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Reduced-order Adaptive Controller Design for Disturbance Attenuation and Asymptotic Tracking for SISO Linear Systems with Noisy Output Measurements

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***Abstract*— In this paper, we study the reduced-order adaptive control design for SISO linear systems under noisy output measurements, as compared to the full-order adaptive control proposed in [1]. We make the same assumption as [1] and follow the same design paradigm, where we simplify the dynamic order of the controllers by using a slightly modified design at a particular step of the control design. The dynamic order of the controllers obtained in this paper is $n-1$ or $n-2$ less than those presented in [1], depending on the eigen-structure of a particular feedback matrix. We prove rigorously that reduced-order controllers admit the same robustness properties as those of [1]. The order reduction will simplify the controller structure enormously without any sacrifice in performance. A numerical example is included in the paper, which demonstrates the improved performance resulting from the order reduction, even though there is no theory justifying the improvement.**

***Index Terms*— adaptive control; nonlinear H^∞ control; cost-to-come function; integrator backstepping.**

I. Introduction

Adaptive control has attracted a lot of research attention in control theory since 1970s. The classical approach for adaptive control design of linear systems has been based on the certainty equivalence principle [2]. In this approach, the controller is designed as if the unknown parameters in the system are known. In implementation, these unknown parameters are replaced by their on-line estimates. The certainty equivalence based design leads to relatively simple controller structure. Many different parameter identifiers can be used as long as they satisfy certain properties independent of the controller design. Any stabilizing control law can be used as well. This approach has shown to be successful for linear systems with or without stochastic disturbance inputs [3], [4], when long term asymptotic performance is considered. Yet, early adaptive control design based on this approach has been shown to be nonrobust when the system to be controlled admits unmodeled dynamics and deterministic exogenous disturbance inputs [5]. This motivates the study of robust adaptive control design in late 1980s and early 1990s. The certainty equivalence based approach fails to generalize to nonlinear systems where there exist severe nonlinearities. This then motivates the study of nonlinear adaptive control design in 1990s.

In 1990s, adaptive control for nonlinear systems was investigated intensely, motivated by the complete characterization of the feedback linearizable or partially feedback linearizable systems [6]. A general nonlinear design tool of integrator backstepping methodology is introduced in [7], which solves the adaptive control design for parametric strict feedback or parametric pure feedback nonlinear systems, where the design is mainly focused on the selection of a Lyapunov function for the closed-loop system and rendering its derivative function nonpositive. This result brings a period of intense research into nonlinear adaptive control design, where a large volume of results flourish, see the book [8] for a list of references on this topic. The integrator backstepping methodology is a systematic design procedure of controllers for nonlinear systems which offers a lot of flexibility, in terms of the choice of the value function and the virtual control law. How to properly choose these design flexibilities remains an open question [9]. It has been shown that a systematically designed nonlinear adaptive control law has better performance for linear systems than that of the certainty equivalence based design when the system is free of disturbance. Yet, this nonlinear design approach stops short of directly addressing the robustness property of the closed-loop system, and may be nonrobust when the system is subject to exogenous disturbance inputs.

The objective of robust adaptive control are to improve the transient performance, to accommodate unmodeled dynamics, and to tolerate exogenous disturbance inputs. These objectives are the same as those that motivates the H^∞-optimal control problem [10]. Intuitively speaking, the adaptive control design makes use of more information about the system under control than a simple robust control design, which suggests that adaptive controllers should have better robustness. Yet earlier adaptive control designs are shown to be nonrobust. Even though robustification of the certainty equivalence based design has been obtained in 1980s through 1990s [11], they still fell short of directly addressing the disturbance attenuation property of the closed-loop system. It has been shown that the objectives of robust control can be achieved by studying the disturbance attenuation property of the closed-loop system in the H^∞-optimal control problem. The game-theoretic approach to H^∞-optimal control further converts the problem into a soft-constrained zero-sum game problem, where the upper value of the game need to be guaranteed to be bounded [12]. This motivates the worst-case analysis based approach to robust adaptive control design [13], [14], [15], [1], where the measures of disturbance attenuation, asymptotic tracking, and transient performance are all incorporated into a single soft-constrained zero-sum game cost function. The unknown parameters of the system are viewed as state variables of an expanded nonlinear system. The robust adaptive control problem is then cast as a nonlinear H^∞-optimal control problem under imperfect state measurements. Applying the game theoretic solution to the nonlinear H^∞-optimal control problem [12], [16], [17], [18], [19], in particular, the cost-to-come function methodology [17], we may obtain a finite dimensional estimator for the expanded state vector in closed form, which further converts the H^∞-optimal control problem

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with imperfect state measurements into one with full information measurements. This full information measurement problem is then solved using the integrator backstepping methodology for a suboptimal solution. The above outlined design paradigm has been applied to identification problems [13], which yields new classes of parameterized robust parameter identifiers for linear and nonlinear systems. It has also been applied to robust adaptive control problems [15], [1], which leads to new classes of parameterized robust adaptive controllers with strong robustness properties. These successes motivate us to further investigate this approach in greater detail.

