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On Model Reduction Using LMI's

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***Abstract*— In this paper, we deal with the problem of approximating a given n-th order LTI system G by an r-th order system G_r where $r < n$. It is shown that lower bounds of the $\mathcal{H}_\infty$ norm of the associated error system can be analyzed by using LMI-related techniques. These lower bounds are given in terms of the Hankel singular values of the system G and coincide with those obtained in the previous studies where the analysis of the Hankel operators plays a central role. Thus, this paper provides an alternative proof for those lower bounds via simple algebraic manipulations related to LMI's. Moreover, when we reduce the system order by the multiplicity of the smallest Hankel singular value, we show that the problem is essentially convex and the optimal reduced-order models can be constructed via LMI optimization.**

I. Introduction

The $\mathcal{H}_\infty$ model reduction problem has been a central topic in control theory. Given a linear time-invariant (LTI) system G of McMillan degree n, the problem is to find a system G_r of McMillan degree r that minimizes the $\mathcal{H}_\infty$ norm $||G - G_r||_\infty$ where $r < n$. Intuitively, model reduction can be done by removing the states from G that are of little effect on the system input-output characteristics. The balanced realization [4], [10], [12], [13] is useful to achieve this, since in this realization the contribution of each state x_i to the input-output characteristics is indicated by the corresponding Hankel singular value $\sigma_i \geq 0$. Thus, the balanced truncation method [4], [10], [12], [13] has been developed, where we first convert the system G to the balanced realization form and second obtain a reduced-order model by truncating the states with small σ_i's. On the other hand, in the optimal Hankel norm approximation method [4], the problem has been dealt with more rigorously by analyzing the Hankel operator of G. It has been shown that the Hankel norm of the error incurred in approximating G by G_r is at least as large as the $(r+1)$-st largest Hankel singular value, and that we can obtain G_r that achieves this lower bound by following the all-pass embedding procedure [4]. These two methods provide constructive ways for model reduction. One significant achievement is that upper bounds and lower bounds of the approximation error have been gained in an analytic form in terms of the Hankel singular values [4], [13].

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From the viewpoints of the LMI-based $\mathcal{H}_\infty$ controller synthesis, the $\mathcal{H}_\infty$ model reduction problem is hard to solve since it can be regarded as a special case of the reduced-order controller synthesis problems. In stark contrast with the full-order controller synthesis, the reduced-order controller synthesis problems are considered to be essentially BMI's and still remain open to this date [2], [7]. Although some effective local algorithms for the computation of the reduced-order $\mathcal{H}_\infty$ controllers have been developed [3], [5], [8], we cannot evaluate the resulting $\mathcal{H}_\infty$ cost in a rigorous fashion due to the lack of analytic results on the achievable performance by the reduced-order controllers. Thus, it is of great importance to establish ways for computing strict lower bounds of the $\mathcal{H}_\infty$ cost attained by the reduced-order controllers.

The goal of this paper is to show that, in dealing with the $\mathcal{H}_\infty$ model reduction problems, we can readily obtain lower bounds of the $\mathcal{H}_\infty$ cost by using the well-established LMI-related techniques. The Elimination Lemma [2], [7], [11], which plays a key role in the LMI-based $\mathcal{H}_\infty$ controller synthesis, leads us to two matrix inequalities that are closely related to the Lyapunov equalities with respect to the controllability and observability Gramians [4], [13]. With these matrix inequalities and the results from the balanced realization [4], [10], [13], it follows that the lower bounds are given in terms of the Hankel singular values. These lower bounds are exactly the same as those obtained in the optimal Hankel approximation method [4]. Thus this paper provides an alternative proof for those lower bounds via simple algebraic manipulations related to the LMI's. Moreover, in the case where we reduce the system order by the multiplicity of the smallest Hankel singular value, we show that the $\mathcal{H}_\infty$ model reduction problem is essentially convex, and that the optimal reduced-order models can be constructed by solving LMI feasibility/optimization problems.

We use the following notations in this paper. I_n and $0_{n,m}$ denote respectively the identity matrix of dimension n and the zero matrix of dimension $n \times m$; the dimensions are omitted when they can be inferred from the context. For a matrix $A \in \mathbf{R}^{n\times n}$, A^{-1} and A^T are the inverse and transpose of the matrix A, respectively. $\mathrm{He}\{A\}$ is a shorthand notation for $A + A^T$. For a symmetric matrix A, we denote by triplet $(\mathrm{In}_-(A), \mathrm{In}_0(A), \mathrm{In}_+(A))$ the numbers of its strictly negative, zero, and strictly positive eigenval-

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ues, respectively. For a matrix $A \in \mathbf{R}^{n\times m}$ with rank r, $A^{\perp} \in \mathbf{R}^{(n-r)\times n}$ is a matrix such that $A^{\perp}A = 0$ and $A^{\perp}(A^{\perp})^T > 0$. Furthermore, $\mathcal{S}_n$ denotes the set of $n \times n$ positive-definite matrices.

