Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Realization from covariances and Markov parameters of a discrete-time periodic system

Patrizio Colaneri and Stefano Pinzoni

***Abstract*— The main contribution of this paper is an algorithm to find a minimal stable state-space realization of a linear discrete-time periodic system from a finite window of (normalized) covariances and Markov parameters. This contribution to the stochastic realization problem for periodic systems can be used for the identification of cyclostationary processes.**

I. Introduction

The state-space realization from input-output data for discrete-time periodic systems is the subject of a number of investigations, [1]-[3], and efficient numerical methods have been devised to this purpose and other theoretical matters, [2]. Streamlines to solve a problem for a discrete-time periodic system are given by the isomorphism between such systems and a class of time-invariant ones. Time-invariant representations for periodic systems are surveyed in [5].

In the stochastic framework, periodic systems are useful to model cyclostationary processes, or periodically correlated signals, [6]. In this paper we provide a simple algorithm to find a minimal stable periodic system from output system covariances and (normalized) Markov parameters. Notice that a minimal realization of a discrete-time periodic system has in general time-periodic dimensions of the state-space, [1]. As such, we are well advised to introduce, from the very beginning, a state-space model of a periodic system with time-periodic dimensions. Namely, consider the model

$$\begin{array}{rcl} x(t+1) & = & A(t)x(t)+B(t)w(t) \\ y(t) & = & C(t)x(t)+D(t)w(t) \end{array} \tag{1}$$

where $x(t) \in I\!R^{n(t)}$ is the state vector, $A(t) \in I\!R^{n(t+1)\times n(t)}$ is the dynamic matrix, $B(t) \in I\!R^{n(t+1)\times m}$, $C(t) \in I\!R^{p\times n(t)}$ are the input and output matrices and $D(t) \in I\!R^{p\times m}$ is the direct feed-through term. The state-space dimension $n(\cdot)$ is periodic of period T and all elements of $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ and $D(\cdot)$ are periodic functions of the same period.

The transition matrix $\Phi_A(t,\tau) \in I\!R^{n(t)\times n(\tau)}$ is defined for $t \geq \tau$ as follows:

$$\Phi_A(t,\tau) = \begin{cases} I_{n(\tau)}, & t=\tau \\ A(t-1)A(t-2)\ldots A(\tau), & t>\tau \end{cases}$$

This paper has been partially supported by IEIIT of the Italian National Research Council (CNR) and by MIUR project "Identification and control of industrial systems".

P. Colaneri is with the Dipartimento di Elettronica e Informazione, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy colaneri@elet.polimi.it

S. Pinzoni is with the Dipartimento di Ingegneria dell'Informazione, Università di Padova, via Gradenigo 6/B, 35131 Padova, Italy pinzoni@dei.unipd.it

where I_n is the identity matrix of dimension n. The monodromy matrix at τ associated with $A(\cdot)$ is the transition matrix over the period, i.e. $\Psi_A(\tau) = \Phi_A(\tau+T,\tau) \in I\!R^{n(\tau)\times n(\tau)}$. It is computed as

$$\Psi_A(\tau) = A(\tau+T-1)A(\tau+T-2)\ldots A(\tau)$$

It is easy to see that only the algebraic multiplicity of the zero eigenvalue of $\Psi_A(\tau)$ may vary with τ. Therefore, the periodic system (1) is asymptotically stable if and only if the characteristic multipliers of $A(\cdot)$, i.e. the eigenvalues of $\Psi_A(\tau)$, are inside the unit disc (for some τ). The realization (1) is minimal if and only if the system is reachable and observable. For a thorough investigation on the structural properties of periodic systems, see [7].