In this paper, we generalize the result of [1] by obtaining reduced-order adaptive controllers, as compared with the full-order controllers proposed in [1], which admits exactly the same robustness properties. We assume that the true system is observable, admits an known upper bound of dynamic order, admits a strictly minimum phase transfer function from the control input to the output, and admits a known sign of high frequency gain, and has a known relative degree. The true system may be uncontrollable from the control input and uncontrollable part is assumed to be stable in the sense of Lyapunov. Furthermore, the critically stable uncontrollable modes must be uncontrollable from the disturbances. These assumptions are the same as those of [1]. We make use of the parameter estimator and state estimator dynamics obtained in [1]. The difference between our design and that of [1] lies in the control design step, in particular, the step 0 of the integrator backstepping procedure. Instead of generating the reference trajectory for the entire state vector of the filter for the measurement output, which is n dimensional, we generate the reference trajectory for a particular linear combination of this state vector, which is essential for the robustness proof, via a one or two dimensional dynamic system. This leads to the reduction of $n-1$ or $n-2$ integrators in the dynamic order of the controller, where n is the known upper bound of the dynamic order of the true system. The specific number of the order reduction depends on the eigen-structure of the feedback matrix, A_f. When A_f has at least one real eigenvalue, the dynamic order of the controller may be reduced by $n-1$. Otherwise, it may only be reduced by $n-2$. Once step 0 of the backstepping procedure is modified, the rest of the backstepping procedure is essentially similar to that of [1], where a controller is formed. It is then proven that the controller achieves a guaranteed level of disturbance attenuation for any continuous exogenous disturbance inputs, total stability for the closed-loop system with respect to the exogenous disturbance inputs and the initial conditions, and asymptotic tracking of the reference trajectory when the disturbance inputs are $\mathcal{L}_\infty$ and $\mathcal{L}_2$. These results are exactly the same as [1]. This completes the preview of our result. Similar to [1], as a result of our assumptions, the closed-loop system may achieve asymptotic tracking even under exogenous disturbance inputs generated by unknown linear exo-system which is stable in the sense of Lyapunov. This result is illustrated by a numerical example in the paper.

The balance of the paper is organized as follows. In Section II, we present the formulation of the problem and discuss the general solution methodology. Next, we summarize the estimation design result of Section 3 of [1] in Section III for the convenience of readers. In Section IV, we present the controller design step using integrator backstepping and the main robustness results for the controllers. An example is included in Section V to illustrate the performance improvement resulting from the order reduction. The paper ends with some concluding remarks in Section VI and an appendix.

II. Problem Formulation

In this paper, the reduced-order controller design is motivated by the results of [1]. The linear system under consideration satisfies the following assumption.

Assumption 1: The linear system is known to be at most n dimensional, $n \in \mathbb{N}$. $\diamond$

By adding additional dynamics if necessary, the true system admits dynamics

$$\dot{\grave{x}} = \grave{A}\grave{x} + \grave{B}u + \grave{D}\grave{w}; \quad \grave{x}(0) = \grave{x}_0 \tag{1a}$$

$$y = \grave{C}\grave{x} + \grave{E}\grave{w} \tag{1b}$$

where $\grave{x}$ is the n-dimensional state vector; u is the scalar control input; y is the scalar system output; $\grave{w}$ is the $\grave{q}$-dimensional disturbance input, $\grave{q} \in \mathbb{N}$; all input and output signals y, u, and $\grave{w}$ are continuous; and the matrices $\grave{A}$, $\grave{B}$, $\grave{C}$, $\grave{D}$, and $\grave{E}$ are generally unknown. System (1) satisfies the following assumption.

Assumption 2: ($\grave{A}$, $\grave{C}$) is observable. The transfer function $H(s) = \grave{C}(sI - \grave{A})^{-1}\grave{B}$ is known to have relative degree $r \in \mathbb{N}$, and is strictly minimum phase. The uncontrollable part (with respect to u) of (1) is stable in the sense of Lyapunov. Any uncontrollable mode corresponding to an eigenvalue of the matrix $\grave{A}$ on the jw-axis is uncontrollable from the disturbance $\grave{w}$. $\diamond$

By Assumption 2, there exist a state diffeomorphism $x = \grave{T}\grave{x}$ and a disturbance transformation $w = \grave{M}\grave{w}$ such that the system is expressed as

$$\dot{x} = Ax + (y\bar{A}_{211} + u\bar{A}_{212})\theta + Dw; \quad x(0) = x_0 \tag{2a}$$

$$y = Cx + Ew \tag{2b}$$

where $\grave{T}$ is unknown; $\grave{M}$ is an unknown $q \times \grave{q}$ dimensional matrix, $q \in \mathbb{N}$; θ is the σ-dimensional vector of unknown parameters, $\sigma \in \mathbb{N}$; and the matrices A, $\bar{A}_{211}$, $\bar{A}_{212}$, D, C, and E admit the following structures: $A = (a_{ij})_{n\times n}$, $a_{i,i+1} = 1$, $a_{ij} = 0$, for $1 \leq i \leq r-1$ and $i+2 \leq j \leq n$; $\bar{A}_{212} = [\mathbf{0}_{\sigma\times(r-1)} \; \bar{A}'_{2120} \; \bar{A}'_{212r}]'$; $\bar{A}_{2120} = [1 \; \mathbf{0}_{1\times(\sigma-1)}]$; $C = [1 \; \mathbf{0}_{1\times(n-1)}]$. The equation (2) is called the design model which satisfies Assumptions $3-5$ described below.

Assumption 3: The matrices D and E are such that $EE' > 0$. Define $\zeta = (EE')^{-\frac{1}{2}}$ and $L = DE'$. $\diamond$

According to the structure of A, $\bar{A}_{212}$, the parameter vector θ is partitioned into $\theta = [b_0 \;\; \theta_s]'$, where b_0 is the high frequency gain of $H(s)$, θ_s is a $(\sigma-1)$-dimensional vector.

Assumption 4: The sign of high frequency gain b_0 is known. W.L.O.G, assume $b_0 > 0$. There exists a known smooth nonnegative radially-unbounded strictly convex function $P(\bar{\theta})$, such that $\theta \in \Theta := \{\bar{\theta} : P(\bar{\theta}) \leq 1\}$. Furthermore, for any $\bar{\theta} \in \Theta$, we have $\bar{b}_0 > 0$. $\diamond$

For the system (2), the control law is generated by $u(t) = \mu(t, y_{[0,t]})$, where $\mu : [0,\infty) \times \mathcal{C} \to \mathbb{R}$.

