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Robust Synchronization of a Class of Nonlinear Systems

David I. Rosas Almeida† and Joaquín Alvarez‡
Scientific Research and Advanced Studies Center of Ensenada (CICESE)
Electronics and Telecommunications Department
Km. 17 Carr. Tij. - Ens., Ensenada, B. C., México
†drosas@cicese.mx, ‡jqalvar@cicese.mx

Abstract— In this paper we present a new algorithm to synchronize two nonlinear systems. The systems may differ in structure, parameter values, and have structural uncertainties; therefore, the algorithm is robust in this sense. The conditions on the systems for applying this algorithm are the following: They must have the same order, and be in integrator chain form. Furthermore, we must know the bounds on the parametric uncertainties and on additional terms due to non modeled dynamics. The algorithm is based on the master/slave synchronization scheme. The coupling signal is designed through the sliding mode control technique. The results are illustrated with an experiment where a Duffing circuit and a simple pendulum are synchronized; both systems exhibit a chaotic attractor when they are not coupled.

Keywords: Synchronization, sliding mode control, chaotic systems.

I. INTRODUCTION

Synchronization, in its most general interpretation means correlated or corresponding in-time behavior of two or more processes [2]. In some situations the synchronization is a natural phenomenon; however, in other cases we must add an interconnection system to obtain this phenomenon, or to improve its transitory characteristic. In this situation the synchronization becomes a control objective and it is called controlled synchronization. In this sense, some algorithms have been proposed to obtain controlled synchronization using a feedback signal. These algorithms can be classified in two schemes: the master/slave or unidirectional scheme and the bi-directional scheme. In the unidirectional scheme, the master system dominates the slave system. Thus, the synchronization is based on the behavior of the master system. In the bi-directional scheme the synchronization is a result from interaction of all systems involved.

In the last years there has been an increasing attention to the synchronization of chaotic systems. The synchronization of this class of systems is a special problem because these systems exhibit a complex, oscillatory, non periodic steady state, althougth their motion is bounded. The state trajectories are sensitive to initial conditions; therefore, it does not seem possible to synchronize them.

In 1991 Pecora and Carroll [9] showed that, under certain conditions, some parts of a chaotic system can be reproduced so that these parts and their duplicates exhibit identical chaotic behavior when they are driven by the same input. Since then, many synchronization algorithms for chaotic systems have been proposed. Among others, we can mention the linear feedback control [10], [7], the adaptive and sliding mode control [4], [14] and observed-based synchronization [5], [11]. The synchronization of chaotic systems has interesting applications in several fields of science and technology; for example, the encriptment of information for private communications systems [3].

There are two very important problems in the controlled synchronization: 1)to ensure its stability, and 2) to guarantee its robustness with respect to parametric and structural uncertainties.

Based on the master/slave scheme, in this paper we propose an algorithm to synchronize two nonlinear systems that can display chaotic behavior. The systems may differ in structure, parameter values and have structural uncertainties; therefore, the algorithm is robust in this sense. The conditions on the systems for applying this algorithm are the following: They must have the same order, and be in integrator chain form. Furthermore, we must know the bounds on the parametric uncertainties and on additional terms due to non modeled dynamics. The proposed algorithm is based on a sliding control technique. It can be applied to piecewise smooth systems. Furthermore, it is easier to design and implement than others reported elsewere (for example [14]).

The paper is organized as follows. The synchronization algorithm is developed in section II. In section III, the proposed algorithm is illustrated with an experiment where two chaotic systems are synchronized. Finally, the conclusions are given in section IV.

II. SYNCHRONIZATION BASED ON A SLIDING MODE CONTROL DESIGN

Consider two dynamical systems in normal form; the master is given by

$$\begin{aligned} \dot{x}_{m_i} &= x_{m_{i+1}}, \qquad (i = 1, \ldots, n-1) \\ \dot{x}_{m_n} &= f_m(t, x_m), \end{aligned} \tag{1}$$

where $x_m \in \Re^n$ is the state vector, $dx_i/dt \equiv \dot{x}_i$, $f_m : \Re^+ \times \Re^n \to \Re$ is a piecewise smooth function and $t \in \Re^+$.