Now, take a T-periodic matrix $\Omega(t) \in I\!R^{n(t)\times n(t)}$, invertible for each t, and perform a periodic change of basis in the state space:

$$z(t) = \Omega(t)x(t)$$

In the new coordinates, the system equations become

$$\begin{array}{rcl} z(t+1) & = & \tilde{A}(t)z(t)+\tilde{B}(t)w(t) \\ y(t) & = & \tilde{C}(t)z(t)+\tilde{D}(t)w(t) \end{array} \tag{2}$$

where

$$\begin{array}{ll} \tilde{A}(t) = \Omega(t+1)A(t)\Omega(t)^{-1} & \tilde{B}(t) = \Omega(t+1)B(t) \\ \tilde{C}(t) = C(t)\Omega(t)^{-1} & \tilde{D}(t) = D(t) \end{array} \tag{3}$$

Systems (1) and (2) are algebraically equivalent and enjoy the same structural properties, namely stability, reachability, observability, etc. Moreover, they share the same input-output properties. When dealing with state-space realizations from input-output data, the structure of the particular realization is often not fixed a-priori.

Stability of system (1) can also be studied via Lyapunov theory, and in particular by resorting to the (filtering-type) difference periodic Lyapunov equation

$$P(t+1) = A(t)P(t)A(t)' + B(t)B(t)' \tag{4}$$

The following result is well known.

Lemma 1.1: If the pair $(A(\cdot), B(\cdot))$ is reachable, then the periodic system (1) is asymptotically stable if and only if equation (4) admits a T-periodic positive definite solution $P(\cdot)$.

Notice that the solution $P(t) \in R^{n(t)\times n(t)}$ is square for each t. If such a positive definite solution $P(\cdot)$ exists, then a change of basis is possible such that, in the new coordinates, the solution of the Lyapunov equation is the identity.

Lemma 1.2: Assume that (1) is stable and that $(A(\cdot), B(\cdot))$ is reachable. Then there exists an algebraically equivalent system (2) such that, for each t,

$$I_{n(t+1)} = \tilde{A}(t)\tilde{A}(t)' + \tilde{B}(t)\tilde{B}(t)' \qquad (5)$$

Proof. The proof is straightforward. Just take $\Omega(t) = P(t)^{-1/2}$ and convert (4) into (5) by letting

$$\begin{aligned} \tilde{A}(t) &= P(t+1)^{-1/2}A(t)P(t)^{1/2} \\ \tilde{B}(t) &= P(t+1)^{-1/2}B(t) \end{aligned}$$

II. Problem formulation

The input-output description of the periodic system can be expressed through the Markov parameters $h_i(t)$ as follows:

$$y(t) = \sum_{i=0}^{\infty} h_i(t)w(t-i) \qquad (6)$$

The Markov coefficients $h_i(\cdot)$ are T-periodic functions which capture the input-output behavior in the sense that the j-th column of $h_i(t) \in I\!\!R^{p\times m}$ is the response of the system at time t to a unit impulse, applied to the j-th component of the input, at time $t-i$. The relation between the Markov parameters and the state-space matrices is self-evident. Indeed, it follows:

$$\begin{aligned} h_0(t) &= D(t) \\ h_i(t) &= C(t)\Phi_A(t, t-i+1)B(t-i), \quad i = 1, 2, \cdots \end{aligned}$$

As easily seen, these Markov parameters are independent of the state-space basis chosen to describe the system.

If the system is stable and $w(\cdot)$ is a white noise with zero mean and unit variance, for any initial condition the state of the system converges to a cyclostationary process $x(\cdot)$ with variance given by $P(\cdot)$, the unique T-periodic solution of the Lyapunov equation (4). The output also converges to a cyclostationary process $y(\cdot)$, whose autocovariance matrices are defined as:

$$r_i(t) := E\left[y(t+i)y(t)'\right] \in I\!\!R^{p\times p}$$

Being the process $y(\cdot)$ cyclostationary, the functions $r_i(\cdot)$ are T-periodic. An easy computation shows that

$$\begin{aligned} r_0(t) &= C(t)P(t)C(t)' + D(t)D(t)' \\ r_i(t) &= C(t+i)\Phi_A(t+i, t+1)S(t),\ i = 1, 2, \ldots \\ S(t) &= A(t)P(t)C(t)' + B(t)D(t)' \end{aligned}$$