Assumption 5: The reference trajectory y_d is r times continuously differentiable, where the signal y_d and the derivatives $y_d^{(1)}, \ldots, y_d^{(r)}$ are uniformly bounded and available for feedback. $\diamond$

The objective of the control design is to make the system output Cx to track the reference trajectory y_d asymptotically while attenuating the effect of x_0, $\grave{w}$, and θ. The uncertainty triple $(x_0, \theta, \grave{w}_{[0,\infty)})$ belongs to the set $\grave{\mathcal{W}} = \mathbb{R}^n \times \Theta \times \mathcal{C}$.

Definition 1: A controller μ is said to achieve *disturbance attenuation level* γ if there exist $l(t, \theta, x, y_{[0,t]}) \geq 0$ and

$l_0(\check{x}_0, \check{\theta}_0) \geq 0$ such that for all $t \geq 0$ the following inequality holds:

$$\sup_{(x_0,\theta,\dot{w}_{[0,\infty)})\in\dot{\mathcal{W}}} J_{\gamma t} \leq 0 \tag{3}$$

where the cost function $J_{\gamma t}$ is

$$\begin{aligned} J_{\gamma t} \;\; &= \;\; \int_0^t ((x_1 - y_d)^2 + l(\tau, \theta, x, y_{[0,\tau]}) - \gamma^2|w|^2)d\tau \\ &\quad -\gamma^2|\theta - \check{\theta}_0|^2_{Q_0} - \gamma^2|x_0 - \check{x}_0|^2_{\Pi_0^{-1}} - l_0(\check{x}_0, \check{\theta}_0) \end{aligned}$$

where $\check{\theta}_0 \in \Theta$ is the initial guess of θ, and $Q_0 > 0$ is the quadratic weighting matrix, quantifying the level of confidence in the estimate $\check{\theta}_0$, $\check{x}_0$ is the initial guess of x_0, and $\Pi_0^{-1} > 0$ is the weighting matrix, quantifying the level of confidence in the estimate $\check{x}_0$.

The definition above intends to guarantee that, $\forall t \geq 0$, the squared $\mathcal{L}_2$ norm of the output tracking error $x_1 - y_d$ on $[0,t]$ is bounded by γ^2 times the squared $\mathcal{L}_2$ norm of $w_{[0,t]}$ plus a constant that depends only on the initial condition of the system.

Let $\check{x}$ denote the estimate of x and $\check{\theta}$ denote the estimate of θ.

In order to bring the adaptive control problem into the framework of H^∞-optimal control for affine-quadratic nonlinear systems with imperfect state measurements, the system dynamics (2) is expanded by adding the simple dynamics of θ: $\dot{\theta} = 0$. Define $\xi := (\theta', x')'$, which satisfies the following dynamics:

$$\begin{aligned} \dot{\xi} \;&=\; \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ y\bar{A}_{211} + u\bar{A}_{212} & A \end{bmatrix} \xi + \begin{bmatrix} \mathbf{0} \\ D \end{bmatrix} w =: \bar{A}\xi + \bar{D}w & \text{(4a)} \\ y \;&=\; \begin{bmatrix} \mathbf{0} & C \end{bmatrix} \xi + Ew =: \bar{C}\xi + Ew & \text{(4b)} \end{aligned}$$

The worst-case optimization of the cost function (3) can be carried out in two steps: first a supremization over $\xi(0)$ and w with known measurement waveform, and then supremization over all possible measurement waveforms, the idea is precisely explained by the following identity for each fixed $t \geq 0$:

$$\sup_{(x_0,\theta,\dot{w}_{[0,\infty)})\in\dot{\mathcal{W}}} J_{\gamma t} \leq \sup_{y_{[0,\infty)}} \sup_{(x_0,\theta,w_{[0,\infty)})|y_{[0,\infty)}} J_{\gamma t} \tag{5}$$

The inner supremization can be interpreted as the evaluation of the worst-case performance with a known output waveform. As a function of output, the control input waveform is independent of the actual disturbance input waveform, and can be viewed as an open-loop time function. This step is actually the identification design step, which is carried out first. In this paper, we make use of the results of [1] for this step.

The outer supremization can be interpreted as the computation of the worst-case measurement waveform against a given control law, which is crucial for the determination of the achievability of the objective (3). This step is the control design step, which we will discuss in detail in Section III.

Now we will turn to the discussion of these two steps in the next two sections.

III. Identifier Design

In this section, we mainly summarize the result of identifier design described in Section 3 of [1] for the convenience of the reader.

We choose $l = |\xi - \hat{\xi}|^2_{\bar{Q}(\tau, y_{[0,\tau]})} + \check{l}$, where $\hat{\xi}$ is the worst-case estimate for ξ; $\bar{Q}$ is the nonnegative-definite weighting function; and $\check{l}$ is part of the weighting function to be designed in the control design step, and is a constant in this step.

Assumption 6: The weighting matrix $\bar{Q}$ is given by

$$\bar{Q} = \bar{\Sigma}^{-1} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Delta \end{bmatrix} \bar{\Sigma}^{-1} + \begin{bmatrix} \epsilon\Phi'C'(\gamma^2\zeta^2 - 1)C\Phi & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

where $\bar{\Sigma}$ is the worst-case covariance matrix defined in (7d). $\Delta > 0$ is $n \times n$ dimensional; ϵ is a scalar function defined by $\epsilon(\tau) = \text{Tr}(\Sigma^{-1}(\tau))/K_c$, $\forall \tau \geq 0$; $K_c \geq \gamma^2 \text{Tr}(Q_0)$ is a constant; and the matrix Σ will play the role of worst-case covariance matrix of the parameter estimation error. ◇

Soft projection is introduced based on Assumption 4.