In the same form it is defined the slave system

$$\begin{aligned} \dot{x}_{s_i} &= x_{s_{i+1}}, \qquad (i=1,\ldots,n-1) \\ \dot{x}_{s_n} &= f_s(t,x_s) + g(t,x_s)\, u(t,x_m,x_s), \end{aligned} \tag{2}$$

where $x_s \in \Re^n$ is the state vector, $f_s : \Re^+ \times \Re^n \to \Re$ is a piecewise smooth function and $g : \Re^+ \times \Re^n \to \Re^n$ is a continuous function that is not zero for all x_s and $t \geq 0$. In this case, $u(\cdot) \in \Re$ is a control input.

The functions $f_m(\cdot)$ and $f_s(\cdot)$ can be structurally different and at the same time they can have parametric and structural uncertainties.

The objective is to design a control signal $u(\cdot)$ so that the slave system is synchronized with the master system. The synchronization criterion is given by

$$\lim_{t\to\infty} \|x_m(t) - x_s(t)\| = 0.$$

To solve this problem, define the synchronization error variables

$$e_i = x_{m_i} - x_{s_i}. \qquad (i=1,\ldots,n)$$

Then the following system describes the dynamics of the synchronization error

$$\begin{aligned} \dot{e}_i &= e_i, \qquad (i=1,\ldots,n-1) \\ \dot{e}_n &= f_m(t,x_m) - f_s(t,x_s) \\ &\quad - g(t,x_s)\, u(t,x_m,x_s). \end{aligned} \tag{3}$$

Now we design the control signal $u(\cdot)$ based on the sliding mode control design.

Consider a discontinuity surface $S = 0$ defined by

$$S = \sum_{i=1}^{n} \gamma_i e_i, \tag{4}$$

where $\gamma_i \in \Re$, $i = 1,\ldots,n$ are positive constants. On the other hand, we define a control $u(\cdot)$ given by

$$u(t,x_m,x_s) = F(t,x_m,x_s)\, sign(S),$$

where $F(\cdot) : \Re^+ \times \Re^n \times \Re^n \to \Re$ is, in general, a piecewise smooth function. In the following subsections we present how to design the surface S and the control $u(\cdot)$ so that the problem will be solved.

A. Design of the sliding surface

The sliding surface must be designed such that, when the trajectories arrived to the discontinuity surface defined by S, they must be directed to the origin of the state space of the error variables; i.e., a sliding mode is presented.

One way to define the behavior of system (3) when its trajectories are in the surface S is through the equivalent control u_{eq}. It can be seen as the average control when the trajectories of the system are in the surface S [12].

The value of u_{eq} is found from the equation $\dot{S} = 0$, which from equation (3) has the form

$$\begin{aligned} \dot{S} = &\sum_{i=2}^{n} \gamma_{i-1} e_i + \gamma_n f_m(t,x_m) \\ - &\gamma_n f_s(t,x_s) - \gamma_n g(t,x_s)\, u_{eq} = 0. \end{aligned}$$

If $\gamma_n = 1$, the equivalent control is given by

$$u_{eq} = \left(\sum_{i=2}^{n} \gamma_{i-1} e_i + f_m(t,x_m) - f_s(t,x_s) \right) / g(t,x_s).$$

Substituting u_{eq} into (3) gives

$$\begin{aligned} \dot{e}_i &= e_i, \qquad (i=1,\ldots,n-1) \\ \dot{e}_n &= -\sum_{i=2}^{n} \gamma_{i-1} e_i. \end{aligned} \tag{5}$$

The last $n-1$ equations form a linear uncoupled system from e_1 with the form

$$\dot{\widetilde{e}} = A\widetilde{e}, \tag{6}$$

where

$$\widetilde{e} = [e_2, e_3, \ldots, e_n]^T$$

and

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -\gamma_1 & -\gamma_2 & -\gamma_3 & \cdots & -\gamma_{n-2} & -\gamma_{n-1} \end{bmatrix}$$

System (6) has a unique equilibrium point at the origin of the error space and is exponentially stable if the constants $\gamma_{i-1} > 0$ for $i = 2,..,n$ are chosen such that matrix A is strictly Hurwitz. Thus, the trajectories in the discontinuity surface will go to the origin of the error space.