It is readily seen that, for all $q = 0, 1, \ldots$

$$\mathcal{R}_q(t) = \mathcal{O}_q(t)P(t)\mathcal{O}_q(t)' + \mathcal{H}_q(t)\mathcal{H}_q(t)' \qquad (7)$$

where

$$\mathcal{R}_q(t) = \begin{bmatrix} r_0(t) & r_1(t) & \cdots & r_q(t) \\ r_1(t) & r_0(t+1) & \cdots & r_{q-1}(t+1) \\ \vdots & \vdots & \ddots & r_1(t+q-1) \\ r_q(t) & r_{q-1}(t+1) & \cdots & r_0(t+q) \end{bmatrix} \qquad (8)$$

$$\mathcal{O}_q(t) = \begin{bmatrix} C(t) \\ C(t+1)A(t) \\ \vdots \\ C(t+q)\Phi_A(t+q, t) \end{bmatrix} \qquad (9)$$

$$\mathcal{H}_q(t) = \begin{bmatrix} h_0(t) & 0 & \cdots & 0 \\ h_1(t+1) & h_0(t+1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_q(t+q) & h_{q-1}(t+q) & \cdots & h_0(t+q) \end{bmatrix} \qquad (10)$$

Notice that $\mathcal{O}_q(\cdot)$, $\mathcal{H}_q(\cdot)$ and $\mathcal{R}_q(\cdot)$ are T-periodic matrices and $\mathcal{O}_q(t) \in I\!\!R^{(q+1)p\times n(t)}$, $\mathcal{H}_q(t) \in I\!\!R^{(q+1)p\times(q+1)m}$, $\mathcal{R}_q(t) \in I\!\!R^{(q+1)p\times(q+1)p}$.

Equation (7) is very important in stochastic realization theory. Indeed, it implies the obvious *data consistency* condition

$$\mathcal{Z}_q(t) := \mathcal{R}_q(t) - \mathcal{H}_q(t)\mathcal{H}_q(t)' \geq 0, \quad \forall t \qquad (11)$$

This means that the autocovariances in $\mathcal{R}_q(t)$ and the Markov parameters in $\mathcal{H}_q(t)$ must satisfy (11) for the existence of a stable state-space realization.

III. Realization for SISO periodic systems

From now on, the attention is restricted to *SISO periodic systems*, i.e. periodic systems with $m = 1$ and $p = 1$. In this case, it is not difficult to find a state-space realization from given periodic matrices $\mathcal{R}_q(t)$ and $\mathcal{H}_q(t)$ of the form (8) and (10), satisfying the data consistency condition (11), with rank deficient $\mathcal{Z}_q(t)$. Indeed, one can always look for a realization with identity state covariance $P(t)$. Let $\mathcal{Z}_q(t) = \mathcal{R}_q(t) - \mathcal{H}_q(t)\mathcal{H}_q(t)'$ be factorized as

$$\mathcal{Z}_q(t) = G(t)G(t)'$$

where $G(t) \in I\!\!R^{(q+1)\times n(t)}$, $n(t)$ being the rank of $\mathcal{Z}_q(t)$, and assume that $n(t) \leq q$, for all t. Hence, $G(t)$ can be partitioned as

$$G(t) = \begin{bmatrix} G_1(t) \\ G_2(t) \end{bmatrix} = \begin{bmatrix} G_3(t) \\ G_4(t) \\ G_5(t) \end{bmatrix} \qquad (12)$$

where $G_1(t) \in I\!\!R^{n(t)\times n(t)}$, $G_2(t) \in I\!\!R^{(q+1-n(t))\times n(t)}$, $G_3(t) \in I\!\!R^{1\times n(t)}$, $G_4(t) \in I\!\!R^{n(t+1)\times n(t)}$ and $G_5(t) \in I\!\!R^{(q-n(t+1))\times n(t)}$. Moreover, select from the first column of $\mathcal{H}_q(t)$ the vector

$$M_1(t) = \begin{bmatrix} h_1(t+1) \\ h_2(t+2) \\ \vdots \\ h_{n(t+1)}(t+n(t+1)) \end{bmatrix} \qquad (13)$$