The following lemma is used in the subsequent discussions.

Lemma 1: [9] For given two symmetric matrices $A \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{n\times n}$, $A < B$ holds only if $\lambda_i(A) < \lambda_i(B)$ $(i = 1, \cdots, n)$ where $\lambda_i(A)$ denotes the i-th largest eigenvalue of A.

II. Balanced Realization and LMI-based Model Reduction

Let us consider a system $G(s) \in \mathcal{RH}_\infty$ of McMillan degree n and its minimal realization

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \qquad A \in \mathbf{R}^{n\times n}, \quad B \in \mathbf{R}^{n\times p}, \quad C \in \mathbf{R}^{q\times n}, \quad D \in \mathbf{R}^{q\times p} \tag{1}$$

The $\mathcal{H}_\infty$ model reduction problem is to find a system $G_r(s)$ of McMillan degree r that minimizes the $\mathcal{H}_\infty$ norm $||G(s)-G_r(s)||_\infty$ where $r < n$. In the sequel, we assume that the realization in (1) is already balanced, i.e., its controllability and observability Gramians are equal and diagonal [4], [10], [13]. Denoting the balanced Gramians by Σ, we have

$$A\Sigma + \Sigma A^T + BB^T = 0 \tag{2a}$$

$$\Sigma A + A^T\Sigma + C^TC = 0 \tag{2b}$$

where

$$\Sigma = \mathrm{diag}\left(\sigma_1 I_{k_1}, \cdots, \sigma_l I_{k_l}, \sigma_{l+1} I_{k_{l+1}}, \cdots, \sigma_m I_{k_m}\right), \qquad \sigma_1 > \cdots > \sigma_l > \sigma_{l+1} > \cdots > \sigma_m > 0 \tag{3}$$

Note that k_i is the multiplicity of σ_i and $k_1+\cdots+k_m = n$.

The diagonal entries of Σ are called the Hankel singular values of the system $G(s)$ [12]. Suppose $\sigma_l \gg \sigma_{l+1}$. Then the balanced realization implies that those states corresponding to $\sigma_{l+1}, \cdots, \sigma_m$ are less controllable and observable than those states corresponding to $\sigma_1, \cdots, \sigma_l$. Thus, truncating those states with small σ_i's will not lose much information about the system input-output characteristics. The balanced truncation method simply applies this truncation operation to the balanced realization of $G(s)$. By partitioning (A, B, C) conformably with $\Sigma_1 = \mathrm{diag}(\sigma_1 I_{k_1}, \cdots, \sigma_l I_{k_l})$ and $\Sigma_2 = \mathrm{diag}(\sigma_{l+1} I_{k_{l+1}}, \cdots, \sigma_m I_{k_m})$, we have

$$G(s) = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \tag{4}$$

Then, the reduced-order model $G_r(s)$ of McMillan degree $r = k_1 + \cdots + k_l$ is constructed by the state-space matrices (A_{11}, B_1, C_1, D). It has been shown that the resulting model $G_r(s)$ is stable. Moreover, the approximation error is proved to be bounded by the following formula [4].

$$||G(s) - G_r(s)||_\infty \leq 2(\sigma_{l+1} + \cdots + \sigma_m) \tag{5}$$

Although the balanced truncation method is promising for the $\mathcal{H}_\infty$ model reduction problems, this method is deficient in the sense that the resulting reduced-order models are not necessarily optimal with respect to the $\mathcal{H}_\infty$ cost. To overcome this, in the framework of the LMI's, the $\mathcal{H}_\infty$ optimal reduced-order models have been sought by means of the bounded real lemma [1]. Indeed, if we denote the state space matrices of the system $G_r(s)$ by (A_r, B_r, C_r, D_r), then the $\mathcal{H}_\infty$ optimal reduced-order models can be sought by minimizing γ^2 subject to the following matrix inequalities.