Now, we will find the conditions so that the discontinuity surface will be a sliding surface; i.e., to find the conditions on $u(\cdot)$ such that the trajectories out of the surface S go to this surface in a finite time.

B. Conditions for the existence of a sliding mode

Consider the following criterion given in [12]

If

$$S\dot{S} < 0 \qquad \forall S \neq 0 \text{ and } \forall t \geq 0,$$

then the surface S is a sliding surface.

In our case, we have the following

$$\begin{aligned} S\dot{S} &= S\left(\sum_{i=2}^{n} \gamma_{i-1} e_i + f_m(t,x_m) - f_s(t,x_s) \right) \\ &\quad - S g(t,x_s)\, F(t,x_m,x_s)\, sign(S) \end{aligned} \tag{7}$$

if $g(t,x_s) \neq 0$ for all x_s and $t \geq 0$, we propose $F(\cdot)$ of the form

$$F(t,x_m,x_s) = g(t,x_s)^{-1} h(t,x_m,x_s), \qquad (8)$$

where $h(t,x_m,x_s) > 0$ for all x_m, x_s and $t \geq 0$. Replacing (8) in (7) we obtain

$$\begin{aligned} S\dot{S} &= S\left(\sum_{i=2}^{n} \gamma_{i-1} e_i + f_m(t,x_m) - f_s(t,x_s)\right) \\ &\quad - h(t,x_m,x_s)|S| \leq \Gamma|S|, \end{aligned}$$

where

$$\begin{aligned} \Gamma &= \sum_{i=2}^{n} \gamma_{i-1}|e_i| + |f_m(t,x_m)| \\ &\quad + |f_s(t,x_s)| - h(t,x_m,x_s). \end{aligned}$$

The problem is solved if we find a function $h(t,x_m,x_s)$ such that

$$\begin{aligned} h(t,x_m,x_s) \geq \sum_{i=2}^{n} \gamma_{i-1}|e_i| \\ + |f_m(t,x_m)| + |f_s(t,x_s)| \end{aligned} \qquad (9)$$

$\forall\ e, x_m, x_s$ and $\forall t \geq 0$.

In the design of the function $h(\cdot)$ it is not necessary to know the functions $f_m(\cdot)$ and $f_s(\cdot)$ in exact form. Therefore, this algorithm is robust to parametric and structural uncertainties.

On the other hand, the condition about the convergence to the surface S in a finite time is also satisfied with the control input given by (8) and (9). We can prove the last statement by using the following criterion given in [12].

If

$$\begin{aligned} \lim_{S\to 0^-} \dot{S} &> 0 \\ \lim_{S\to 0^+} \dot{S} &< 0 \end{aligned} \qquad (10)$$

then the convergence to the surface S is in a finite time.

For this case, $\dot{S}$ is given by

$$\begin{aligned} \dot{S} &= \sum_{i=2}^{n} \gamma_{i-1} e_i + f_m(t,x_m) - f_s(t,x_s) \\ &\quad - h(t,x_m,x_s)\, sign(S) \end{aligned}$$

and by design of $h(t,x_m,x_s)$ (equation (9)) both conditions in (10) are satisfied in global or local form, this depends on the function $h(t,x_m,x_s)$.

In the next section we illustrate this synchronization algorithm for chaotic systems through an example where a Duffing circuit and a pendulum are synchronized.