Theorem 3.1: Let $q \geq 0$, $\mathcal{R}_q(t)$ and $\mathcal{H}_q(t)$ be given for $t \in [0, T-1]$ and assume that the data consistency condition (11) is satisfied, with $n(t) := \text{rank}\mathcal{Z}_q(t) \leq q$. Let $G(t)$ be a full-rank factor of $\mathcal{Z}_q(t)$ and define $G_1(t)$, $G_3(t)$, $G_4(t)$ and $M_1(t)$ as in (12), (13). Then, the T-periodic quadruple

$$\tilde{A}(t) = G_1(t+1)^{-1}G_4(t) \tag{14}$$
$$\tilde{B}(t) = G_1(t+1)^{-1}M_1(t) \tag{15}$$
$$\tilde{C}(t) = G_3(t) \tag{16}$$
$$\tilde{D}(t) = h_0(t) \tag{17}$$

defines a T-periodic realization of the given Markov coefficients and covariance data. Moreover, if the pair $(\tilde{A}(\cdot), \tilde{B}(\cdot))$ is reachable, then $\tilde{A}(\cdot)$ is asymptotically stable.

Proof. The proof follows from the special structure of $\mathcal{R}_q(t)$ and $\mathcal{H}_q(t)$. In particular,

$$G_4(t)G_4(t)' + M_1(t)M_1(t)' = G_1(t+1)G_1(t+1)'$$

An argument generalizing the proof of Lemma 2.1 in [8] to the periodic case shows that $G_1(t+1)$ is non-singular for all t. This, together with (14)-(15), leads to (5). The fact that the quadruple $(\tilde{A}(\cdot), \tilde{B}(\cdot), \tilde{C}(\cdot), \tilde{D}(\cdot))$ is indeed a realization of the input-output data can be proved by inspection. Finally, by Lemma 1.1, the reachability of $(\tilde{A}(\cdot), \tilde{B}(\cdot))$ implies the stability of $\tilde{A}(\cdot)$.

A. Realization from normalized data

In this section, we aim at investigating the realization problem for cyclostationary processes under the assumption that the Markov parameters are normalized. Notice that this situation is quite common in practice since the variance of the input is often unknown. We have found a motivating study for our work in the paper [8], where stationary processes are dealt with.

Define the normalized Markov and covariance parameters as:

$$\hat{h}_i(t) = \frac{h_i(t)}{h_0(t-i)}, \quad \hat{r}_i(t) = \frac{r_i(t)}{\sqrt{r_0(t)r_0(t+i)}} \tag{18}$$

Accordingly, one can define the normalized matrices

$$\hat{\mathcal{H}}_q(t) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \hat{h}_1(t+1) & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \hat{h}_q(t+q) & \cdots & \hat{h}_1(t+q) & 1 \end{bmatrix}, \tag{19}$$

$$\hat{\mathcal{R}}_q(t) = \begin{bmatrix} 1 & \hat{r}_1(t) & \cdots & \hat{r}_q(t) \\ \hat{r}_1(t) & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \hat{r}_q(t) & \cdots & \hat{r}_1(t+q-1) & 1 \end{bmatrix} \tag{20}$$

The obvious relation with $\mathcal{H}_q(t)$ and $\mathcal{R}_q(t)$ is

$$\mathcal{H}_q(t) = \hat{\mathcal{H}}_q(t)\Delta_{H,q}(t), \quad \mathcal{R}_q(t) = \Delta_{R,q}(t)\hat{\mathcal{R}}_q(t)\Delta_{R,q}(t)$$

where

$$\Delta_{H,q}(t) = \begin{bmatrix} h_0(t) & 0 & \cdots & 0 \\ 0 & h_0(t+1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_0(t+q) \end{bmatrix}$$

$$\Delta_{R,q}(t) = \begin{bmatrix} \sqrt{r_0(t)} & 0 & \cdots & 0 \\ 0 & \sqrt{r_0(t+1)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{r_0(t+q)} \end{bmatrix}$$

Now define:

$$\hat{\mathcal{Z}}_q(t) := \hat{\mathcal{R}}_q(t) - \Delta_{R,q}(t)^{-1}\hat{\mathcal{H}}_q(t)\Delta_{H,q}(t)^2\hat{\mathcal{H}}_q(t)'\Delta_{R,q}(t)^{-1} \tag{21}$$

In order to ensure the existence of a realization one has to find T-periodic sequences $h_0(\cdot)$, $r_0(\cdot)$ such that, for all t, $\hat{\mathcal{Z}}_q(t) \geq 0$ with $\det(\hat{\mathcal{Z}}_q(t)) = 0$. If a rank minimization objective is also included, this problem turns out to be a slightly modified version of the well-known Frisch problem, for which no analytical solution is known. However, it is always possible to find a solution via heuristic methods. For more details the interested reader is referred to [9].