Define $\rho := \min_{\theta_s \in \mathbb{R}^{\sigma-1}} P(0, \theta_s)$, then, $\rho > 1$. The design will try to guarantee that the estimate $\check{\theta}$ lies in the open set $\Theta_o := \{\bar{\theta} : P(\bar{\theta}) < (1+\rho)/2\}$. then, we have that $\check{b}_0 \geq c_0 > 0$ always.

Define the smooth projection function

$$P_r(\check{\theta}) \;\; := \;\; \begin{cases} \frac{\exp((1-P(\check{\theta}))^{-1})}{(\frac{1+\rho}{2} - P(\check{\theta}))^3} \left(\frac{\partial P}{\partial \theta}(\check{\theta})\right)'; & \forall \check{\theta} \in \Theta_o \backslash \Theta \\ \mathbf{0}_{\sigma \times 1}; & \forall \check{\theta} \in \Theta \end{cases} \tag{6}$$

then, it is obvious that $P_r(\check{\theta})$ is smooth on the set Θ_o, and $(\theta - \check{\theta})' P_r(\check{\theta}) \leq 0$, $\forall \check{\theta} \in \Theta_o$. Then the identifier dynamics are summarized as follows, which is the result of [1].

$$\begin{aligned} &(A - \zeta^2 LC)\Pi + \Pi(A - \zeta^2 LC)' - \Pi C'(\zeta^2 - \gamma^{-2})C\Pi \\ &\quad + DD' - \zeta^2 LL' + \gamma^2 \Delta = 0 & \text{(7a)} \\ &\dot{\Sigma} = -(1-\varepsilon)\Sigma\Phi'C'(\gamma^2\zeta^2 - 1)C\Phi\Sigma;\; \Sigma(0) = \gamma^{-2}Q_0^{-1} & \text{(7b)} \\ &\dot{s}_\Sigma = -s_\Sigma^2(\gamma^2\zeta^2 - 1)(1-\varepsilon)C\Phi\Phi'C';\; s_\Sigma(0) = \frac{1}{\gamma^2 \text{Tr}(Q_0)} & \text{(7c)} \\ &\epsilon = K_c^{-1} s_\Sigma^{-1};\quad \bar{\Sigma} = \begin{bmatrix} \Sigma & \Sigma\Phi' \\ \Phi\Sigma & \frac{1}{\gamma^2}\Pi + \Phi\Sigma\Phi' \end{bmatrix} & \text{(7d)} \\ &A_f = A - \zeta^2 LC - \Pi C'C(\zeta^2 - \gamma^{-2}) & \text{(7e)} \\ &\dot{\Phi} = A_f\Phi + y\bar{A}_{211} + u\bar{A}_{212};\quad \Phi(0) = \mathbf{0}_{n\times\sigma} & \text{(7f)} \\ &\dot{\check{\theta}} = -\Sigma P_r(\check{\theta}) - \Sigma\Phi'C'(y_d - C\check{x}) - \begin{bmatrix} \Sigma & \Sigma\Phi' \end{bmatrix} \bar{Q}\xi_c \\ &\quad + \gamma^2\zeta^2\Sigma\Phi'C'(y - C\check{x});\; \check{\theta}(0) = \check{\theta}_0 & \text{(7g)} \\ &\dot{\check{x}} = -\Phi\Sigma P_r(\check{\theta}) + A\check{x} - (\gamma^{-2}\Pi + \Phi\Sigma\Phi')C'(y_d - C\check{x}) \\ &\quad - \begin{bmatrix} \Phi\Sigma & \gamma^{-2}\Pi + \Phi\Sigma\Phi' \end{bmatrix} \bar{Q}\xi_c + (y\bar{A}_{211} + u\bar{A}_{212})\check{\theta} \\ &\quad + \zeta^2(\Pi C' + \gamma^2\Phi\Sigma\Phi'C' + L)(y - C\check{x});\; \check{x}(0) = \check{x}_0 & \text{(7h)} \\ &W(t, \xi, \check{\xi}, \bar{\Sigma}) = |\theta - \check{\theta}|^2_{\Sigma^{-1}} + \gamma^2|x - \check{x} - \Phi(\theta - \check{\theta})|^2_{\Pi^{-1}} & \text{(7i)} \\ &\dot{W} = -|x_1 - y_d|^2 - \gamma^4|x - \hat{x} - \Phi(\theta - \hat{\theta})|^2_{\Pi^{-1}\Delta\Pi^{-1}} + |\xi_c|^2_{\bar{Q}} \\ &\quad -\epsilon(\gamma^2\zeta^2 - 1)|\theta - \hat{\theta}|^2_{\Phi'C'C\Phi} + \gamma^2|w|^2 - \gamma^2|w - w_*|^2 \\ &\quad + |C\check{x} - y_d|^2 - \gamma^2\zeta^2|y - C\check{x}|^2 + 2(\theta - \check{\theta})'P_r(\check{\theta}) & \text{(7j)} \end{aligned}$$

where $s_\Sigma(\tau) := 1/\text{Tr}(\Sigma^{-1}(\tau))$, which is introduced to avoid the computation of Σ^{-1} on line; Φ is a filtered signal of y and u; W is the value function for the identifier design step; $\xi_c = \hat{\xi} - \check{\xi}$ and $\check{\xi} = (\check{\theta}', \check{x}')'$; w_* denotes the worst-case disturbance.

