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Hopf Bifurcation Control for Affine Systems

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***Abstract*—In this paper we establish conditions to control the Hopf bifurcation of nonlinear systems with two uncontrollable modes on the imaginary axis. We use the center manifold to reduce the system dynamics to dimension two, and find expressions in terms of the original vector fields.**

I. INTRODUCTION

There exists a great interest to analyze control systems that can exhibit complex dynamics. An emerging research field that has become very stimulating is the bifurcation control which, for example, tries to modify the dynamical behavior of a system around bifurcation points, generate a new bifurcation in a desirable parameter value [3], delay the onset of an inherent bifurcation [10], or stabilize a bifurcated solution [1], [2]. In [6] an overview of this field is included.

There are many works that study the bifurcation control problem. In [1], [2], [11] this problem is analyzed using state feedback control. In [9], [5], [8] the problem is investigated using normal forms and invariant.

In this paper, we analyze control systems with two uncontrollable modes on the imaginary axes. We propose a state feedback control $u = u(z;\mu,\beta_1,\beta_2)$ such that μ causes the Hopf bifurcation, β_1 determines the stability of the equilibrium point, and β_2 establishes the orientation and stability of the periodic orbit. This analysis is based on the Hopf bifurcation and center manifold theorems [7], [4].

II. STATEMENT OF THE PROBLEM

Consider the nonlinear system

$$\dot{\xi} = F(\xi) + G(\xi)u, \tag{1}$$

where $\xi \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}$ is the control input. The vector fields $F(\xi)$ and $G(\xi)$ are assumed to be sufficiently smooth, with $F(0) = 0$. Assume that

$$J = DF(0) = \begin{pmatrix} J_H & 0 \\ 0 & J_S \end{pmatrix}$$

with $J_H = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix}_{2\times 2}$, and $J_S \in \mathbb{R}^{(n-2)\times(n-2)}$ a Hurwitz matrix. Suppose that $F(\xi) = \begin{pmatrix} F_1(\xi) \\ F_2(\xi) \end{pmatrix}$, $G(\xi) = \begin{pmatrix} G_1(\xi) \\ G_2(\xi) \end{pmatrix}$, and $\xi = \begin{pmatrix} z \\ w \end{pmatrix}$, with $z \in \mathbb{R}^2$, $w \in \mathbb{R}^{n-2}$, $F_1, G_1 : \mathbb{R}^2 \times \mathbb{R}^{n-2} \to \mathbb{R}^2$, and $F_2, G_2 : \mathbb{R}^2 \times \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$. Then, expanding system (1) around $\xi = 0$ yields

$$\begin{aligned} \dot{z} &= J_H z + F_{21}(z,w) + F_{31}(z,w) + \cdots \\ &\quad + (b_1 + M_1 z + M_2 w + G_{21}(z,w) + \cdots)u, \\ \dot{w} &= J_S w + F_{22}(z,w) + F_{32}(z,w) + \cdots \\ &\quad + (b_2 + M_3 z + M_4 w + G_{22}(z,w) + \cdots)u, \end{aligned} \tag{2}$$

where $G(0) = b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, $DG(0) = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$, with $b_1 \in \mathbb{R}^2$, $b_2 \in \mathbb{R}^{n-2}$, and

$$\begin{aligned} F_{2j}(z,w) &= \frac{1}{2} z^T \frac{\partial^2 F_j}{\partial z^2}(0,0) z + z^T \frac{\partial^2 F_j}{\partial z \partial w}(0,0) w \\ &\quad + \frac{1}{2} w^T \frac{\partial^2 F_j}{\partial w^2}(0,0) w, \\ G_{2j}(z,w) &= \frac{1}{2} z^T \frac{\partial^2 G_j}{\partial z^2}(0,0) z + z^T \frac{\partial^2 G_j}{\partial z \partial w}(0,0) w \\ &\quad + \frac{1}{2} w^T \frac{\partial^2 G_j}{\partial w^2}(0,0) w, \\ F_{3j}(z,w) &= \frac{1}{6} \frac{\partial^3 F_j}{\partial z^3}(0,0)(z,z,z) + \cdots , \end{aligned}$$

for $j = 1, 2$.

We wish to design a control law $u = u(z,\mu)$, with μ a real parameter, such that the original system (1) undergoes a Hopf bifurcation at $\xi = 0$ and $\mu = 0$, and that we could control it, *i.e.*, that we could decide the stability and direction of the emerging periodic solution.

We suppose that

H1 $\operatorname{rank}\big(b \;\, Jb \;\, \cdots \;\, J^{n-1}b\big) = n - 2.$

There are many ways to satisfy the condition **H1**; in this paper we analyze the case where $b_1 = 0$ and $b_{2j} \neq 0$ for $j = 1, 2, \ldots, n-2$, where $b_2 = (b_{21}, b_{22}, \ldots, b_{2,n-2})^T$. This corresponds to the case where the linear approximation of (1) has two uncontrollable modes, $\pm i\omega_0$, at $\xi = 0$.