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} > 0,$$

$$\begin{bmatrix} P_{11}A + A^TP_{11} & A^TP_{12} + P_{12}A_r & P_{11}B + P_{12}B_r & C^T \\ * & P_{22}A_r + A_r^TP_{22} & P_{12}^TB + P_{22}B_r & -C_r^T \\ * & * & -\gamma^2 I & D^T - D_r^T \\ * & * & * & -I \end{bmatrix} < 0 \tag{6}$$

Unfortunately, however, the above inequalities are not LMI's with respect to the matrix variables P_{11}, P_{12}, P_{22} and A_r, B_r, C_r, D_r since bilinear terms occur. Thus, the $\mathcal{H}_\infty$ model reduction problems are likely to be essentially non-convex problems represented by BMI's and computing globally optimal solutions remains open to this date.

Nevertheless, the formulation (6) is still useful to obtain suboptimal solutions via the coordinate-based decent methods [6], [8]. Indeed, by constraining the variables A_r and B_r to be constant, the inequalities in (6) are linear with respect to P, C_r and D_r. Also, if we fix P_{12} and P_{22} to be constant, the inequalities in (6) come to be LMI's with respect to P_{11}, A_r, B_r, C_r and D_r. By minimizing γ^2 using the freedom of unfixed variables iteratively, we can obtain suboptimal solutions for the $\mathcal{H}_\infty$ model reduction problems.

III. Main Results

A. *Analysis of Lower Bounds Using LMI-related Techniques*

Now we are in a position to state the main results of the paper. The first result concerns lower bounds of the $\mathcal{H}_\infty$ cost incurred in approximating $G(s)$ by $G_r(s)$. To derive the lower bounds, we follow the standard procedure for the LMI-based $\mathcal{H}_\infty$ controller synthesis. Applying the Elimination Lemma [2], [7], [11] to (6), we readily obtain the following theorem that forms an important basis for the analysis of the lower bounds.

Theorem 1: Let us consider a system $G(s) \in \mathcal{RH}_\infty$ of McMillan degree n and its minimal realization

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{7}$$

Then, there exist a $G_r(s) \in \mathcal{RH}_\infty$ of McMillan degree at most r that satisfies $||G(s) - G_r(s)||_\infty < \gamma$ if and only if there exist $X_{11} \in \mathcal{S}_n$, $P_{11} \in \mathcal{S}_n$, $P_{12} \in \mathbf{R}^{n\times r}$ and $P_{22} \in \mathcal{S}_r$ satisfying the following matrix inequalities.

$$AX_{11} + X_{11}A^T + \frac{1}{\gamma^2}BB^T < 0, \tag{8a}$$

$$P_{11}A + A^TP_{11} + C^TC < 0, \tag{8b}$$

$$X_{11} = (P_{11} - P_{12}P_{22}^{-1}P_{12}^T)^{-1} \tag{8c}$$

Proof: See the appendix section for the proof. ■

The condition (8) is still non-convex with respect to the decision variables due to the equality constraint (8c). This equality constraint commonly arises in the general reduced-order $\mathcal{H}_\infty$ controller synthesis [2], [7] and prevents us from reducing those synthesis problems into LMI's. It is known that this equality constraint can be recast into a rank constraint on the variables X_{11} and P_{11} and hence, in the previous works, research efforts have been made mainly on establishing efficient computation methods for solving those rank-constrained-LMI's [3], [5], [8]. On the other hand, studies on seeking for analytic results deduced by the rank-constrained-LMI's are rare, and research in this direction would be an important topic in the future.

In this paper we are dealing with a special case of the reduced-order $\mathcal{H}_\infty$ controller synthesis problems, i.e., the $\mathcal{H}_\infty$ model reduction problem. It follows that we can fully rely on the results from the balanced realization [4], [10], [13]. Indeed, by noting that the first two inequalities in (8) are closely related to the Lyapunov equalities (2) for the balanced controllability and observability Gramian, we can show that lower bounds of the $\mathcal{H}_\infty$ cost incurred in the approximation of $G(s)$ by $G_r(s)$ can be given in terms of the Hankel singular values. In the following corollary, we neglect the multiplicity of the Hankel singular values of $G(s)$ given in (3) and denote them by $\sigma_1 \geq \cdots \geq \sigma_r \geq \sigma_{r+1} \geq \cdots \geq \sigma_n > 0$ for the ease of our statements.

