

Fast Reduced Order Modeling Technique for Large Scale LTV Systems

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Abstract—Linear time varying (LTV) systems are commonly applied in commercial PDE solvers. Large scale nonlinear PDE-based models are usually discretised by computational techniques that lead to LTV formulation. Proper Orthogonal Decomposition has been largely employed to reduce numerical PDE-based models, however computational saving is often far below the expected rate in spite of the dramatic reduction of the original order. In this paper, we address a practical solution to this problem by only conducting Galerkin projection onto pre-selected state variables and estimate the rest by the known POD basis vectors. The technique saves considerable computational effort needed to obtain a reduced order model and enables fast prediction of the future states, which is essential for control design.

I. INTRODUCTION

Proper Orthogonal Decomposition (POD) or also known as Karhunen-Loève Expansion is acknowledged as one of the most prominent model reduction techniques for large scale models, [6], [2],[7]. The strength of this method lies on the use of data, either from experiments or from a rigorous simulation model. The data is collected to extract the main underlying dynamics or patterns of the ensemble. The main dynamics is then incorporated into a set of optimal, orthonormal basis functions. Typically the number of basis functions that span the original data is very few compared to the original order of the data and still can capture the original dynamics within reasonable accuracy. This method is thus very favorable for data obtained from high-fidelity models, especially those originated from numerical simulation of PDE-based models, which generally suffer from high order due to the fine spatial discretisation.

A continuous PDE-based model for a function $T : \mathbb{X} \times \mathbb{T} \rightarrow \mathbb{R}$ with $D(\cdot)$ as the operator may be written as:

$$\frac{\partial T}{\partial t} = D(T) \quad (1)$$

Then a residual $R(x, t)$ is defined as:

$$R(T(x, t)) = \frac{\partial T}{\partial t} - D(T) \quad (2)$$

In proper orthogonal decomposition, it is postulated that $T(x, t)$ can be expressed as Fourier expansion[8].

$$T(x, t) = \sum_{i=1}^{\infty} a_i(t)\varphi_i(x), \quad x \in \mathbb{X}, t \in \mathbb{T} \quad (3)$$

with basis functions $\varphi_i(x)$ are orthonormal and $(\varphi_i(x), \varphi_j(x)) = \delta_{ij}$.

Truncated expansion of $T(x, t)$ to n -th order is given by:

$$T_n(x, t) = \sum_{i=1}^n a_i(t)\varphi_i(x), \quad x \in \mathbb{X}, t \in \mathbb{T} \quad (4)$$

The POD method requires that the Galerkin projection of the residual $R_n(x, t)$ on the space spanned by $\varphi_i(\cdot)$, $i = 1, \dots, n$ vanishes.

$$(R(T(x, t)), \varphi_i(x)) = 0, \quad i = 1, \dots, n \quad (5)$$

From (5), the reduced order model is derived as n -th order ODE which solves the unknown POD basis coefficients $a_i(t)$:

$$\dot{a}_i(t) = \left(D \left(\sum_{i=1}^n a_i(t)\varphi_i(x) \right), \varphi_i(x) \right) \quad (6)$$

In practice, the computational domain $\mathbb{X} \times \mathbb{T}$ is discretised as $\hat{\mathbb{X}} \times \hat{\mathbb{T}}$, with $\hat{\mathbb{X}} \subseteq \mathbb{X}$ and $\hat{\mathbb{T}} \subseteq \mathbb{T}$. The resulted dimension of $\hat{\mathbb{X}}$, K is typically high. Numerically (1) is solved as discrete K -th order model with the solution vector at every time step $\mathbb{T} \in \mathbb{R}^K$.

In the reduced order model computation (6), $n \ll K$, so dramatic computational saving is expected. This is true when we have an LTI system as $D(\cdot)$ is fixed. However, when we have nonlinear or LTV system, the Galerkin projection procedure is still expensive because $D(\cdot)$ varies and has to be updated constantly. Thus often the achieved computational saving is below our expectation, especially in the case when fast prediction of a long horizon is required to for optimal control design. It is not surprising that even though theoretically POD is applicable to both linear and nonlinear models, linearization in large-scale case is still mandatory to obtain reasonably fast prediction.