Consider the control law

$$u(z,\mu) = \beta_1 \mu + \beta_2 (z_1^2 + z_2^2) = \beta_1 \mu + \beta_2 z^T z, \tag{3}$$

where $\beta_1, \beta_2 \in \mathbb{R}$.

Now, using the control law (3) in system (2) we obtain the closed-loop system

$$\begin{aligned} \dot{z} &= J_H z + \mathcal{F}_1(z,w,\mu), \\ \dot{w} &= \beta_1 b_2 \mu + J_S w + \mathcal{F}_2(z,w,\mu), \end{aligned} \tag{4}$$

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where

$$\begin{aligned}\mathcal{F}_1(z,w,\mu) &= \beta_1\mu M_1 z + \beta_1\mu M_2 w + F_{21}(z,w)\\ &\quad +\beta_1\mu G_{21}(z,w) + \beta_2 z^T z\,(M_1 z + M_2 w)\\ &\quad +F_{31}(z,w) + \cdots,\\ \mathcal{F}_2(z,w,\mu) &= \beta_1\mu M_3 z + \beta_1\mu M_4 w + F_{22}(z,w)\\ &\quad +\beta_2 z^T z\,(b_2 + M_3 z + M_4 w)\\ &\quad +\beta_1\mu G_{22}(z,w) + F_{32}(z,w) + \cdots.\end{aligned}$$

Then, our goal is to find β_1 and β_2 such that system (4) undergoes a Hopf bifurcation and can be controllable. For this, we use the center manifold theory.

III. Center manifold

A. Quadratic terms

Equation (4) represents a μ-parameterized family of systems, which we can write as an extended system

$$\begin{pmatrix}\dot z\\ \dot\mu\\ \dot w\end{pmatrix} = \begin{pmatrix}J_H & 0 & 0\\ 0 & 0 & 0\\ 0 & \beta_1 b_2 & J_S\end{pmatrix}\begin{pmatrix}z\\ \mu\\ w\end{pmatrix} + \begin{pmatrix}\mathcal{F}_1(z,w,\mu)\\ 0\\ \mathcal{F}_2(z,w,\mu)\end{pmatrix}.$$

In this form, the system has a three-dimensional center manifold through the origin. To find this manifold, we need to change coordinates to put the linear part in diagonal form. Then, using the transformation matrix

$$\begin{pmatrix}z\\ \mu\\ w\end{pmatrix} = \mathcal{P}\begin{pmatrix}x\\ \mu\\ y\end{pmatrix},$$

where

$$\mathcal{P} = \begin{pmatrix}J_H & 0 & 0\\ 0 & 1 & 0\\ 0 & -\beta_1 J_S^{-1} b_2 & J_S\end{pmatrix}$$

and

$$\mathcal{P}^{-1} = \begin{pmatrix}J_H^{-1} & 0 & 0\\ 0 & 1 & 0\\ 0 & \beta_1 J_S^{-2} b_2 & J_S^{-1}\end{pmatrix},$$

we can put (4) into standard form

$$\begin{pmatrix}\dot x\\ \dot\mu\\ \dot y\end{pmatrix} = \begin{pmatrix}J_H & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & J_S\end{pmatrix}\begin{pmatrix}x\\ \mu\\ y\end{pmatrix} + \begin{pmatrix}f(x,\mu,y)\\ 0\\ g(x,\mu,y)\end{pmatrix},$$

or

$$\begin{aligned}\dot x &= J_H x + f(x,\mu,y),\\ \dot\mu &= 0,\\ \dot y &= J_S y + g(x,\mu,y),\end{aligned}\tag{5}$$

where

$$f(x,\mu,y) = J_H^{-1}\mathcal{F}_1(J_H x,\mu,-\beta_1 J_S^{-1} b_2\mu + J_S y),\tag{6}$$
$$g(x,\mu,y) = J_S^{-1}\mathcal{F}_2(J_H x,\mu,-\beta_1 J_S^{-1} b_2\mu + J_S y).\tag{7}$$

We seek a center manifold

$$y = h(x,\mu) = \frac{1}{2}x^T H_1 x + x^T H_2\mu + \frac{1}{2}H_3\mu^2 + \cdots\tag{8}$$

such that $h(0,0)=0$, $Dh(0,0)=0$ and

$$h_i(x,\mu) = \frac{1}{2}x^T H_{1i} x + x^T H_{2i}\mu + \frac{1}{2}H_{3i}\mu^2 + \cdots$$

for $i = 1,2,\ldots,n-2$. Substituting (8) into (5) and using the chain rule, we obtain

$$\begin{aligned}&\frac{\partial h(x,\mu)}{\partial x}\left[J_H x + f(x,\mu,h(x,\mu))\right]\\ &\quad -J_S h(x,\mu) - g(x,\mu,h(x,\mu)) \equiv 0.\end{aligned}\tag{9}$$