Corollary 1: Let us consider a system $G(s) \in \mathcal{RH}_\infty$ of McMillan degree n with the Hankel singular values $\sigma_1 \geq \cdots \geq \sigma_r \geq \sigma_{r+1} \geq \cdots \geq \sigma_n > 0$. Then, for all $G_r(s) \in \mathcal{RH}_\infty$ of McMillan degree less than or equal to r, we have

$$||G(s) - G_r(s)||_\infty \geq \sigma_{r+1} \tag{9}$$

Proof: To prove the assertion, we show that the condition (8) does not hold if $\gamma \leq \sigma_{r+1}$. From (2) and the first two inequalities in (8), we readily obtain

$$\begin{aligned} &A(X_{11} - \frac{1}{\gamma^2}\Sigma) + (X_{11} - \frac{1}{\gamma^2}\Sigma)A^T < 0, \\ &(P_{11} - \Sigma)A + A^T(P_{11} - \Sigma) < 0 \end{aligned} \tag{10}$$

Since A is stable, it follows that

$$X_{11} - \frac{1}{\gamma^2}\Sigma > 0, \quad P_{11} - \Sigma > 0 \tag{11}$$

With these inequalities and (8c), it turns out that the following condition is necessary for the condition (8) to hold.

$$\Sigma - \gamma^2\Sigma^{-1} < P_{12}P_{22}^{-1}P_{12}^T \tag{12}$$

If $\gamma \leq \sigma_{r+1}$, however, we see from the diagonal entries of $\Sigma - \gamma^2\Sigma^{-1}$ that $\mathrm{In}_-(\Sigma - \gamma^2\Sigma^{-1}) \leq n - r - 1$ whereas it is apparent that $\mathrm{In}_0(P_{12}P_{22}^{-1}P_{12}^T) \geq n - r$. Thus, from Lemma 1, the condition (12) cannot be satisfied if $\gamma \leq \sigma_{r+1}$. This completes the proof. ■

The lower bound given in Corollary 1 is exactly the same as those obtained in the optimal Hankel norm approximation method [4], [12]. In these previous works, the Hankel operator of $G(s)$ and its Hankel norm is analyzed in detail and the lower bound is derived for approximation errors measured by the Hankel norm. In stark contrast, we derive here the lower bound by directly working on the $\mathcal{H}_\infty$ norm of the associated error systems. Simple algebraic manipulations related to the LMI's and basic results form linear algebra are enough to arrive at the lower bound.

B. Optimal $\mathcal{H}_\infty$ Model Reduction via LMI Optimization

In the preceding subsection, we have shown that $||G(s) - G_r(s)|| \geq \sigma_{r+1}$ holds for all $G_r(s) \in \mathcal{RH}_\infty$ of McMillan degree less than or equal to r. The goal of this subsection is to show that, in the case where we reduce the system order by the multiplicity of the smallest Hankel singular value, i.e., if $r = n - k_m$, this lower bound is indeed the infimum and the optimal reduced-order model that attains this infimum can be obtained via LMI optimization. To this end, let us again focus on the Lyapunov equalities in (2). Then, it is a direct consequence that the pair $(\frac{1}{\sigma_m^2}\Sigma, \Sigma)$ satisfies the following equalities corresponding to (8a) and (8b) with $\gamma = \sigma_m$, respectively.

$$A\frac{1}{\sigma_m^2}\Sigma + \frac{1}{\sigma_m^2}\Sigma A^T + \frac{1}{\sigma_m^2}BB^T = 0, \tag{13a}$$

$$\Sigma A + A^T\Sigma + C^TC = 0 \tag{13b}$$

Moreover, in relation to the equality condition (8c), it is important to note that the pair $(\frac{1}{\sigma_m^2}\Sigma, \Sigma)$ satisfies

$$\frac{1}{\sigma_m^2}\Sigma = (\Sigma - P_{12}P_{22}^{-1}P_{12}^T)^{-1} \tag{14}$$

with

$$P_{12} = \begin{bmatrix} I_{n-k_m} \\ 0_{k_m,n-k_m} \end{bmatrix},$$

$$P_{22} = \mathrm{diag}\left(\left(\sigma_1 - \frac{\sigma_m^2}{\sigma_1}\right)^{-1} I_{k_1}, \cdots, \left(\sigma_{m-1} - \frac{\sigma_m^2}{\sigma_{m-1}}\right)^{-1} I_{k_{m-1}}\right) > 0 \tag{15}$$

The equalities in (13) and (14) imply that, in the case where $r = n - k_m$, the conditions in (8) will be satisfied for $\gamma = \sigma_m$ with $X_{11} = \frac{1}{\sigma_m^2}\Sigma$, $P_{11} = \Sigma$ and P_{12} and P_{22} given in (15), provided that we replace the inequalities in (8) to equalities. Although these arguments are not enough to conclude that σ_m is the infimum of $||G(s) - G_{n-k_m}(s)||_\infty$, the above discussions can be made more rigorous and we are led to the following results.