Some work has been done to improve numerical tractability of large scale model reduction for modest scale (of dimension $\approx 10^2$) as well as linear subspace approach of balanced model reduction for nonlinear systems. The main problem is, model reduction methods attempt to separate dominant dynamics from the less dominant ones. Truncation of states is in principle truncation of dynamics, not truncation of the original states. From computational point of view, if the system under investigation has a lot of different dynamics (the transformation matrix is of high rank) and/or the model is large scale and varying, there is barely significant computational gain because the effort to

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obtain a reduced order model could be almost the same as finding the original solution with some numerical iterations.

In this paper, we propose a practical solution to overcome this problem by conducting Galerkin projection on pre-selected states only instead of the whole dimension of the state variables. We show that by conducting such procedure we managed to reach the expected computational saving and eventually to control an LTV system. The LTV system itself is obtained from a discretisation of a 2D non linear heat conduction problem.

The paper is organized as follows. First, the technique used to estimate the whole states of the system based on limited information is discussed, then the discretised 2D non linear heat conduction model is presented. Subsequently, the application of the acceleration technique to the 2D non linear heat conduction model is shown and finally, the control design based on the reduced order model is elaborated. The last section covers the conclusion and outlook.

II. MISSING POINT ESTIMATION

Consider the solution of K -th order model resulted from the discretisation of (1) of a particular time step $T \in \mathbb{R}^K$. As written in (4), we can approximate T by its truncated expansion T_n .

$$T \approx T_n = \sum_{i=1}^n a_i(t) \varphi_i(x) \quad (7)$$

If a vector as T which initially resides in K -dimensional space can be well-approximated by n -th order expansion, then T is an element of an ensemble of intrinsic low dimension. It follows, that generally, only n points of information is required to reconstruct the original vector [6].

Suppose that only limited entries of T are known, and denote this incomplete copy of vector T as \tilde{T} . This can be expressed as

$$\tilde{T} = \mathbf{m} \cdot T \quad (8)$$

with $\mathbf{m} \in \mathbb{R}^K$ is a mask vector with one at the index of the available data and zero at the index of missing data. The product notation in (8) represents pointwise multiplication.

Our main objective is to find the estimates of $a_i(t)$, $i = 1, \dots, n$ based on the incomplete data \tilde{T} . The estimates of the POD basis coefficients are denoted as $\tilde{a}_i(t)$ to distinguish these coefficients to those obtained when the data is intact. If $\tilde{a}_i(t)$ is known, then the complete approximation based on limited information can be reconstructed as

$$\tilde{T}_D = \sum_{i=1}^n \tilde{a}_i(t) \varphi_i(x) \quad (9)$$

The vector \tilde{T}_D approximates \tilde{T} on the mask \mathbf{m} because it approximates both the available and the missing entries of \tilde{T} . As the non-missing entries of \tilde{T} equals to the corresponding elements on T , then \tilde{T}_D also approximates T .

This relation can be defined in an inner product on the mask \mathbf{m}

$$\|\tilde{T} - \tilde{T}_D\|_{\mathbf{m}} = \|T - \tilde{T}_D\|_{\mathbf{m}} \quad (10)$$

with $\|T\|_{\mathbf{m}}^2 = (\mathbf{m} \cdot T, \mathbf{m} \cdot T)$ indicates that $\|\cdot\|_{\mathbf{m}}^2$ is defined only on the non-missing data.

By the formulation of the missing-point inner product (10), we can find $\tilde{a}_i(t)$ by requiring that the error function

$$E = \|\tilde{T} - \tilde{T}_D\|_{\mathbf{m}}^2 \quad (11)$$

to be minimum.

Expansion of the E leads to

$$\begin{aligned} E &= \left(\tilde{T} - \sum_{i=1}^D \tilde{a}_i(t) \varphi_i(x), \tilde{T} - \sum_{j=1}^D \tilde{a}_j \varphi_j(x) \right) \\ &= \|\tilde{T}\|_{\mathbf{m}}^2 - 2 \sum_{i=1}^D \tilde{a}_i(t) \left(\tilde{T}, \varphi_i(x) \right)_{\mathbf{m}} \\ &\quad + \sum_{i,j=1}^D \tilde{a}_j(t) \tilde{a}_i(t) \left(\varphi_j(x), \varphi_i(x) \right)_{\mathbf{m}} \end{aligned} \quad (12)$$