This partial differential equation for h will be solved in the simplest case, that is, when J_S is diagonal, *i.e.*,

$$J_S = \begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_{n-2}\end{pmatrix},$$

with $\lambda_j < 0$ for each j. Besides, we are just interested to calculate H_1 because we will make $\mu = 0$ when we calculate the first Lyapunov coefficient a. Now, if

$$g_2(x,\mu) = \frac{1}{2}x^T N_1 x + x^T N_2\mu + \frac{1}{2}N_3\mu^2,\tag{10}$$

with $g_{2i}(x,\mu) = \frac{1}{2}x^T N_{1i} x + x^T N_{2i}\mu + \frac{1}{2}N_{3i}\mu^2$, for $i = 1,\ldots,n-2$, represents the quadratic terms of $g(x,\mu,h(x,\mu))$, then from (9) we obtain,

$$\frac{\partial h_i(x,\mu)}{\partial x}J_H x - \lambda_i h_i(x,\mu) - g_{2i}(x,\mu) + \text{h.o.t.} \equiv 0 \;\Leftrightarrow$$

$$\begin{aligned}&\left(x^T H_{1i} + H_{2i}^T\mu\right) J_H x\\ &-\lambda_i\left(\tfrac{1}{2}x^T H_{1i} x + x^T H_{2i}\mu + \tfrac{1}{2}H_{3i}\mu^2\right)\\ &-\tfrac{1}{2}\left(x^T N_{1i} x + x^T N_{2i}\mu + \tfrac{1}{2}N_{3i}\mu^2\right) + \text{h.o.t.} \equiv 0 \;\Leftrightarrow\end{aligned}$$

$$\begin{aligned}&x^T\left(H_{1i}J_H - \tfrac{1}{2}\lambda_i H_{1i} - \tfrac{1}{2}N_{1i}\right)x\\ &+x^T\left(J_H^T H_{2i} - \lambda_i H_{2i} - N_{2i}\right)\mu\\ &-\tfrac{1}{2}\left(\lambda_i H_{3i} + N_{3i}\right)\mu^2 + \text{h.o.t.} \equiv 0,\end{aligned}$$

for $i = 1,\ldots,n-2$, where we consider only the quadratic terms. Then

$$\begin{aligned}H_{1i} &= \frac{1}{2}N_{1i}\left(J_H - \frac{1}{2}\lambda_i I_2\right)^{-1}\\ &= N_{1i}\mathcal{R}_i\end{aligned}$$

where $I_2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and

$$\begin{aligned}\mathcal{R}_i &= \frac{1}{2}\left(J_H - \frac{1}{2}\lambda_i I_2\right)^{-1}\\ &= \frac{-1}{\lambda_i^2 + 4\omega_0^2}\begin{pmatrix}\lambda_i & -2\omega_0\\ 2\omega_0 & \lambda_i\end{pmatrix}\end{aligned}\tag{11}$$

Now we are going to calculate N_1. Observe that, from (7),

$$\begin{aligned}g(x,\mu,h(x,\mu)) &= J_S^{-1}\mathcal{F}_2(J_H x,\mu,-\beta_1 J_S^{-1} b_2\mu + J_S h(x,\mu))\\ &= \frac{1}{2}J_S^{-1}\left(x^T J_H^T F_{2zz}(0,0) J_H x\right)\\ &\quad +\beta_2\omega_0^2 x^T x J_S^{-1} b_2 + \cdots,\end{aligned}$$

but

$$\frac{1}{2}J_S^{-1}\left(x^TJ_H^TF_{2zz}(0,0)J_Hx\right)=\frac{1}{2}x^T\mathcal{A}x$$

where $\mathcal{A}=\mathcal{A}(\omega_0,\lambda_i,\frac{\partial^2F_2}{\partial z^2}(0,0))$, and

$$\begin{aligned}\beta_2\omega_0^2x^TxJ_S^{-1}b_2 &= \beta_2\omega_0^2x^Tx\begin{pmatrix}\frac{b_{21}}{\lambda_1}\\ \vdots\\ \frac{b_{2,n-2}}{\lambda_{n-2}}\end{pmatrix}\\ &= \beta_2\omega_0^2\begin{pmatrix}\frac{b_{21}}{\lambda_1}x^Tx\\ \vdots\\ \frac{b_{2,n-2}}{\lambda_{n-2}}x^Tx\end{pmatrix}\\ &= \beta_2\omega_0^2\begin{pmatrix}x^T\left(\frac{b_{21}}{\lambda_1}I_2\right)x\\ \vdots\\ x^T\left(\frac{b_{2,n-2}}{\lambda_{n-2}}I_2\right)x\end{pmatrix}\\ &= \beta_2\omega_0^2x^T\mathcal{B}x\end{aligned}$$

where $\mathcal{B}_i=\left(\frac{b_{2i}}{\lambda_i}I_2\right)_{2\times 2}$. Therefore,