Lemma 2: Let us consider a system $G(s) \in \mathcal{RH}_\infty$ of McMillan degree n with the Hankel singular values given in (3). Then, for arbitrary $\gamma > \sigma_m$, there exists a $G_{n-k_m}(s) \in \mathcal{RH}_\infty$ of McMillan degree at most $n - k_m$ that satisfies $||G(s) - G_{n-k_m}(s)||_\infty < \gamma$.

Proof: See the appendix section for the proof. ■

From Lemma 2 and Corollary 1 we can conclude that σ_m is the infimum of $||G(s) - G_{n-k_m}(s)||_\infty$. The proof of the above lemma heavily relies on the equalities (13) and (14) (see the appendix section). These equalities are obtained particularly for $r = n - k_m$, and unfortunately, similar equalities are not easily available in other cases. Due to this fact, our discussion here is rather restrictive, and we cannot say anything on the strictness of the lower bounds given in Corollary 1 when $r < n - k_m$.

The results in Lemma 2 coincide with those obtained in the optimal Hankel norm approximation method (see, e.g., [12]). In that method, the way to construct the optimal reduced-order model $G_{n-k_m}(s)$ that achieves the infimal approximation error has been given by means of the all-pass embedding procedure. In the rest of section, we show that the optimal reduced-order models can be constructed also via LMI optimization. One important implication of the proof of Lemma 2 is that, in the case where $r = n - k_m$, we can fix the matrix variable P_{12} in (8) to be constant as in (15) without introducing any conservatism. If P_{12} is fixed, however, the matrix inequalities in (8) turn out to LMI's. Once the matrix variables (P_{11}, P_{12}, P_{22}) that satisfy (8) can be found, the optimal reduced-order models can be reconstructed by solving (6) for (A_r, B_r, C_r, D_r). To summarize, the $\mathcal{H}_\infty$ optimal reduced-order models can be obtained by solving LMI optimization/feasibility problems.

Theorem 2: The reduced-order model $G_{n-k_m}(s)$ of McMillan degree at most $n - k_m$ that minimizes $||G(s) - G_{n-k_m}(s)||_\infty$ can be obtained by the two-step procedure:

1. Minimize γ subject to the LMI's:

$$\begin{bmatrix} P_{11} & P_{12}Q_{22} \\ Q_{22}P_{12}^T & Q_{22} \end{bmatrix} > 0,$$

$$\begin{bmatrix} \mathrm{He}\{(P_{11} - P_{12}Q_{22}P_{12}^T)A\} & (P_{11} - P_{12}Q_{22}P_{12}^T)B \\ B^T(P_{11} - P_{12}Q_{22}P_{12}^T) & -\gamma^2 I \end{bmatrix} < 0, \tag{16}$$

$$P_{11}A + A^T P_{11} + C^T C < 0$$

where $P_{11} \in \mathcal{S}_n$ and $Q_{22} \in \mathcal{S}_{n-k_m}$ are matrix variables whereas $P_{12} \in \mathbf{R}^{n\times(n-k_m)}$ is a constant matrix given by $P_{12} = \begin{bmatrix} I_{n-k_m} \\ 0_{k_m,n-k_m} \end{bmatrix}$. For the subsequent step, define $\tilde{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & Q_{22}^{-1} \end{bmatrix}$ and denote the optimal value of γ by γ_{opt}.

2. Obtain (A_r, B_r, C_r, D_r) by solving the LMI given in (6), where P is fixed to $\tilde{P}$ and γ to γ_{opt}.

The LMI (16) in the first step follows from (8) by defining $Q_{22} := P_{22}^{-1}$. Analytic formulas provided in [7], [11] are also useful for the reconstruction of $G_r(s)$ in the second step.

It should be noted that the results in Theorem 2 are valid only in the case where (A, B, C) is balanced, since the choice of P_{12} depends on the state space realizations. Thus, in other cases, the fixation $P_{12} = \begin{bmatrix} I_{n-k_m} \\ 0_{k_m,n-k_m} \end{bmatrix}$ could be a source of conservatism and the optimal reduced-order models might not be obtained.