Differentiating E with respect to the i -th coefficient gives

$$\frac{\partial E}{\partial a_i(t)} = 0 - 2 \left(\tilde{T}, \varphi_i(x) \right)_{\mathbf{m}} + 2 \sum_{j=1}^D \tilde{a}_j \left(\varphi_j(x), \varphi_i(x) \right)_{\mathbf{m}} = 0 \quad (13)$$

from which it follows the equation to solve the coefficient estimates $\tilde{a}_i(t)$

$$\sum_{j=1}^D \tilde{a}_j \left(\varphi_j(x), \varphi_i(x) \right)_{\mathbf{m}} = \left(\tilde{T}, \varphi_i(x) \right)_{\mathbf{m}} \quad (14)$$

which can be expressed as linear system

$$M \tilde{\mathbf{a}} = \mathbf{f} \quad (15)$$

where

$$M_{ij} = \left(\varphi_j(x), \varphi_i(x) \right)_{\mathbf{m}}$$

and

$$f_i = \left(\tilde{T}, \varphi_i(x) \right)_{\mathbf{m}} \quad (16)$$

Thus, with the knowledge of limited information, we can estimate the basis coefficients $\tilde{a}_i(t)$ and in this way estimate the other unknown data. This technique, Missing Point Estimation (MPE) has been used for data compression of static images. In this paper, this technique is going to be implemented for dynamic systems.

III. NONLINEAR HEAT CONDUCTION MODEL

The computational domain \mathbb{X} for the nonlinear heat conduction model of a thin plate is depicted in Figure 1. The thickness of the plate is 1cm. The continuous model for the nonlinear heat conduction model is given by:

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + S \quad (17)$$

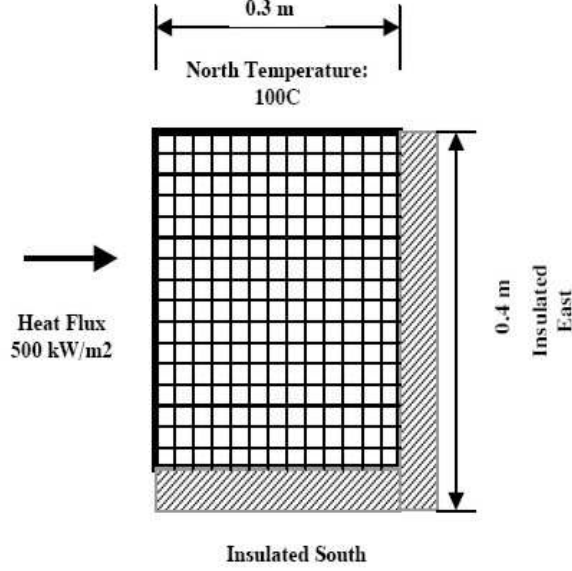


Fig. 1. Computational Domain of the Heat Conduction Model

where $T(x, t)$ is the temperature distribution along the plate, k is the conductivity constant, and S represents the external sources, namely the incoming heat flux along the west boundary and the constant temperature distribution along the north boundary. In (17), the conductivity constant is temperature dependent where refractive index $n = 1.48$, Boltzman constant $s = 5.67 \times 10^{-8} \text{W}/(\text{m}^2\text{K}^4)$, average normal absorption coefficient $a = 33.4$ and some physical constants are $\alpha = 1.059$, $B = -80$, $R = 8.12$.

$$k = \alpha e^{B/RT} + 2 \frac{16n^2 s T^3}{3a} \quad (18)$$

The model is thus originally a nonlinear PDE-based model. To simulate (17), the computational domain \mathbb{X} is discretised into 44 orthogonal grid cells in the y direction and 33 orthogonal grid cells in the x direction. The method used to discretise (17) is Computational Fluid Dynamics Finite Volume Method [9].