Lemma 1: Consider the dynamic equation (7b). Let Assumption 6 hold and $\gamma \geq \zeta^{-1}$. Then, $K_c^{-1} I_\sigma \leq \Sigma(\tau) \leq \gamma^{-2} Q_0^{-1}$; $\gamma^2 \text{Tr}(Q_0) \leq \text{Tr}(\Sigma^{-1}(\tau)) \leq K_c$, $\forall \tau \in [0, t]$.

We make the following assumptions.

Assumption 7: If the matrix $A - \zeta^2 LC$ is Hurwitz, then the desired disturbance attenuation level $\gamma \geq \zeta^{-1}$. Otherwise, the desired disturbance attenuation level $\gamma > \zeta^{-1}$. ◇

Assumption 8: The matrix Π_0 is chosen as the unique positive-definite solution to the algebraic Riccati equation (7a). ◇

Assumption 7 makes the observations that the quantity ζ^{-1} is the ultimate lower bound on the achievable performance level for the adaptive system using the proposed design method. Assumption 8 is made to simplify the identifier structure, which implies that the matrix A_f is Hurwitz.

To simplify the estimator dynamics, we may generate Φ by $2n$-dimensional prefiltering system for y and u: $\dot{\eta} = A_f\eta + p_n y$, $\eta(0) = 0$; $\dot{\lambda} = A_f\lambda + p_n u$, $\lambda(0) = 0$, where p_n is an n-dimensional vector such that (A_f, p_n) is controllable. Let $M_f = [A_f^{n-1}p_n \quad \ldots \quad A_f p_n \quad p_n]$. Then, Φ is given by

$$\begin{aligned}\Phi &= \left[\begin{array}{cccc} A_f^{n-1}\eta & \ldots & A_f\eta & \eta \end{array}\right] M_f^{-1}\bar{A}_{211} \\ &\quad + \left[\begin{array}{cccc} A_f^{n-1}\lambda & \ldots & A_f\lambda & \lambda \end{array}\right] M_f^{-1}\bar{A}_{212}\end{aligned}$$

This completes the summary of the identification design of [1].

IV. Control Design

We will discuss the reduced-order controller design in detail. The controller design will be based on the following inequality.

$$\begin{aligned}&\sup_{(x_0,\theta,\dot{w}_{[0,\infty)})\in\dot{\mathcal{W}}} J_{\gamma t} \le \sup_{y_{[0,\infty)}} \sup_{(x_0,\theta,w_{[0,\infty)})|y_{[0,\infty)}} J_{\gamma t} \le \\ &\sup_{y_{[0,\infty)}\in\mathcal{C}} \left\{ \int_0^t (|C\check{x} - y_d|^2 + |\xi_c|^2_{\bar{Q}} - \check{l} - \gamma^2\zeta^2|y - C\check{x}|^2)d\tau \right. \\ &\qquad \left. -l_0 \right\} \qquad (8)\end{aligned}$$

Our purpose is to guarantee that the supremum is less than or equal to zero for all measurement waveforms. Instead of y, we can equivalently supremize over the transformed variable $v = \zeta(y - C\check{x})$. The problem is then a nonlinear H^∞ control problem with full information. The variables u and ξ_c will be designed at this stage. The control design for u will make use of the integrator backstepping methodology.

On the basis of the cost function (8), we only need to achieve γ level of disturbance attenuation with respect to the equivalent disturbance v. The variables to be controlled in the control design step are Σ, s_Σ, η, λ, $\check{\theta}$, and $\check{x}$. The control variables to be designed are u and ξ_c. Since there is a nonnegative definite weighting on ξ_c in (8), we cannot use integrator backstepping methodology to design feedback law for ξ_c. Then, we will set $\xi_c = 0$ in the backstepping design for u, and then optimize the choice of ξ_c based on the value function derived in the backstepping procedure at the end. Σ, s_Σ, and $\check{\theta}$ are always bounded by the identifier design, so we will treat their dynamics as part of the zero dynamics in backstepping. Denote the elements of $\check{x}$ by $[\check{x}_1, \ldots, \check{x}_n]$. Observe the lower triangular structure in the dynamics of η, $\check{x}_1$, …, $\check{x}_r$, we will use integrator backstepping methodology to stabilize them. Since, the system (1) is strictly minimum phase, we expect that $\check{x}_{r+1}$, …, $\check{x}_n$ will be bounded once y is bounded. λ is not necessarily bounded without control. But the dynamics of λ is directly affected by u, and therefore, can not be stabilized in conjunction with η, $\check{x}_1$, …, $\check{x}_r$ using backstepping. Then, we assume that λ is bounded and treat its dynamics as part of the zero dynamics in the backstepping design. Theoretically, it can be shown that λ is bounded with the derived control law.

Based on the above discussion, we start the backstepping procedure.

Step 0: The dynamics of η are given by:

$$\dot{\eta} = A_f\eta + p_n\check{x}_1 + p_n(v/\zeta)$$

According to [1], we need to generate a linear combination of η, η_L, whose transfer function from y is strictly minimum phase and has relative degree 1. Then, we need to generate the signal η_{Ld} which η_L must track. In [1], η_{Ld} is generated by an n-dimensional dynamics with state η_d. In this paper, we explore method to reduce the dimension of the dynamics required to generate η_{Ld} by choosing η_L to be some specific linear combination of η. It will be shown that if the matrix A_f has at least one real eigenvalue, then η_{Ld} can be generated via a scalar dynamic equation. Otherwise, η_{Ld} can be generated via a second-order dynamics. This way, the order of the adaptive controller is reduced by $n-1$ or $n-2$. Next, we will discuss these two cases separately.