$$\begin{aligned}g(x,\mu,h(x,\mu)) &= \frac{1}{2}x^T\mathcal{A}x+\beta_2\omega_0^2x^T\mathcal{B}x+\cdots\\ &= \frac{1}{2}x^T\left(\mathcal{A}+2\beta_2\omega_0^2\mathcal{B}\right)x+\cdots,\end{aligned}$$

then, from (10), $N_1=\mathcal{A}+2\beta_2\omega_0^2\mathcal{B}$. Now then, from (11), we obtain

$$\begin{aligned}H_{1i} &= N_{1i}\mathcal{R}_i\\ &= \left(\mathcal{A}_i+2\beta_2\omega_0^2\mathcal{B}_i\right)\mathcal{R}_i\\ &= \bar{\mathcal{A}}_i+2\beta_2\omega_0^2\bar{\mathcal{B}}_i,\end{aligned}$$

where

$$\bar{\mathcal{B}}_i=-\frac{b_{2i}}{\lambda_i\left(\lambda_i^2+4\omega_0^2\right)}\begin{pmatrix}\lambda_i & -2\omega_0\\ 2\omega_0 & \lambda_i\end{pmatrix}. \quad (12)$$

Finally, from (8),

$$h(x,\mu=\frac{1}{2}x^TH_1x+\cdots,$$

where $H_1=\bar{\mathcal{A}}+2\beta_2\omega_0^2\bar{\mathcal{B}}$.

B. Dynamics on the center manifold

On the center manifold the dynamics is given by

$$\dot{x}=J_Hx+f(x,\mu,h(x,\mu)), \quad (13)$$

where, from (6),

$$\begin{aligned}f(x,\mu,h(x,\mu)) =& J_H^{-1}\mathcal{F}_1(J_Hx,\mu,-\beta_1J_S^{-1}b_2\mu+J_Sh(x,\mu))\\ =&\beta_1\mu J_H^{-1}\left(M_1-b_2^T(J_S^{-1})^TF_{1wz}(0,0)\right)J_Hx\\ &+\frac{1}{2}J_H^{-1}\left(x^TJ_H^TF_{1zz}(0,0)J_Hx\right)\\ &+J_H^{-1}\left(x^TJ_H^TF_{1zw}(0,0)J_Sh(x,\mu)\right)\\ &+\beta_2J_H^{-1}(x^TJ_H^TJ_Hx)M_1J_Hx\\ &+\frac{1}{6}J_H^{-1}\left(x^TJ_H^TF_{1zzz}(0,0)J_Hx\right)J_Hx+\cdots,\end{aligned}$$

but, we define

$$\beta_1\mu\mathcal{M}x=\beta_1\mu J_H^{-1}\left(M_1-b_2^T(J_S^{-1})^TF_{1wz}(0,0)\right)J_Hx$$

where

$$\mathcal{M}=J_H^{-1}\left(M_1-b_2^T\left(J_S^{-1}\right)^TF_{1wz}(0,0)\right)J_H; \quad (14)$$

$$\frac{1}{2}x^T\mathcal{Q}x=\frac{1}{2}J_H^{-1}\left(x^TJ_H^TF_{1zz}(0,0)J_Hx\right)$$

where $\mathcal{Q}=\mathcal{Q}(\omega_0,\frac{\partial^2F_1}{\partial z^2}(0,0))$, and

$$\begin{aligned}\frac{1}{6}\mathcal{C}(x,x,x)=&J_H^{-1}\left(x^TJ_H^TF_{1zw}(0,0)J_Sh(x,\mu)\right)\\ &+\beta_2J_H^{-1}(x^TJ_H^TJ_Hx)M_1J_Hx\\ &+\frac{1}{6}J_H^{-1}\left(x^TJ_H^TF_{1zzz}(0,0)J_Hx\right)J_Hx. \quad (15)\end{aligned}$$

Observe that

$$\frac{1}{6}J_H^{-1}\left(x^TJ_H^TF_{1zzz}(0,0)J_Hx\right)J_Hx=\frac{1}{6}\mathcal{C}_0(x,x,x)$$

with $\mathcal{C}_0=\mathcal{C}_0(\omega_0,\frac{\partial^3F_1}{\partial z^3}(0,0))$;

$$\begin{aligned}\beta_2J_H^{-1}(x^TJ_H^TJ_Hx)M_1J_Hx &= \beta_2\omega_0^2x^Tx\left(J_H^{-1}M_1J_H\right)x\\ &= \beta_2\omega_0^2\mathcal{C}_M(x,x,x),\end{aligned}$$

with $M_1=\begin{pmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}$ and

$$\mathcal{C}_M=\begin{pmatrix}m_{22}I_2 & -m_{21}I_2\\ -m_{12}I_2 & m_{11}I_2\end{pmatrix}, \quad (16)$$