In closing this section, we show that it is possible also to obtain the optimal reduced-order model $G_{n-k_m}(s)$ in Theorem 2 via a one-step LMI optimization procedure. By the similarity transformation $\bar{A}_r := P_{22}A_rP_{22}^{-1}$, $\bar{B}_r := P_{22}B_r$ and $\bar{C}_r := C_rP_{22}^{-1}$, we see that there exist (A_r, B_r, C_r, D_r) that satisfy (6) for some $P > 0$ if and only if

$$\begin{bmatrix} P_{11}A + A^T P_{11} & A^T P_{12} + P_{12}P_{22}^{-1}\bar{A}_r P_{22} & P_{11}B + P_{12}P_{22}^{-1}\bar{B}_r & C^T \\ * & \bar{A}_r P_{22} + P_{22}\bar{A}_r^T & P_{12}^T B + \bar{B}_r & -P_{22}\bar{C}_r^T \\ * & * & -\gamma^2 I & D^T - D_r^T \\ * & * & * & -I \end{bmatrix} < 0 \tag{17}$$

Furthermore, by the congruence transformation with $\mathrm{diag}(I, Q_{22}, I, I)$ where $Q_{22} := P_{22}^{-1}$, we have

$$\begin{bmatrix} P_{11}A + A^T P_{11} & A^T P_{12}Q_{22} + P_{12}Q_{22}\bar{A}_r & P_{11}B + P_{12}Q_{22}\bar{B}_r & C^T \\ * & Q_{22}\bar{A}_r + \bar{A}_r^T Q_{22} & Q_{22}P_{12}^T B + Q_{22}\bar{B}_r & -\bar{C}_r^T \\ * & * & -\gamma^2 I & D^T - D_r^T \\ * & * & * & -I \end{bmatrix} < 0 \tag{18}$$

If the matrix variable P_{12} is fixed to be constant, the above inequality is an LMI with respect to the matrix variables P_{11}, Q_{22} and $\tilde{A}_r := Q_{22}\bar{A}_r$, $\tilde{B}_r := Q_{22}\bar{B}_r$, $\bar{C}_r$, D_r. Once these variables have been found, the optimal reduced-order models can be reconstructed by

$$G_r(s) = \left[\begin{array}{c|c} Q_{22}^{-1}\tilde{A}_r & Q_{22}^{-1}\tilde{B}_r \\ \hline \bar{C}_r & D_r \end{array}\right] \tag{19}$$

The matrix inequality (18) as well as (8) clearly indicate that the non-convexity of the problem stems from the bilinear terms with respect to the matrix variable P_{12}. Hence, if we can fix P_{12} without introducing any conservatism as in Theorem 2, we are able to obtain globally optimal solutions via LMI optimization.

IV. Conclusion

In this paper, we applied the well-established LMI-related techniques to the $\mathcal{H}_\infty$ model reduction problems so that we can obtain lower bounds of the $\mathcal{H}_\infty$ cost incurred in the approximation. Following the standard procedure for the LMI-based $\mathcal{H}_\infty$ controller synthesis [2], [7], [11], we arrived at two matrix inequalities with non-convex equality constraints that commonly occur in the general reduced-order $\mathcal{H}_\infty$ controller synthesis. With these inequalities and the particular results from the balanced realization, it turns out that the lower bounds are given in terms of the Hankel singular values. Moreover, in the case where we reduce the system order by the multiplicity of the smallest Hankel singular value, we prove that the problem is essentially convex and the $\mathcal{H}_\infty$ optimal reduced-order models can be obtained by solving LMI optimization problems. These results are not completely new and coincide with those obtained in the optimal Hankel norm approximation method [4]. Our novel contribution is showing alternative proofs for those results via recently developed LMI-related techniques.

Recall that the $\mathcal{H}_\infty$ model reduction problem is a special case of the reduced-order $\mathcal{H}_\infty$ controller synthesis problems. It should be noted that those results on the lower bounds of the $\mathcal{H}_\infty$ cost and the optimal solutions for a specific order case have not been gained in the general reduced-order $\mathcal{H}_\infty$ controller synthesis setting. It is not yet clear to us whether the LMI-based techniques explored in this paper can be extended to the general reduced-order $\mathcal{H}_\infty$ controller synthesis. This topic is currently under investigation.