After the T-periodic diagonal matrices $\Delta_{H,q}(\cdot)$ and $\Delta_{R,q}(\cdot)$ have been selected, factorize $\hat{\mathcal{Z}}_q(t)$ as follows

$$\hat{\mathcal{Z}}_q(t) = \hat{G}(t)\hat{G}(t)'$$

where

$$\hat{G}(t) = \begin{bmatrix} \hat{G}_1(t) \\ \hat{G}_2(t) \end{bmatrix} = \begin{bmatrix} \hat{G}_3(t) \\ \hat{G}_4(t) \\ \hat{G}_5(t) \end{bmatrix} \tag{22}$$

and $\hat{G}_1(t) \in I\!R^{n(t)\times n(t)}$, $\hat{G}_2(t) \in I\!R^{(q+1-n(t))\times n(t)}$, $\hat{G}_3(t) \in I\!R^{1\times n(t)}$, $\hat{G}_4(t) \in I\!R^{n(t+1)\times n(t)}$ and $\hat{G}_5(t) \in I\!R^{(q-n(t+1))\times n(t)}$. Finally, select from $\hat{\mathcal{H}}_q(t)$ the vector

$$\hat{M}_1(t) = \begin{bmatrix} \hat{h}_1(t+1) \\ \hat{h}_2(t+2) \\ \vdots \\ \hat{h}_{n(t+1)}(t+n(t+1)) \end{bmatrix} \tag{23}$$

We are now in a position to provide the main result below, whose proof is similar to that of Theorem 3.1 and is therefore omitted.

Theorem 3.2: Let $q \geq 0$, $\hat{\mathcal{R}}_q(t)$ and $\hat{\mathcal{H}}_q(t)$ be given for $t \in [0, T-1]$ and assume that $\hat{\mathcal{R}}_q(t) > 0$ for all t. Moreover, define $\hat{\mathcal{Z}}_q(t)$ as in (21). Let $\Delta_{H,q}(t)$ and $\Delta_{R,q}(t)$ be such that $\hat{\mathcal{Z}}_q(t) \geq 0$ with $\det(\hat{\mathcal{Z}}_q(t)) = 0$. Let $\hat{G}(t)$ be a full-rank factor of $\hat{\mathcal{Z}}_q(t)$ and define $\hat{G}_1(t)$, $\hat{G}_3(t)$, $\hat{G}_4(t)$ and $\hat{M}_1(t)$ as in (22), (23). Then, the T-periodic quadruple

$$\tilde{A}(t) = \hat{G}_1(t+1)^{-1}\hat{G}_4(t)$$
$$\tilde{B}(t) = \hat{G}_1(t+1)^{-1}\Delta_{R,n(t+1)-1}(t+1)\hat{M}_1(t)\Delta_{H,0}(t)$$
$$\tilde{C}(t) = \hat{G}_3(t)$$
$$\tilde{D}(t) = \Delta_{H,0}(t)$$

defines a T-periodic realization of the given normalized Markov coefficients and covariance data. Moreover, if $(\tilde{A}(\cdot), \tilde{B}(\cdot))$ is reachable, then $\tilde{A}(\cdot)$ is asymptotically stable. ∎

The general formulation of Theorem 3.2 can be particularized into two specific problems of interest in applications. In fact, when only the Markov parameters are normalized, then $R_q(t)$ is completely known and hence $\Delta_{R,q}(t)$ is fixed. For example, this situation occurs when we want to model via stochastic realization a given cyclostationary process. In this case, the free parameters to be adjusted to meet the data consistency condition are the Markov parameters $h_0(\cdot)$. Analogous comments hold for the dual problem of finding a T-periodic realization based on a finite window of Markov parameters and normalized covarainces. completely known.