and

$$\begin{aligned}&J_H^{-1}\left(x^TJ_H^TF_{1zw}(0,0)J_Sh(x,\mu)\right)\\ &=J_H^{-1}\left(x^TJ_H^TF_{1zw}(0,0)J_S\left(\tfrac{1}{2}x^T\left(\bar{\mathcal{A}}+2\beta_2\omega_0^2\bar{\mathcal{B}}\right)x\right)\right)\\ &=\tfrac{1}{2}J_H^{-1}\left(x^TJ_H^TF_{1zw}(0,0)J_S\left(x^T\bar{\mathcal{A}}x\right)\right)\\ &\quad+\beta_2\omega_0^2J_H^{-1}\left(x^TJ_H^TF_{1zw}(0,0)J_S\left(x^T\bar{\mathcal{B}}x\right)\right)\\ &=\tfrac{1}{2}\mathcal{C}_{\bar{\mathcal{A}}}(x,x,x)+\beta_2\omega_0^2\mathcal{C}_{\bar{\mathcal{B}}}(x,x,x)\end{aligned}$$

with $\mathcal{C}_{\bar{\mathcal{A}}}=\mathcal{C}_{\bar{\mathcal{A}}}(\omega_0,\lambda_i,\frac{\partial^2F_2}{\partial z^2}(0,0),\frac{\partial^2F_1}{\partial z\partial w})$ and

$$\mathcal{C}_{\bar{\mathcal{B}}}(x,x,x) = J_H^{-1}\left(x^TJ_H^TF_{1zw}(0,0)J_S\left(x^T\bar{\mathcal{B}}x\right)\right).$$

We are going to calculate $\mathcal{C}_{\bar{\mathcal{B}}}$. Observe that

$$F_{1zw}(0,0)=\begin{pmatrix}F_{1zw}^1(0,0)\\ F_{1zw}^2(0,0)\end{pmatrix}, \quad (17)$$

where

$$\begin{aligned}F_{1zw}^j(0,0) &= \begin{pmatrix}F_{1z_1w}^j(0,0)\\ F_{1z_2w}^j(0,0)\end{pmatrix}\\ &= \begin{pmatrix}\frac{\partial^2F_1^j(0,0)}{\partial w_1\partial z_1}, & \ldots, & \frac{\partial^2F_1^j(0,0)}{\partial w_{n-2}\partial z_1}\\ \frac{\partial^2F_1^j(0,0)}{\partial w_1\partial z_2}, & \ldots, & \frac{\partial^2F_1^j(0,0)}{\partial w_{n-2}\partial z_2}\end{pmatrix},\end{aligned}$$

and

$$J_S\left(x^T\bar{\mathcal{B}}x\right)=\begin{pmatrix}x^T\left(\lambda_1\bar{\mathcal{B}}_1\right)x\\ \vdots\\ x^T\left(\lambda_{n-2}\bar{\mathcal{B}}_{n-2}\right)x\end{pmatrix},$$

then

$$F_{1zw}(0,0)J_S\left(x^T\bar{\mathcal{B}}x\right) = \begin{pmatrix} F^1_{1zw}(0,0)J_S\left(x^T\bar{\mathcal{B}}x\right) \\ F^2_{1zw}(0,0)J_S\left(x^T\bar{\mathcal{B}}x\right) \end{pmatrix},$$

where

$$F^j_{1zw}(0,0)J_S\left(x^T\bar{\mathcal{B}}x\right) = \begin{pmatrix} x^T\left(\sum_{k=1}^{n-2}\frac{\partial^2 F^j_1(0,0)}{\partial w_k\partial z_1}\lambda_k\bar{\mathcal{B}}_k\right)x \\ x^T\left(\sum_{k=1}^{n-2}\frac{\partial^2 F^j_1(0,0)}{\partial w_k\partial z_2}\lambda_k\bar{\mathcal{B}}_k\right)x \end{pmatrix} = x^T\mathcal{S}^jx,$$

with

$$\begin{aligned} \mathcal{S}^j_i &= \sum_{k=1}^{n-2}\frac{\partial^2 F^j_1(0,0)}{\partial w_k\partial z_i}\lambda_k\bar{\mathcal{B}}_k \\ &= -\sum_{k=1}^{n-2}\frac{\partial^2 F^j_1(0,0)}{\partial w_k\partial z_i}\frac{b_{2k}}{(\lambda_k^2+4\omega_0^2)}\begin{pmatrix} \lambda_k & -2\omega_0 \\ 2\omega_0 & \lambda_k \end{pmatrix} \end{aligned}$$

for $i,j=1,2$.