Appendix

Proof of Theorem 1: In order to prove Theorem 1, we follow the standard procedure for the LMI-based $\mathcal{H}_\infty$ controller synthesis [2], [7]. Let us first write the state space realization of the error system $E(s) := G(s) - G_r(s)$ as follows:

$$\begin{aligned} E(s) &= \left[\begin{array}{c|c} A_e & B_e \\ \hline C_e & D_e \end{array}\right] \\ &= \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B}_1 \\ \hline \mathcal{C}_1 & \mathcal{D}_{11} \end{array}\right] + \left[\begin{array}{c} \mathcal{B}_2 \\ \hline \mathcal{D}_{12} \end{array}\right] \mathcal{G} \left[\begin{array}{c|c} \mathcal{C}_2 & \mathcal{D}_{21} \end{array}\right] \end{aligned} \tag{20}$$

where

$$\left[\begin{array}{c|c:c} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \hline \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} \\ \hdashline \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \end{array}\right] = \left[\begin{array}{cc|c:cc} A & 0 & B & 0 & 0 \\ 0 & 0 & 0 & I_r & 0 \\ \hline C & 0 & D & 0 & -I_q \\ \hdashline 0 & I_r & 0 & 0 & 0 \\ 0 & 0 & I_p & 0 & 0 \end{array}\right], \tag{21}$$

$$\mathcal{G} = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}$$

Then, the matrix inequality (6) comes to

$$\begin{bmatrix} \mathrm{He}\{P\mathcal{A}\} & P\mathcal{B}_1 & \mathcal{C}_1^T \\ \mathcal{B}_1^T P & -\gamma^2 I & \mathcal{D}_{11}^T \\ \mathcal{C}_1 & \mathcal{D}_{11} & -I \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} P\mathcal{B}_2 \\ 0_{p,r+q} \\ \mathcal{D}_{12} \end{bmatrix} \mathcal{G} \begin{bmatrix} \mathcal{C}_2 & \mathcal{D}_{21} & 0_{r+p,q} \end{bmatrix} \right\} < 0 \tag{22}$$

The conditions in (8) is now derived from (22) by eliminating the variable $\mathcal{G}$. Indeed, we see from the Elimination Lemma [2], [7], [11] that (22) holds if and only if there exist $P \in \mathcal{S}_{n+r}$ such that

$$\begin{bmatrix} P\mathcal{B}_2 \\ 0_{p,r+q} \\ \mathcal{D}_{12} \end{bmatrix}^{\perp} \mathcal{L}(P) \left(\begin{bmatrix} P\mathcal{B}_2 \\ 0_{p,r+q} \\ \mathcal{D}_{12} \end{bmatrix}^{\perp} \right)^T < 0, \tag{23a}$$

$$\begin{bmatrix} \mathcal{C}_2^T \\ \mathcal{D}_{21}^T \\ 0_{q,r+p} \end{bmatrix}^{\perp} \mathcal{L}(P) \left(\begin{bmatrix} \mathcal{C}_2^T \\ \mathcal{D}_{21}^T \\ 0_{q,r+p} \end{bmatrix}^{\perp} \right)^T < 0 \tag{23b}$$

where $\mathcal{L}(P)$ denotes the first term in (22). Here, we have from (21) that

$$\begin{bmatrix} P\mathcal{B}_2 \\ 0_{p,r+q} \\ \mathcal{D}_{12} \end{bmatrix}^{\perp} = \begin{bmatrix} [\ I_n\ 0_{n,r}\ ]P^{-1} & 0_{n,p} & 0_{n,q} \\ 0_{p,n+r} & I_p & 0_{p,q} \end{bmatrix},$$
$$\begin{bmatrix} \mathcal{C}_2^T \\ \mathcal{D}_{21}^T \\ 0_{q,r+p} \end{bmatrix}^{\perp} = \left[\begin{array}{cc:cc} I_n & 0_{n,r} & 0_{n,p} & 0_{n,q} \\ 0_{q,n} & 0_{q,r} & 0_{q,p} & I_q \end{array}\right] \tag{24}$$

Thus, by partitioning P as in (6), the inequalities (23a) and (23b) reduce respectively to

$$\begin{bmatrix} \mathrm{He}\{A(P_{11}-P_{12}P_{22}^{-1}P_{12}^T)^{-1}\} & B \\ B^T & -\gamma^2 I \end{bmatrix} < 0,$$
$$\begin{bmatrix} P_{11}A + A^TP_{11} & C^T \\ C & -I \end{bmatrix} < 0 \tag{25}$$

Applying the Schur Complement technique [1] to the inequalities in (25) lead to (8) with $X_{11} = (P_{11} - P_{12}P_{22}^{-1}P_{12}^T)^{-1}$. This completes the proof. ■

Proof of Lemma 2: Let us define $\varepsilon := \gamma - \sigma_m > 0$ and consider the following matrix inequalities that correspond to (8) in Theorem 1.