III. EXPERIMENTAL ILLUSTRATION OF THE PROPOSED ALGORITHM

A. The pendulum model

Consider a forced simple pendulum with viscous and Coulomb friction,

$$\left(I + ml_c^2\right)\ddot{\theta} + \rho\dot{\theta} + \alpha \text{sign}\left(\dot{\theta}\right) + mgl_c \sin(\theta) = \tau(t), \qquad (11)$$

where I =0.0085 [kg$\cdot$ m^2] represents the inertia, m =1.6365 [kg] is the mass of the pendulum, l_c =0.0762 [m] is the center of masses, ρ =0.00054 [Nm] is the viscous frictional coefficient, α =0.05492 [Nm] is the Coulomb frictional coefficient, g is the gravity coefficient, $\tau(t)$ is the applied torque, and sign$(\cdot)$ is the signum function defined by

$$\text{sign}(x) = \begin{cases} 1, & x > 0 \\ [-1,1], & x = 0 \\ -1, & x < 0 \end{cases}.$$

These parameter values correspond to a pendubot manufactured by Mechatronic Systems Inc [8]. The state space form of system (11) is given by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -k_1 x_2 - k_2 \text{sign}(x_2) - k_3 \sin(x_1) + k_4 \tau(t), \end{aligned}$$

where $k_1 = 2.9996^{-2}$, $k_2 = 3.0507$, $k_3 = 67.912$ and $k_4 = 55.549$. If

$$\tau(t) = A\sin(\omega t)$$

it has been shown that it exhibits a chaotic behavior for some values of A and ω [1].

B. The Duffing circuit

Duffing's system is defined by the following equations

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_1 - x_1^3 - ax_2 + b\sin(wt), \end{aligned} \qquad (12)$$

with $a = 0.25$, $b = 0.3$ and $w = 1$ it presents a chaotic atractor [6].

We made a time scaling to facilitate the implementation of the circuit. We define a new time variable $\tau = t/\alpha$; if $\alpha = 6.43$ then the equation (12) can be written in the following form

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= 41.345z_1 - 41.345z_1^3 - 1.6075z_2 \\ &\quad + 12.403\sin(6.43t), \end{aligned} \qquad (13)$$

these equations define the master system in the synchronization scheme.

The Duffing circuit is shown in figure 1.

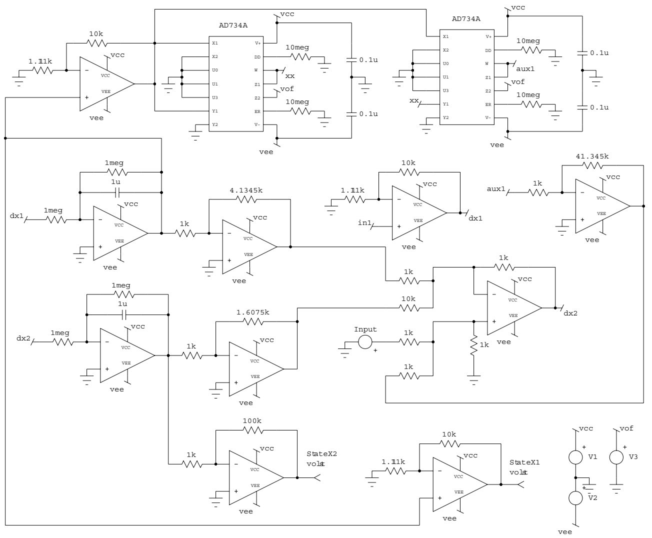

Fig. 1. The Duffing circuit.