Now then, $x^TJ^T_H=\omega_0(-x_2,x_1)$, then,

$$\begin{aligned} & x^TJ^T_HF_{1zw}(0,0)J_S\left(x^T\bar{\mathcal{B}}x\right) \\ &= \omega_0\begin{pmatrix} (-x_2,x_1)x^T\mathcal{S}^1x \\ (-x_2,x_1)x^T\mathcal{S}^2x \end{pmatrix} \\ &= \omega_0\begin{pmatrix} -x_2\left(x^T\mathcal{S}^1_1x\right)+x_1\left(x^T\mathcal{S}^1_2x\right) \\ -x_2\left(x^T\mathcal{S}^2_1x\right)+x_1\left(x^T\mathcal{S}^2_2x\right) \end{pmatrix}, \end{aligned}$$

therefore,

$$\begin{aligned} & J^{-1}_H\left(x^TJ^T_HF_{1zw}(0,0)J_S\left(x^T\bar{\mathcal{B}}x\right)\right) \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -x_2\left(x^T\mathcal{S}^1_1x\right)+x_1\left(x^T\mathcal{S}^1_2x\right) \\ -x_2\left(x^T\mathcal{S}^2_1x\right)+x_1\left(x^T\mathcal{S}^2_2x\right) \end{pmatrix} \\ &= \begin{pmatrix} -x_2\left(x^T\mathcal{S}^2_1x\right)+x_1\left(x^T\mathcal{S}^2_2x\right) \\ x_2\left(x^T\mathcal{S}^1_1x\right)-x_1\left(x^T\mathcal{S}^1_2x\right) \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{S}^2_2 & -\mathcal{S}^2_1 \\ -\mathcal{S}^1_2 & \mathcal{S}^1_1 \end{pmatrix}(x,x,x), \end{aligned}$$

then

$$\mathcal{C}_{\bar{\mathcal{B}}} = \begin{pmatrix} \mathcal{S}^2_2 & -\mathcal{S}^2_1 \\ -\mathcal{S}^1_2 & \mathcal{S}^1_1 \end{pmatrix} \tag{18}$$

Now, we re-write (15)

$$\begin{aligned} \frac{1}{6}\mathcal{C} &= \frac{1}{6}\mathcal{C}_0+\beta_2\omega_0^2\mathcal{C}_M+\frac{1}{2}\mathcal{C}_{\bar{\mathcal{A}}}+\beta_2\omega_0^2\mathcal{C}_{\bar{\mathcal{B}}} \\ &= \frac{1}{6}\left(\mathcal{C}_0+3\mathcal{C}_{\bar{\mathcal{A}}}\right)+\frac{1}{6}\left(6\beta_2\omega_0^2\left(\mathcal{C}_M+\mathcal{C}_{\bar{\mathcal{B}}}\right)\right) \\ &= \frac{1}{6}\left(\mathcal{C}_1+\mathcal{C}_2\right), \end{aligned} \tag{19}$$

where $\mathcal{C}_1=\mathcal{C}_1(\omega_0,\lambda_i,\frac{\partial^3F_1(0,0)}{\partial z^3},\frac{\partial^2F_2(0,0)}{\partial z^2},\frac{\partial^2F_1(0,0)}{\partial z\partial w})$ and

$$\mathcal{C}_2 = 6\beta_2\omega_0^2\begin{pmatrix} m_{22}I_2+\mathcal{S}^2_2 & -m_{21}I_2-\mathcal{S}^2_1 \\ -m_{12}I_2-\mathcal{S}^1_2 & m_{11}I_2+\mathcal{S}^1_1 \end{pmatrix} \tag{20}$$

Finally, the dynamics on the center manifold is given by

$$\dot{x} = J_\mu x+\frac{1}{2}\mathcal{Q}(x,x)+\frac{1}{6}\mathcal{C}(x,x,x)+\cdots, \tag{21}$$

where $J_\mu = J_H+\beta_1\mu\mathcal{M}$, and $\mathcal{M}$ and $\mathcal{C}$ are given by (14,19,20).

Remark *Remember that we just consider those quadratic and cubic terms in (21) that do not depend on μ because we put $\mu=0$ to find the first Lyapunov coefficient. At the same time, we have just found expressions for those terms that depend on β_2 and that are needed to find the mentioned coefficient.*

IV. Control of the Hopf Bifurcation

In this section we will find conditions to ensure that system (21) undergoes a Hopf bifurcation that can be controlled.

A. Hopf bifurcation theorem

Theorem 1: ([7]) Suppose that the system

$$\dot{x} = f(x,\mu)$$

with $x\in\mathbb{R}^n$, $\mu\in\mathbb{R}$ has an equilibrium (x_0,μ_0) at which the following properties are satisfied:

(A1) $D_xf(x_0,\mu_0)$ has a simple pair of pure imaginary eigenvalues and no other eigenvalues with zero real parts.

(A2) Let $\lambda(\mu)$, $\bar{\lambda}(\mu)$ be the eigenvalues of $D_xf(x_0,\mu_0)$ which are imaginary at $\mu=\mu_0$, such that

$$\frac{d}{d\mu}\left(Re(\lambda(\mu))\right)|_{\mu=\mu_0} = d\neq 0. \tag{22}$$

Then there is a unique three-dimensional center manifold passing through $(x_0,\mu_0)\in\mathbb{R}^n\times\mathbb{R}$ and a smooth system of coordinates for which the Taylor expansion of degree three on the center manifold, in polar coordinates, is given by

$$\begin{aligned} \dot{r} &= (d\mu+ar^2)r, \\ \dot{\theta} &= \omega+c\mu+br^2. \end{aligned}$$

If $a\neq 0$, there is a surface of periodic solutions in the center manifold which has quadratic tangency with the eigenspace of $\lambda(\mu_0)$, $\bar{\lambda}(\mu_0)$ agreeing to second order with the paraboloid $\mu=-\frac{a}{d}r^2$. If $a<0$, then these periodic solutions are stable limit cycles, while if $a>0$, are repelling.