By Finite Volume Method, in every grid cell, (17) is integrated over a specified time horizon and a finite grid cell volume $\Delta V = \Delta x \times \Delta y \times 1\text{cm}$. The algorithm is implemented in MATLAB, but generally CFD commercial packages which use Finite Volume also implement the same code. For algorithm details please refer to [9].

$$\int_t^{t+\Delta t} \int_{\Delta V} \rho c \frac{\partial T}{\partial t} dV dt = \int_t^{t+\Delta t} \int_{\Delta V} \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) dV dt \quad (19)$$

$$+ \int_t^{t+\Delta t} \int_{\Delta V} \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) dV dt + \int_t^{t+\Delta t} \int_{\Delta V} S dV dt$$

If temperature at specific grid point and at the future time step is denoted by $T_P(k+1)$, then based on (19), we can

write the discretised equation as

$$\rho c \frac{\Delta x \Delta y}{\Delta t} T_P(k+1) = \rho c \frac{\Delta x \Delta y}{\Delta t} T_P(k+1) + \frac{k_E(k+1)AT_E(k+1) - k_P(k+1)AT_P(k+1)}{\delta_{PE}} + \frac{k_P(k+1)AT_P(k+1) - k_W(k+1)T_W(k+1)}{\delta_{PW}} + \frac{k_N(k+1)AT_N(k+1) - k_P(k+1)AT_P(k+1)}{\delta_{PN}} + \frac{k_P(k+1)AT_P(k+1) - k_S(k+1)AT_S(k+1)}{\delta_{PS}} + S(k)\Delta V$$

where E, W, N, S denote the eastern, western, northern, and southern neighboring grid cells and $\delta_{PE}, \delta_{PW}, \delta_{PN}, \delta_{PS}$ are the distances from a particular grid cell to its eastern, western, northern, and southern grid cells, respectively.

Recursive formulation of the discretised equation is shown in (20), where all the unknown temperature of the whole grid cells are collected within $\mathbf{T}(k+1) \in \mathbb{R}^{1452}$ and the coefficients are also functions of the temperature itself due to the dependency of the conductivity constant. If the time step is chosen small enough, the conductivity constant at specified grid point $k_P(k+1)$ doesn't differ too much from $k_P(k)$, and we can use conductivity constant at the current time step to evaluate the temperature distribution at the future time step.

$$\mathbf{A}(k)\mathbf{T}(k+1) = \mathbf{A}_0(k)\mathbf{T}(k) + \mathbf{S}(k) \quad (20)$$

Equation (20) is the numerical model for the nonlinear 2D heat conduction model and represents a discrete LTV system.

IV. APPLICATION OF MPE

The recursive equation (20) is the basis for the numerical model. POD-based reduced order model is derived as the projection of (20) onto a set of orthonormal basis functions, $\Phi = [\varphi_1(x), \varphi_2(x), \dots, \varphi_n]$.

$$\begin{aligned} \mathbf{A}(k)\mathbf{T}(k+1) &= \mathbf{A}_0(k)\mathbf{T}(k) + \mathbf{S}(k) \\ \Phi^T \mathbf{A}(k)\Phi \mathbf{a}(k+1) &= \Phi^T \mathbf{A}_0(k)\Phi \mathbf{a}(k) + \Phi^T \mathbf{S}(k) \\ \mathbf{a}(k+1) &= \mathbf{A}_r(k)\mathbf{a}(k) + \mathbf{S}_r(k) \end{aligned} \quad (21)$$

It is clear from (21), the reduced order model needs to be updated at every time step and in fact still a function of the whole temperature field, as $\mathbf{A}_r(k)$ and $\mathbf{S}_r(k)$ are temperature-dependent. This process is generally quite expensive.

To derive reduced order model based on limited information, first rewrite (20) for these chosen points only.

$$\begin{aligned} \tilde{\mathbf{a}}_P \tilde{\mathbf{T}}(k+1) &= \tilde{\mathbf{a}}_0 \mathbf{T}(k) + \sum_{j=W,E,S,N} \tilde{\mathbf{a}}_j \tilde{\mathbf{T}}_j(k+1) + \tilde{\mathbf{S}}(k) \\ \tilde{\mathbf{T}}(k+1) &= \frac{\tilde{\mathbf{a}}_0}{\tilde{\mathbf{a}}_P} \mathbf{T}(k) + \sum_{j=W,E,S,N} \frac{\tilde{\mathbf{a}}_j}{\tilde{\mathbf{a}}_P} \tilde{\mathbf{T}}_j(k+1) + \frac{1}{\tilde{\mathbf{a}}_P} \tilde{\mathbf{S}}(k) \end{aligned} \quad (22)$$

Referring to (15), we can plug $\tilde{T}(k+1)$ as our "limited information" into the term f .

$$M\tilde{\mathbf{a}}(k+1) = \tilde{\Phi}^T \tilde{T}(k+1) \quad (23)$$

$$\tilde{\mathbf{a}}(k+1) = M^{-1} \tilde{\Phi}^T \tilde{T}(k+1) \quad (24)$$

The explicit expression for $\tilde{\mathbf{a}}(k+1)$ involves inversion of M , but this can be calculated offline because M is only the inner product of the incomplete basis vectors. The term $\tilde{\Phi}$ refers to the set of incomplete basis vectors.