C. Synchronization of a pendulum and a Duffing circuit

Consider the Duffing circuit (13) and the pendulum model given by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -(k_1+\Delta_1)\,x_2-(k_2+\Delta_2)\,sign\,(x_2) \\ &\quad -(k_3+\Delta_3)\sin(x_1)+(k_4+\Delta_4)\,(\tau+u\,(x,z)), \end{aligned} \tag{14}$$

where Δ_i, $i=1,...,4$, are possible parametric variations due to parametric uncertainties; they are not known, but we know their maximum bounds given by[1].

$$\begin{aligned} |\Delta_1| &\le \delta_1 = 1.3768\times10^{-2}, \\ |\Delta_2| &\le \delta_2 = 1.6241, \\ |\Delta_3| &\le \delta_3 = 9.368, \\ |\Delta_4| &\le \delta_4 = 23.576. \end{aligned}$$

The error dynamics betwen the master and slave system is given by

$$\begin{aligned} \dot{e}_1 &= e_2, \\ \dot{e}_2 &= 41.345z_1-41.345z_1^3-1.6075z_2 \\ &\quad +12.403\sin(6.43t)+(k_1+\Delta_1)\,x_2 \\ &\quad +(k_2+\Delta_2)\,sign\,(x_2)+(k_3+\Delta_3)\sin(x_1) \\ &\quad -(k_4+\Delta_4)\,\tau-(k_4+\Delta_4)\,u\,(x,z). \end{aligned} \tag{15}$$

Now, we define the discontinuity surface S by

$$S = me_1+e_2.$$

[1]We consider variations of $\pm20\%$ from the nominal values of the parameters.

Thus, the equivalent control u_{eq} is given by

$$u_{eq}\,(x,z)=\frac{\Phi}{(k_4+\Delta_4)},$$

where

$$\begin{aligned} \Phi = {} & me_2+41.345z_1-41.345z_1^3 \\ & -1.6075z_2+(k_1+\Delta_1)\,x_2 \\ & +12.403\sin(6.43t) \\ & +(k_2+\Delta_2)\,sign\,(x_2) \\ & +(k_3+\Delta_3)\sin(x_1)-(k_4+\Delta_4)\,\tau. \end{aligned}$$

Replacing u_{eq} in (15) we obtain

$$\begin{aligned} \dot{e}_1 &= e_2, \\ \dot{e}_2 &= -me_2. \end{aligned} \tag{16}$$

If $m>0$ we obtain a sliding mode.

Now, we propose an input control u given by

$$\begin{aligned} u &= (k_4-\Delta_{4\,\max})\,F\,(x,z)\,sign\,(s) \\ &= a_1\,|e_2|+a_2\,|z_1|+a_3\,|z_1^3| \\ &\quad +a_4\,|z_2|+a_5\,|x_2|+a_6 \end{aligned} \tag{17}$$

where $a_1 > m/31.973$, $a_2 > 1.2931$, $a_3 > 1.2931$, $a_4 > 0.050277$, $a_5 > 0.00136$, $a_6 > 4.2132$. This input control satisfies the conditions (8) and (9); therefore, the problem is solved.

Some numerical results are presented in figures 2 and 3. The control parameters are the following: $a_1 = (m/31.973)1.2$, $a_2 = 1.3$, $a_3 = 1.3$, $a_4 = 0.1$, $a_5 = 0.01$, $a_6 = 4.4$ and $m = 30$. As we can see in figure 3, the synchronization algorithm is robust. We applied variations of 0, 10 and 20 percent in all parameters of the pendulum, the error between state variables of the master and slave systems for each case are practically the same.

D. Experimental results

By security, in the first stage of experiments to illustrate the proposed synchronization algorithm, we did not include the real mechanical system; it has been emulated in a RT card. In the following experiment stage, we will use the real mechanical system.

The experiment was made in the following form. The pendulum's equations and the input control were designed in Simulink® and were loaded in the real time card of the DSpace® development system. With the same card and program were acquired the state variables of the Duffing circuit to close the control loop.

The results are the following. The behavior of the master and slave systems with no control input is shown in figure 4. As we can see,there is not natural synchronization; their corresponding behavior is very different.