For bidimensional systems, there exists an expression to find the called first Lyapunov coefficient a. Consider the system

$$\dot{x} = Jx+F(x),$$

where $J=\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$, $F(x)=\begin{pmatrix} F_1(x) \\ F_2(x) \end{pmatrix}$, $F(0)=0$ and $DF(0)=0$. Then

$$a = \frac{1}{16\omega}(R_1+\omega R_2), \tag{23}$$

where

$$\begin{aligned} R_1 &= F_{1x_1x_2}(F_{1x_1x_1}+F_{1x_2x_2}) \\ &\quad -F_{2x_1x_2}(F_{2x_1x_1}+F_{2x_2x_2}) \\ &\quad -F_{1x_1x_1}F_{2x_1x_1}+F_{1x_2x_2}F_{2x_2x_2} \\ R_2 &= F_{1x_1x_1x_1}+F_{1x_1x_2x_2}+F_{2x_1x_1x_2}+F_{2x_2x_2x_2}. \end{aligned}$$

There exists another way to express R_2. If

$$F(x) = \frac{1}{2}\mathcal{Q}(x,x) + \frac{1}{6}\mathcal{C}(x,x,x) + \cdots$$

where $\mathcal{Q} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$, $\mathcal{C} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$ with $Q_i, C_{ij} \in \mathbb{R}^{2\times 2}$, then

$$R_2 = tr(C_{11} + C_{22}), \tag{24}$$

with $tr(\cdot) = \text{trace}(\cdot)$.

B. Control law design

In this section we are going to prove, using the theorem 1, that system (21) undergoes the Hopf bifurcation at $x = 0$ and $\mu = 0$. First, we are going to prove that the eigenvalues of J_μ cross the imaginary axes when $\mu = 0$, and second, we will show that the first Lyapunov coefficient a is different of zero.

1) Eigenvalues of J_μ: The characteristic equation of the linear part of (21) is given by

$$\lambda^2 - tr(J_\mu)\lambda + det(J_\mu) = 0$$

where $tr(J_\mu) = \beta_1 \mu tr(\mathcal{M})$ and $det(J_\mu) = \omega_0^2 + \beta_1\mu\omega_0(\mathcal{M}_{21} - \mathcal{M}_{12}) + \beta_1^2\mu^2 det(\mathcal{M})$, with $\mathcal{M} = (\mathcal{M}_{ij})$. Then, for μ sufficiently small, the eigenvalues are given by

$$\lambda(\mu) = \frac{1}{2}tr(J_\mu) \pm i\sqrt{det(J_\mu) - \left(\frac{1}{2}tr(J_\mu)\right)^2}.$$

Then, $\lambda(0) = \pm i\omega_0$ and

$$Re(\lambda(\mu)) = \frac{1}{2}tr(J_\mu) = \frac{1}{2}\beta_1\mu tr(\mathcal{M})$$

but, from (14),

$$\begin{aligned} tr(\mathcal{M}) &= tr(M_1 - b_2^T \left(J_S^{-1}\right)^T F_{1wz}(0,0)) \\ &= tr(M_1) - tr(b_2^T \left(J_S^{-1}\right)^T F_{1wz}(0,0)) \\ &= tr(M_1) - tr(b_2^T \left(J_S^{-1}\right)^T F_{1wz}(0,0))^T \\ &= tr(M_1) - tr(F_{1zw}^T(0,0)J_S^{-1}b_2), \end{aligned}$$

and from (17),

$$F_{1zw}^T(0,0)J_S^{-1}b_2 = \begin{pmatrix} F_{1zw}^1 J_S^{-1}b_2 & F_{1zw}^2 J_S^{-1}b_2 \end{pmatrix}$$

where

$$\begin{aligned} F_{1zw}^j(0,0)J_S^{-1}b_2 &= \begin{pmatrix} F_{1z_1w}^j(0,0)J_S^{-1}b_2 \\ F_{1z_2w}^j(0,0)J_S^{-1}b_2 \end{pmatrix} \\ &= \begin{pmatrix} \sum_{k=1}^{n-2} \frac{\partial^2 F_1^j(0,0)}{\partial w_k \partial z_1}\frac{b_{2k}}{\lambda_k} \\ \sum_{k=1}^{n-2} \frac{\partial^2 F_1^j(0,0)}{\partial w_k \partial z_2}\frac{b_{2k}}{\lambda_k} \end{pmatrix}, \end{aligned}$$