Case 1: The matrix A_f has at least a real negative eigenvalue a_0. Let the real vector a be a left eigenvector of A_f associated with a_0. Define $\eta_R = a'\eta$, then, we obtain $\dot{\eta}_R = A_R\eta_R + \bar{p}_n y$, where$A_R = a_0$ and $\bar{p}_n = a'p_n$. By Lemma 3, we have that $a'p_n$ is non-zero. The reference trajectory for η_R to track is η_{Rd}, which can be generated by a scalar system $\dot{\eta}_{Rd} = A_R\eta_{Rd} + \bar{p}_n y_d$. Then, the signal $\eta_L = \eta_R = a'\eta$, and $\eta_{Ld} = \eta_{Rd}$.

Case 2: The matrix A_f has no real eigenvalue. Let μ_1 and $\bar{\mu}_1$ be a pair of complex conjugate eigenvalues of A_f; Let u_1 be the left eigenvector of A_f associated with μ_1, then the left eigenvector of A_f associated with $\bar{\mu}_1$ is $\bar{u}_1$, which is the complex conjugate of u_1. In this case, we define $\eta_R = [\Re(u_1) \ \Im(u_1)]'\eta$. Then, we obtain $\dot{\eta}_R = A_R\eta_R + \bar{p}_n y$, where $\bar{p}_n \neq 0$ by Lemma 3,

$$A_R = \begin{bmatrix} \Re(\mu_1) & -\Im(\mu_1) \\ \Im(\mu_1) & \Re(\mu_1) \end{bmatrix}; \qquad \bar{p}_n = \begin{bmatrix} \Re(u_1') \\ \Im(u_1') \end{bmatrix} p_n$$

Generate η_{Rd} by $\dot{\eta}_{Rd} = A_R\eta_{Rd} + \bar{p}_n y_d$. Then, the signal η_L[1] can be chosen as $\eta_L = [1 \quad 1]P^{-1}\eta_R$, where the invertible matrix P is given by $P = \left[\begin{array}{cc} A_R\bar{p}_n & \bar{p}_n \end{array}\right] \begin{bmatrix} 1 & 0 \\ -2\Re(\mu_1) & 1 \end{bmatrix}$. Clearly, η_L is minimum phase and has relative degree 1 with respect to the input y. Then, $\eta_{Ld} = [1 \quad 1]P^{-1}\eta_{Rd}$.

According to both two cases, the matrix A_R is a Hurwitz matrix, then there exist a positive definite matrix Y such that the following generalized algebraic Riccati equation admits a positive-definite solution Z, by Lemma 2 of [1].

$$A_R'Z + ZA_R' + \gamma^{-2}\zeta^{-2}Z\bar{p}_n\bar{p}_n'Z + Y = 0 \qquad (9)$$

In terms of the matrix Z, we define the value function for the dynamics of $\tilde{\eta}_R := \eta_R - \eta_{Rd}$ to be $V_0(\tilde{\eta}_R) = |\tilde{\eta}_R|^2_Z$. Then, the derivative of V_0 is given by, where $\nu_0(\tilde{\eta}_R) = \gamma^{-2}\zeta^{-1}\bar{p}_n'Z\tilde{\eta}_R$.

$$\dot{V}_0 = -|\tilde{\eta}_R|^2_Y + \gamma^2 v^2 - \gamma^2(v - \nu_0)^2 + 2\tilde{\eta}_R'Z\bar{p}_n(\check{x}_1 - y_d)$$

This shows that the dynamics of $\tilde{\eta}_R$ achieves attenuation level γ from the disturbance v to the output $Y^{1/2}\tilde{\eta}_R$, if $\check{x}_1$ is the control variable and is set to $\check{x}_1 = y_d$. However, $\check{x}_1$ is a state variable, therefore, $\check{x}_1$ is called the virtual control input, and the desired control law $\check{x}_1 = y_d$ is called the virtual control law.

This completes the virtual control design for the η_R dynamics.

The construction of steps after Step 0 will closely follow the steps in [1], and hence will not be included here. The resulting control law and value function are

$$u = \bar{\mu} = -\frac{1}{\check{b}_0}(a_{r,r+1}\check{x}_{r+1} + \ldots + a_{rn}\check{x}_n - y_d^{(r)} - \bar{\alpha}_r) \qquad (10a)$$

$$\bar{V} = |\tilde{\eta}_R|^2_Z + \sum_{j=1}^{r} \frac{1}{2} z_j^2 \qquad (10b)$$

[1]Note that the vector $[1 \quad 1]$ can be generally chosen a $[k_1 \quad 1]$ where $k_1 > 0$.

and the time derivative of $\bar{V}$ is

$$\dot{\bar{V}}=-z_1^2-|\tilde{\eta}_R|_Y^2-\sum_{j=1}^{r}\bar{\beta}_j z_j^2+\bar{\varsigma}_r'\bar{Q}\xi_c+\gamma^2 v^2-\gamma^2(v-\bar{\nu}_r)^2$$

where z_i, $i=1,\ldots,r$ are transformed variables introduced in the design, $\bar{\alpha}_r$, $\bar{\nu}_r$, $\bar{\beta}_i$, $i=1,\ldots,r$, and $\bar{\varsigma}_r$ are known functions defined in detail in [20]. This completes the backstepping design procedure.