Figures 5 and 6 show the results when the input control (17) is applied to the slave system. As we can see, in

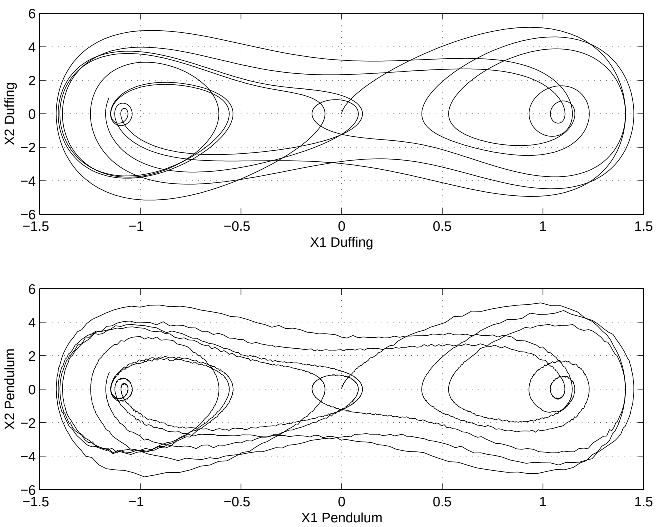

Fig. 2. State trajectories of the master and slave systems of the controlled case (Simulated).

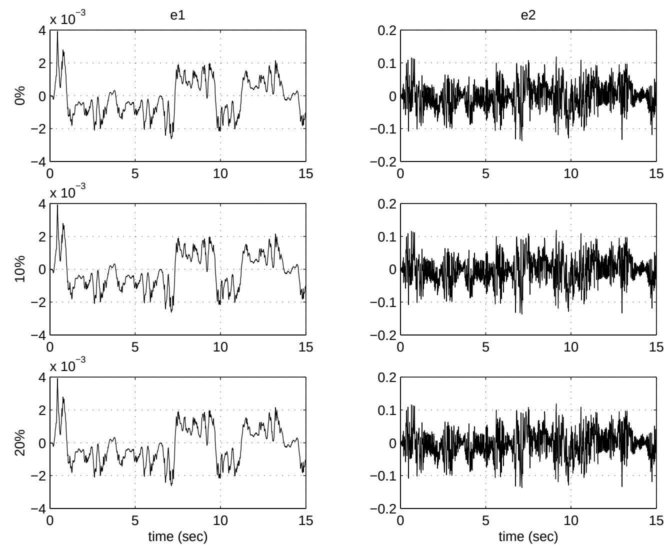

Fig. 3. Synchronization errors for different cases for variations in the pendulum parameters (Simulated).

this case the error variables e_1 and e_2 have values near to zero but they do not stay at zero. This is due to the chattering in the control input, a natural phenomenon when the sliding mode control is applied. Therefore, we can say that the systems are approximately synchronized [2], i.e.

$$\lim_{t\to\infty} \|x_m(t) - x_s(t)\| \leq \varepsilon.$$

for a sufficiently small $\varepsilon > 0$. In this case, ε is the magnitude of the chattering in the error state, which in general, is directly related with unmodelled dynamics in the closed loop system and with the magnitude of the control input [13]. In our example, the unmodelled dynamics are delays in the real time card and the magnitude of the control input is directly related with function $h(\cdot)$. However, in many applications these errors can be permisible and the synchronization algorithm can be applied.

IV. CONCLUSIONS

In this paper we have proposed a new algorithm to synchronize two nonlinear systems with same relative degree, and minimum phase. This algorithm is based on the master/slave synchronization scheme. The systems involved may differ in structure, parameter values and have structural uncertainties; therefore, the algorithm is robust in this sense. We note that, with the proposed algorithm, the problem of controlled synchronization is solved; that is, to find a feedback control input u as a function of the states and the time. If we do not have full access to the state, we need to design an observer to estimate the unknown states. On the other hand, due to using the sliding mode control technique the control input produces a small synchronization error; therefore, we obtain approximate synchronization. However, this synchronization error may be permissible in some applications. The algorithm has been illustrated through synchronization of a Duffing circuit (master system) and a simple pendulum (slave system), both systems exhibiting chaotic behavior. In this experiment were considered only the uncertainties in the slave system; however, they can be also considered in the master system. Finally, the algorithm proposed can be straightforward extended to synchronize multi-systems under the master/slave synchronization scheme.