To complete the derivation, we express $\tilde{T}(k+1) = \tilde{\Phi} \tilde{\mathbf{a}}$ as in (22), and after rearranging we obtain a recursive model for $\tilde{\mathbf{a}}(k+1)$

$$\tilde{\mathbf{a}}(k+1) = \tilde{\mathbf{A}}_r(k) \tilde{\mathbf{a}}(k) + \tilde{\mathbf{S}}_r(k) \quad (25)$$

where $\tilde{\mathbf{A}}_r(k)$ and $\tilde{\mathbf{S}}_r$ are still functions of time, but updated based on limited number of data instead of the whole data at a particular time step.

Point Selection

The next natural question will be how to select points in \mathbb{R}^{1452} to have a good approximation despite dramatic information reduction? First, since the reduced order model is going to be used for control design, it is imperative to include points adjacent to control inputs to accommodate changes in manipulated variables. In this case, the control inputs are western and northern boundaries. Further, there is boundary condition applied to the insulated eastern and southern boundaries, and the information of the zero temperature gradient across these boundaries may be lost if the points adjacent to them are excluded. Hence, in the first step, we take points which have direct connection to the boundary conditions and control inputs. Figure 5 depicts grid points that belong to this category marked by red line. The next step is selection of additional points. When we consider the coefficients obtained from complete data \mathbf{a} , then a group of selected points can be described as:

$$\tilde{T} = \tilde{\Phi} \mathbf{a}$$

and we know the estimates of a based on incomplete data can be obtained as:

$$\tilde{\Phi}^T \tilde{\Phi} \tilde{\mathbf{a}} = \tilde{\Phi}^T \tilde{\Phi} \mathbf{a} \quad (26)$$

We can use (26) to require the coefficients obtained from incomplete data $\tilde{\mathbf{a}} \approx \mathbf{a}$. This is equivalent by requiring

$$\tilde{\Phi}^T \tilde{\Phi} \approx I \quad (27)$$

To distinguish more important points to less important points, we observe the contribution of basis vector components of a point to the inner product $\Phi^T \Phi$. This is conducted by removing one point and see the effect how far the new incomplete inner product $\tilde{\Phi}^T \tilde{\Phi}$ in the absence of this point to the complete inner product $\Phi^T \Phi$.

$$\|\Phi^T \Phi\| \leq \|\tilde{\Phi}^T \tilde{\Phi}\| + \|\Delta\| \quad (28)$$

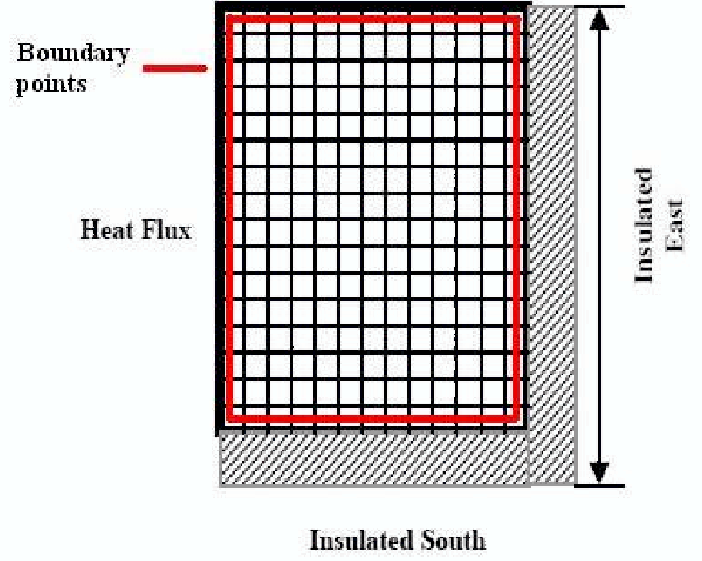


Fig. 2. Selected Boundary Points

The point with large $\|\Delta\|$ is more important than the points with small $\|\Delta\|$. Once the points are ranked, then we take the point with the largest $\|\Delta\|$ as the first candidate. The second candidate is the second largest, and we see whether $\tilde{\Phi}^T \tilde{\Phi}$ with two missing points and missing boundary points are still well defined. This procedure is continued until n_{add} points are selected.