then

$$\begin{aligned} & tr(F_{1zw}^T(0,0)J_S^{-1}b_2) \\ &= \sum_{k=1}^{n-2}\frac{b_{2k}}{\lambda_k}\frac{\partial^2 F_1^1(0,0)}{\partial w_k\partial z_1} + \sum_{k=1}^{n-2}\frac{b_{2k}}{\lambda_k}\frac{\partial^2 F_1^2(0,0)}{\partial w_k\partial z_2} \\ &= \sum_{k=1}^{n-2}\frac{b_{2k}}{\lambda_k}\left(\frac{\partial^2 F_1^1(0,0)}{\partial w_k\partial z_1} + \frac{\partial^2 F_1^2(0,0)}{\partial w_k\partial z_2}\right) \\ &= \sum_{k=1}^{n-2}\frac{b_{2k}}{\lambda_k}\frac{\partial}{\partial w_k}\left(\frac{\partial F_1^1}{\partial z_1}(0,0) + \frac{\partial F_1^2}{\partial z_2}(0,0)\right) \\ &= \sum_{k=1}^{n-2}\frac{b_{2k}}{\lambda_k}\frac{\partial}{\partial w_k}\left(div_z F_1\right)(0,0), \end{aligned}$$

therefore,

$$Re(\lambda(\mu)) = \frac{\beta_1\mu}{2}\mathcal{K}_1,$$

where

$$\mathcal{K}_1 = tr(M_1) - \sum_{k=1}^{n-2}\frac{b_{2k}}{\lambda_k}\frac{\partial}{\partial w_k}\left(div_z F_1\right)(0,0), \tag{25}$$

and from (22),

$$\begin{aligned} d &= \frac{d}{d\mu}Re(\lambda(\mu))|_{\mu=0} \\ &= \frac{\beta_1}{2}\mathcal{K}_1 \end{aligned} \tag{26}$$

2) First Lyapunov coefficient: From (23-24),

$$a = \frac{1}{16\omega_0}(R_1 + \omega_0 R_2)$$

where, for our system (21),

$$\begin{aligned} R_2 &= tr(\mathcal{C}_1 + \mathcal{C}_2) \\ &= tr(\mathcal{C}_1) + tr(\mathcal{C}_2) \\ &= \delta_1 + \delta_2 \end{aligned}$$

where

$$\delta_1 = \delta_1(\omega_0, \lambda_i, F_{2zz}(0,0), F_{1zzz}(0,0)) = tr(\mathcal{C}_1)$$

and

$$\begin{aligned} \delta_2 &= tr(\mathcal{C}_2) \\ &= tr\left(6\beta_2\omega_0^2\begin{pmatrix} m_{22}I_2 + \mathcal{S}_2^2 & -m_{21}I_2 - \mathcal{S}_1^2 \\ -m_{12}I_2 - \mathcal{S}_2^1 & m_{11}I_2 + \mathcal{S}_1^1 \end{pmatrix}\right) \\ &= 12\beta_2\omega_0^2\mathcal{K}_2, \end{aligned}$$

where

$$\mathcal{K}_2 = tr(M_1) - \sum_{k=1}^{n-2}\frac{b_{2k}\lambda_k}{\lambda_k^2 + 4\omega_0^2}\frac{\partial}{\partial w_k}\left(div_z F_1\right)(0,0). \tag{27}$$

Then,

$$a = \frac{3}{4}\beta_2\omega_0^2\mathcal{K}_2 + \delta, \tag{28}$$

We have then proved the next result.

Theorem 2: Consider the system

$$\dot{\xi} = F(\xi) + G(\xi)u,$$

with $F(0) = 0$ and $DF(0) = J = \begin{pmatrix} J_H & 0 \\ 0 & J_S \end{pmatrix}$, with $J_H = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix}$ and $J_S = diag\{\lambda_1, \ldots, \lambda_{n-2}\}$ Hurwitz. If $\mathcal{K}_1$ and $\mathcal{K}_2$, given by (25) and (27), respectively, are different of zero, $G(0) = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, with $b_1 = 0$, and $\text{rank}\big(b \ Jb \ \cdots \ J^{n-1}b\big) = n-2$, then there exists β_1, β_2 such that, with the control law

$$u = \beta_1\mu + \beta_2(z_1^2 + z_2^2),$$

the system undergoes the Hopf bifurcation at $\mu = 0$. Moreover, it is possible to control the stability and direction of the emerging periodic solution near the origin, by selecting the signs of d and a in (26) and (28), respectively.

For the case $n = 2$, $\mathcal{K}_1 = \mathcal{K}_2 = tr(M_1)$. This case was reported in [11]

V. Conclusions

In this paper we have derived sufficient conditions to ensure the control of the Hopf bifurcation in nonlinear systems with two uncontrollable modes in the imaginary axes. We have used the center manifold theorem to reduce the analysis to dimension two; nevertheless, we have obtained expressions in terms of the original vector fields. The control law designed has a constant term, which establish the stability of the equilibrium point, and a quadratic term, which determines the orientation and stability of the periodic solution near the origin.

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