V. ACKNOWLEDGMENTS

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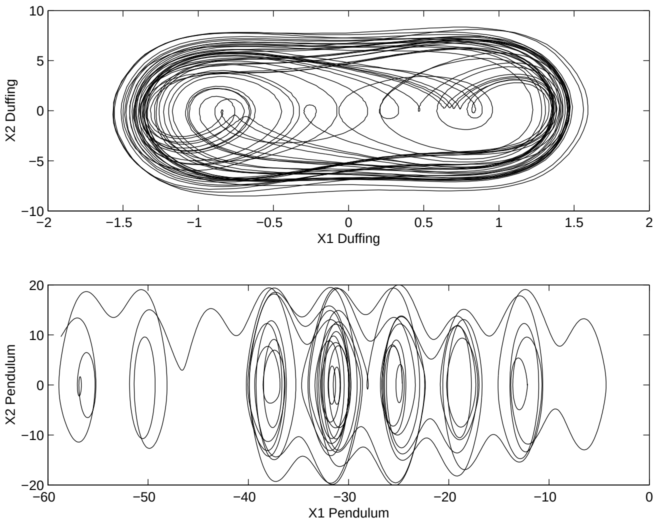

Fig. 4. State trajectories of the master and slave systems of the uncontrolled case (Experimental).

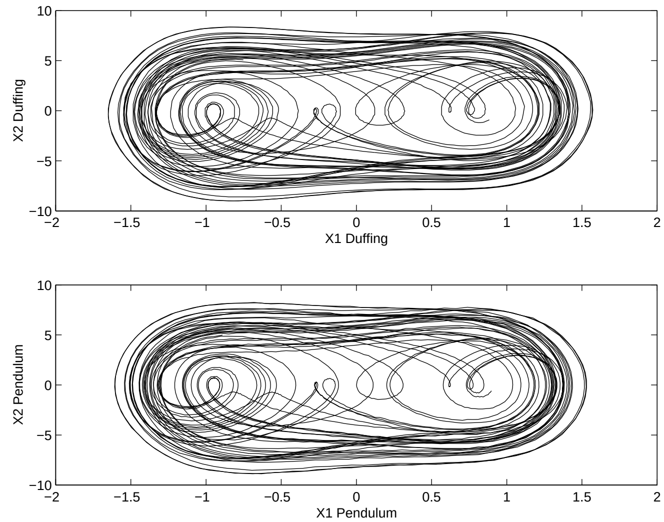

Fig. 6. State trajectories of the master and slave systems of the controlled case (Experimental).

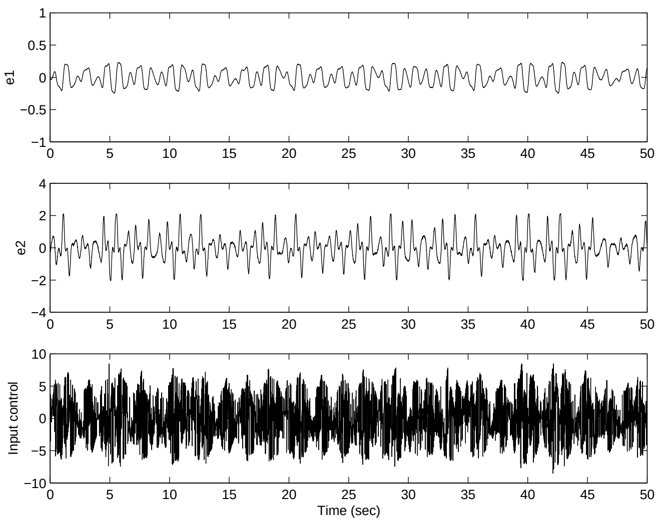

Fig. 5. The errors and control signals of the controlled case (Experimental).

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