The theory presented above is now illustrated by a simple example.

Example 3.1: Consider $q = 1$, $T = 3$ and the (normalized) data:

$$\begin{aligned} r_0(0) &= 1,\ r_1(0) = 0.5 \\ r_0(1) &= 1,\ r_1(1) = -0.2 \\ r_0(2) &= 1,\ r_1(2) = 0.75 \\ \hat{h}_1(0) &= 5,\ \hat{h}_1(1) = 2,\ \hat{h}_1(2) = 1 \end{aligned}$$

It is easy to see that $\mathcal{R}_q(t)$ is positive definite for each t. The problem is to find

$$\alpha := h_0(0)^2,\ \beta := h_0(1)^2,\ \gamma := h_0(2)^2$$

such that the three matrices

$$\begin{aligned} \hat{Z}_q(0) &= \begin{bmatrix} 1-\alpha & 0.5-2\alpha \\ 0.5-2\alpha & 1-4\alpha-\beta \end{bmatrix} \\ \hat{Z}_q(1) &= \begin{bmatrix} 1-\beta & -0.2-\beta \\ -0.2-\beta & 1-\beta-\gamma \end{bmatrix} \\ \hat{Z}_q(2) &= \begin{bmatrix} 1-\gamma & 0.75-5\gamma \\ 0.75-5\gamma & 1-25\gamma-\alpha \end{bmatrix} \end{aligned}$$

are positive semidefinite with (minimum) rank one.

The only solution is

$$\alpha = 0.136,\ \beta = 0.3959, \gamma = 0.0164$$

so that

$$\begin{aligned} \hat{G}(0) &= \begin{bmatrix} -0.9295 \\ -0.2453 \end{bmatrix}, \quad \hat{G}(1) = \begin{bmatrix} -0.7773 \\ 0.7666 \end{bmatrix} \\ \hat{G}(2) &= \begin{bmatrix} -0.9917 \\ -0.6738 \end{bmatrix} \end{aligned}$$

and

$$\begin{array}{lll} \tilde{A}(0) = \ \ 0.3156 & \tilde{A}(1) = -0.773 & \tilde{A}(2) = \ \ 0.724 \\ \tilde{B}(0) = -0.9489 & \tilde{B}(1) = -0.6345 & \tilde{B}(2) = -0.6885 \\ \tilde{C}(0) = -0.9295 & \tilde{C}(1) = -0.7773 & \tilde{C}(2) = -0.9917 \\ \tilde{D}(0) = \ \ 0.3688 & \tilde{D}(1) = \ \ 0.6292 & \tilde{D}(2) = \ \ 0.128 \end{array}$$

References

[1] P. Colaneri, S. Longhi, "The realization problem for linear periodic systems", *Automatica*, vol. 31, n. 5, pp. 775–779, 1995.

[2] A. Varga, "Balancing related methods for minimal realization of periodic systems", *Systems and Control Letters*, vol. 36, pp. 339–349, 1999.

[3] M. Kujiper, "A periodically time-varying minimal partial realization algorithm based on twisting", *Automatica*, vol. 35, pp. 1543–1548, 1999.

[4] A. Varga, P. Van Dooren, "Computational methods for periodic systems - An overview", Prepr. *IFAC Workshop on Periodic Control Systems*, Como, Italy, 2001.

[5] S. Bittanti, P. Colaneri, "Invariant representations of discrete-time periodic systems - A survey", *Automatica*, vol. 36, n. 12, pp. 1777–1793, 2000.

[6] W.A. Gardner (Ed.), *Cyclostationarity in Communications and Signal Processing*, IEEE Press, New York, 1994.

[7] S. Bittanti, "Deterministic and stochastic linear periodic systems", in *Time Series and Linear Systems*, S. Bittanti ed., Springer-Verlag, Berlin, 1986, pp. 141–182.

[8] P. Enqvist, "Pole-zero identification based on simultaneous realization of normalized covariances and Markov parameters", Proc. *41st IEEE Conference on Decision and Control*, Las Vegas, NV, 2002.

[9] M. Fazel, "Matrix rank minimization with applications", *Ph.D. Thesis*, Stanford University, 2002.