Based on the value functions of identification design and control design, we obtain the following value function for the closed-loop adaptive nonlinear system: $U=V+W$, which satisfies

$$\begin{aligned}\dot{U}=&-|x_1-y_d|^2-\gamma^4|x-\hat{x}-\Phi(\theta-\hat{\theta})|^2_{\Pi^{-1}\Delta\Pi^{-1}}-\sum_{j=1}^{r}\bar{\beta}_j z_j^2\\&-\epsilon(\gamma^2\zeta^2-1)|\theta-\hat{\theta}|^2_{\Phi'C'C\Phi}+2(\theta-\check{\theta})'P_r(\check{\theta})-\frac{1}{4}|\bar{\varsigma}_r|^2_{\bar{Q}}\\&+|\xi_c+\frac{1}{2}\bar{\varsigma}_r|^2_{\bar{Q}}-|\tilde{\eta}_R|^2_Y+\gamma^2|w|^2-\gamma^2|w-w_{opt}|^2\end{aligned}\quad(11)$$

where the worst-case disturbance is

$$\begin{aligned}w_{opt}=&\zeta E'\bar{\nu}_r+\gamma^{-2}(I-\zeta^2E'E)\bar{D}'\bar{\Sigma}^{-1}(\xi-\check{\xi})\\&+\zeta^2E'C(\check{x}-x)\end{aligned}$$

From (11), the optimal choice for $\hat{\xi}$ is $\hat{\xi}_*=\check{\xi}-\frac{1}{2}\bar{\varsigma}_r$, The optimal choice may be too complicated for implementation, we can also choose $\hat{\xi}=\check{\xi}$ as a suboptimal choice. Both of these choices yield that the closed-loop systems are dissipative with storage function U and supply rate $-|x_1-y_d|^2+\gamma^2|w|^2$.

It is seen that the controller structure can be finally simplified by $n-1$ integrators or by $n-2$ integrators.

Till now, we have finished the design for the entire closed-loop system, which involves states, x, $\check{x}$, $\check{\theta}$, Σ, s_Σ, η, η_{Rd}, and λ. We turn to study the robustness of the closed-loop system, which is made precise in the following theorem.

Theorem 1: Consider the robust adaptive control problem formulated in Section II, with Assumptions $1-8$ holding. Then, the robust adaptive controller $\bar{\mu}$ defined by (10a), with either optimal worst-case estimate $\hat{\xi}_*$ or the suboptimal choice $\hat{\xi}=\check{\xi}$, achieves the following strong robustness properties for the closed-loop system.

1) The controller $\bar{\mu}$ achieves disturbance attenuation level γ for any uncertainty triple $(x_0,\theta,\grave{w}_{[0,\infty)})\in\grave{\mathcal{W}}$.
2) Given a $c_w>0$, there exists a constant $c_c>0$ and a compact set $\Theta_c\subset\Theta_o$, such that, $\forall(x_0,\theta,\grave{w}_{[0,\infty)})\in\grave{\mathcal{W}}$ with $|x_0|\le c_w$; $|\grave{w}(t)|\le c_w$; $\forall t\in[0,\infty)$, all closed-loop state variables are bounded as follows: $\forall t\in[0,\infty)$, $|x(t)|\le c_c$, $|\check{x}(t)|\le c_c$, $\check{\theta}(t)\in\Theta_c$, $|\eta(t)|\le c_c$, $|\eta_{Rd}(t)|\le c_c$, $|\lambda(t)|\le c_c$, $K_c^{-1}I_\sigma\le\Sigma(t)\le\gamma^{-2}Q_0^{-1}$, $K_c^{-1}\le s_\Sigma(t)\le\gamma^{-2}(\mathrm{Tr}(Q_0))^{-1}$.
3) For all $(x_0,\theta,\grave{w}_{[0,\infty)})\in\grave{\mathcal{W}}$ with $\grave{w}_{[0,\infty)}\in\mathcal{L}_2\cap\mathcal{L}_\infty$, $\lim_{t\to\infty}(x_1(t)-y_d(t))=0$.

Proof: see the full version [20]. ■

V. An Example

To illustrate the performance of the reduced-order controller designed in this paper, we present an example in this section.

We consider the same system as Example 2 of [1]. The true system is given by

$$\begin{aligned}\begin{bmatrix}\dot{\grave{x}}_1\\\dot{\grave{x}}_2\\\dot{\grave{x}}_3\end{bmatrix}&=\begin{bmatrix}1&1&0\\0&0&1\\0&\theta&0\end{bmatrix}\begin{bmatrix}\grave{x}_1\\\grave{x}_2\\\grave{x}_3\end{bmatrix}+\begin{bmatrix}1\\0\\0\end{bmatrix}u+\begin{bmatrix}0.1&0\\0&0\\0&0\end{bmatrix}\grave{w}\\y&=\begin{bmatrix}1&0&0\end{bmatrix}\grave{x}+\begin{bmatrix}0&0.1\end{bmatrix}\grave{w}\end{aligned}$$

where the initial condition for the state $\grave{x}$ and the true value of the parameter θ are set to be $\grave{x}_1(0)=1$, $\grave{x}_2(0)=1$, $\grave{x}_3(0)=2$, and $\theta^*=-4$. The design model (see [1]) is easy to obtain via the state and disturbance transformations. The reference trajectory is generated by the following differential equation: $\dot{x}_d=-x_d+d$; $y_d=x_d$, where d is the command signal. The initial state $x_d(0)$ is chosen to be 0. The parameter θ is assumed to belong to the set $[20,0]$ with the projection function $P(\theta)$ chosen as $P(\theta)=0.01(\theta+10)^2$. The initial estimates for the parameter θ and the state variables were chosen as $\check{\theta}_0=-1$ and $\check{x}_0=[1\ \ 0\ \ 0]'$. For the adaptive controller design, we choose $\gamma=0.4$, $Q_0=1$, $K_c=0.4$, and $\Delta=0.6I_3$.

Then, the calculation shows that there exists a real eigenvalue of the matrix A_f which is equal to -3.170. Therefore, we can achieve the controller structure simplification of $n-1$ integrators.

