9169$$

If the parameters k_a, k_b, k_c are such that

$$k_a > \frac{c_2 - \lambda(1 - c_3)}{(\frac{k_c \lambda}{\lambda_2}) + 1 - c_3} \qquad (37)$$

$$k_a k_c < \lambda_1 + \frac{\lambda_1(1 - c_3^2)}{2 c_2 M} k_b^2 \qquad (38)$$

then stability along the pass holds. The design task here is to design a control law of the form (22) such that stability along the pass holds under this control law with H_∞ norm bound less than γ (i.e. an application of Theorem 4).

The numerical data used here is $\lambda_1 = 600$, $\lambda_2 = 2000$, $M = 100$, $k_a = 50$, $k_b = 1$, $k_c = 6$. This yields $\lambda = 461.54$ and

$$A = \left[\begin{array}{cc|c} -0.0050 & -5.8077 & 0 \\ 1 & -0.0050 & 0.0494 \\ \hline 1 & 0 & 0.7692 \end{array}\right]$$

$$B_1 = \begin{bmatrix} 1.3827 \\ 1.1073 \end{bmatrix}, \quad D_1 = 0.2644$$

Then for the particular choice of $\gamma = 1.4$, the design procedure of (23) (implemented using LMI Control Toolbox [4]) gives the solution

$$\begin{aligned} W_1 &= \begin{bmatrix} 16.9993 & 0.9563 \\ 0.9563 & 1.2216 \end{bmatrix}, \ W_2 = 18.1714 \\ N_1 &= \begin{bmatrix} -11.0732 - 6.5747 \end{bmatrix}, \ N_2 = -5.7055 \end{aligned} \qquad (39)$$

and the corresponding control law matrices are

$$K_1 = \begin{bmatrix} -0.3647 - 5.0965 \end{bmatrix}, \ K_2 = -0.3140 \qquad (40)$$

This controller guarantees stability along the pass and ensures that the H_∞ norm bound is never greater than γ.

VI. Conclusions

This paper has developed substantial new results on the relatively open problem of the control of differential linear repetitive processes which are a distinct class of 2D linear systems of both systems theoretic and applications interest. The result is physically based control laws in an H_∞ setting where the required computations are LMI based. Also it has been shown that these results can be extended to the case of uncertainty in the model where here this is assumed to be norm bounded in both the state and pass profile updating equations of the defining state space model. Extensions to other uncertainty representations are also possible and will be reported elsewhere.

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