$$\begin{aligned}(P_{11} - P_{12}P_{22}^{-1}P_{12}^T)A + A^T(P_{11} - P_{12}P_{22}^{-1}P_{12}^T)\\ +\frac{1}{(\sigma_m+\varepsilon)^2}(P_{11} - P_{12}P_{22}^{-1}P_{12}^T)BB^T\\ \times(P_{11} - P_{12}P_{22}^{-1}P_{12}^T) < 0,\end{aligned} \tag{26a}$$

$$P_{11}A + A^TP_{11} + C^TC < 0, \tag{26b}$$

$$P_{11} - P_{12}P_{22}^{-1}P_{12}^T > 0 \tag{26c}$$

Then, to prove Lemma 2, it is enough to show that for any $\varepsilon > 0$, ther exists $P_{11} \in \mathcal{S}_n$ satisfying (26) with P_{12} and P_{22} given in (15). To this end, let us first consider the stabilizing solution $\Pi > 0$ of the following Riccati equation, which does exist if $Q > 0$ is small enough.

$$\Pi A + A^T\Pi + \frac{1}{2\sigma_m}\Pi BB^T\Pi + Q = 0 \tag{27}$$

Then, we see that $P_{11} := \Sigma + \varepsilon\Pi$ satisfies (26b), since we have from (13b) and (27) that

$$\begin{aligned}(\Sigma+\varepsilon\Pi)A + A^T(\Sigma+\varepsilon\Pi) + CC^T\\ = -\varepsilon(\frac{1}{2\sigma_m}\Pi BB^T\Pi + Q) < 0\end{aligned} \tag{28}$$

On the other hand, the left-hand side of (26a) comes to be

$$\begin{aligned}(\Sigma + \varepsilon\Pi - P_{12}P_{22}^{-1}P_{12}^T)A + A^T(\Sigma + \varepsilon\Pi - P_{12}P_{22}^{-1}P_{12}^T)\\ +\frac{1}{(\sigma_m+\varepsilon)^2}(\Sigma + \varepsilon\Pi - P_{12}P_{22}^{-1}P_{12}^T)BB^T\\ \times(\Sigma + \varepsilon\Pi - P_{12}P_{22}^{-1}P_{12}^T)\end{aligned} \tag{29a}$$

$$\begin{aligned}= \frac{\varepsilon}{(\sigma_m+\varepsilon)^2}(\sigma_m^2\Sigma^{-1}BB^T\Pi + \Pi BB^T\sigma_m^2\Sigma^{-1})\\ -\frac{2\sigma_m\varepsilon+\varepsilon^2}{\sigma_m^2(\sigma_m+\varepsilon)^2}\sigma_m^2\Sigma^{-1}BB^T\sigma_m^2\Sigma^{-1}\\ +\frac{\varepsilon^2}{(\sigma_m+\varepsilon)^2}\Pi BB^T\Pi + \varepsilon(\Pi A + A^T\Pi)\end{aligned} \tag{29b}$$

$$\begin{aligned}= -\frac{2\sigma_m\varepsilon+\varepsilon^2}{\sigma_m^2(\sigma_m+\varepsilon)^2}\left(\frac{\sigma_m^2}{2\sigma_m+\varepsilon}\Pi - \sigma_m^2\Sigma^{-1}\right)BB^T\\ \times\left(\frac{\sigma_m^2}{2\sigma_m+\varepsilon}\Pi - \sigma_m^2\Sigma^{-1}\right)\\ -\frac{\varepsilon^2}{2\sigma_m(2\sigma_m+\varepsilon)}\Pi BB^T\Pi - \varepsilon Q\end{aligned} \tag{29c}$$

$$< 0$$

where in deriving (29b) from (29a) we use $\Sigma + \varepsilon\Pi - P_{12}P_{22}^{-1}P_{12}^T = \varepsilon\Pi + \sigma_m^2\Sigma^{-1}$ that follows from (14) and the following equality condition resulting from (13a).

$$\sigma_m^2\Sigma^{-1}A + A^T\sigma_m^2\Sigma^{-1} + \sigma_m^2\Sigma^{-1}BB^T\Sigma^{-1} = 0 \tag{30}$$

Further, (29c) is readily derived from (29b) by using (27) and completing the square. It remains to show that the condition (26c) is satisfied, which is a simple task since the left-hand side of (26c) reduces to $\varepsilon\Pi + \sigma_m^2\Sigma^{-1} > 0$ as shown in the above discussion. Thus, by observing that $P_{11} = \Sigma + \varepsilon\Pi > 0$ satisfies (26) with P_{12} and P_{22} given in (15), the proof is completed. ■

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