Other design parameters are chosen as $Y=1.068\times10^4$ and $\bar{\beta}_1=0.1$. The command signal $d(t)$ and the disturbance input $\grave{w}=[\grave{w}_1\ \grave{w}_2]'$ are set to be $d(t)=4\sin(t)$, $\grave{w}_1(t)=${Band-limited white noise with power 0.03, seed 6000, and sampling period 1 sec}, and $\grave{w}_2(t)=0.2\sin(3t+\frac{\pi}{2})$.

The simulation results are shown in Figure 1. We simulate the closed-loop performance under reduced-order and full-order controller. Graph (a), (c), and (e) show the system response with the reduced-order controller. Graph (b), (d), and (f) show the system response with the full-order controller. We observe that, for both cases, the tracking errors satisfy the desired attenuation level, the parameter estimates asymptotically oscillate around the true value -4 due to the sinusoidal disturbance, the control inputs are bounded by 8. We also observe that the transient response and steady-state behavior under the reduced-order controller are much better than those under the full-order controller. It shows that the reduced-order controller achieves a better closed-loop performance, which is observed in most of the simulations that we have done. Furthermore, the improvement in performance is more pronounced for higher-order systems.

VI. Conclusion

In this paper, we present the reduced-order adaptive controller design for SISO linear systems with noisy output measurements. The assumption for our study are exactly the same as that of [1], and we use the same design paradigm as well. The adaptive controller design is proceeded in two steps. The first step is estimation design which is the same as [1]. We summarized their results here for convenience of the readers. The controller design step uses integrator backstepping methodology, where the step 0 is different from [1]. Instead of generating η_d, which is n-dimensional as in [1], we generate a reduced-order signal η_{Rd}. When the matrix A_f has at least one real eigenvalue, η_{Rd} is a scalar signal, which results in an order reduction of $n-1$. On the other hand, when the matrix A_f has only complex eigenvalues, η_{Rd} is two dimensional, which results in an order reduction of $n-2$. The rest of the steps of the integrator backstepping procedure are essentially similar to that of [1]. The controller designed is

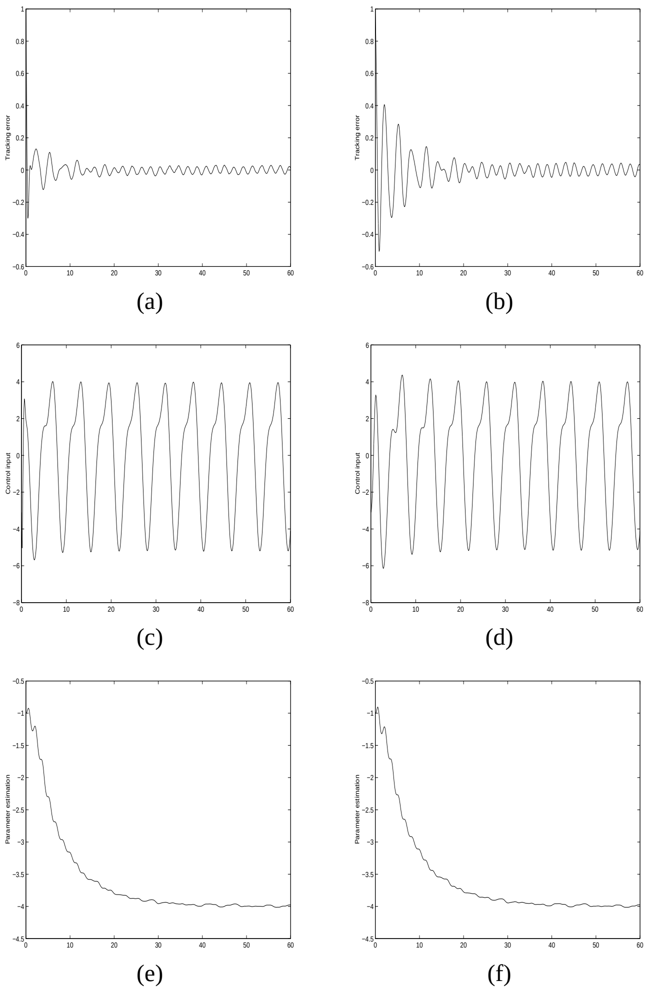

Fig. 1. Comparison of system responses under reduced-order controller and system responses under full-order controller, with command input $d(t) = 4\sin(t)$ and arbitrarily varying disturbances.
(a), (c), (e): reduced-order controller; (b), (d), (f): full-order controller. (a), (b): Tracking error; (c), (d): Control input; (e), (f): Parameter estimate.

proven to achieve strong robustness properties: achievement of the desired disturbance attenuation level with respect to continuous exogenous disturbance inputs with the ultimate lower bound for the attenuation level being ζ^{-1}; total stability with respect to exogenous disturbance input and the initial condition for the closed-loop system; asymptotic tracking of the reference trajectory when the disturbance input belong to $\mathcal{L}_2 \bigcap \mathcal{L}_\infty$. These results are the same as those of [1]. A numerical example demonstrates the performance improvement resulting from the order reduction, even though there is no theoretical justification for the improvement.

Extensions of this result may be worked out for SISO linear systems with partly measured disturbance, and for SISO linear systems with repeated noisy measurement. These extensions are straightforward, based on this result, and will not be pursued in the immediate future.

VII. Appendix

Lemma 2: Consider a $n \times n$ real matrix A, there exists a $1 \times n$ real vector C such that the pair (A, C) is observable. Then, there exists a $n \times 1$ real vector P such that the pair (A, P) is controllable.

Proof: See full version [20]. ■

Lemma 3: Consider a $n \times n$ real matrix A and $n \times 1$ real vector P, the pair (A, P) is controllable. Let the matrix A has an eigenvalue λ_1. Let a be a left eigenvector of A associated with λ_1. Then, $a'P$ is non-zero.

Proof: See full version [20]. ■

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