A more formal discussion on well-posedness of sub domain has been discussed in [8] which indeed relates to the condition number of the incomplete POD basis inner product, unfortunately there has not been significant attention to the potential of implementing this concept in model reduction for dynamical systems.

In the case of heat conduction model, 300 snapshots $\mathbf{T}_{snap} = (T(1), \dots, T(300))$ are collected from an initial condition of zero temperature to be subjected to incoming west heat flux of 500kW/m^2 and north temperature of 100°C . The orthonormal POD basis vectors are obtained as singular value decomposition of the snapshots

$$\mathbf{T}_{snap} = \Phi \Sigma \Psi^T \quad \Phi \in \mathbb{R}^{1452 \times 1452}, \quad \Psi \in \mathbb{R}^{1452 \times 300}$$

There are 6 basis vectors corresponding to 6 largest singular values taken, so in the reduced order modeling, $\Phi \in \mathbb{R}^{1452 \times 6}$. To simulate the reduced order model based on limited data, 150 boundary points are taken, and $n_{add} = 50$ additional points are included from the criterion of $\|\Delta\|$. Thus there are 200 points used as basis of information to update the reduced order model (25). The plots of the most dominant basis vector (the one correspond to the largest singular value) of the complete data and the incomplete data are shown in Figure 3. At the locations where the

points are not selected, the basis vector component is zero. Simulation with this MPE-based reduced order model takes

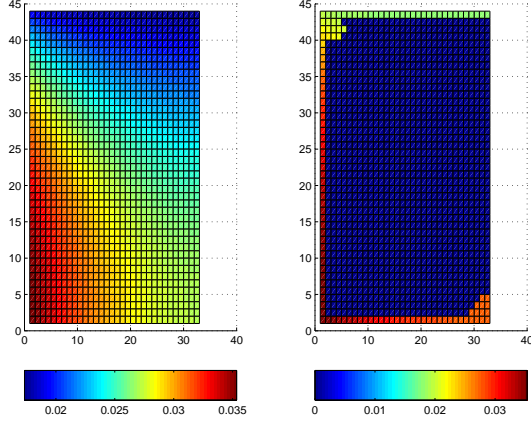


Fig. 3. Left: most dominant basis vector with complete data, right: most dominant basis vector with incomplete data

about 35 seconds, while the original simulation takes about 120 minutes. The required computational gain is at least 100 times faster than the original model to enable fast, long horizon prediction of 100 time steps ahead. Further, when the original model is reduced by classic POD procedure, the simulation takes about 10 minutes. Hence the MPE-based reduced order model is 20 times faster than the classic POD based procedure and more than 200 times faster than the original model.

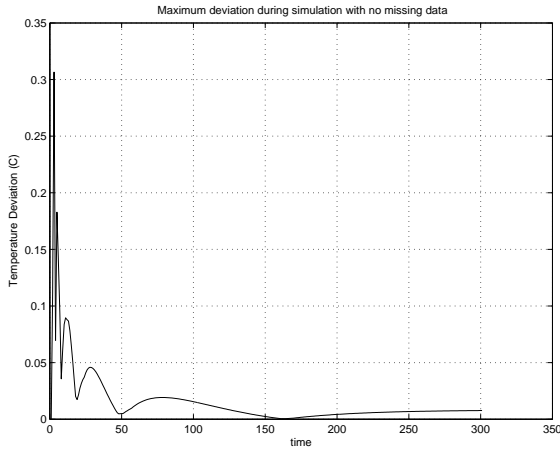


Fig. 4. Maximum deviation of every time step of reduced order model by classic POD

As shown in Figure 4 and 5, the temperature deviation when we only use limited data is quite reasonable, accounts to about 2°C, which is about 1% of the plate temperature range which lies in the 100°C to 200°C. Indeed the approximation by classic POD is more superior, but this is compensated by a lot more computing effort.

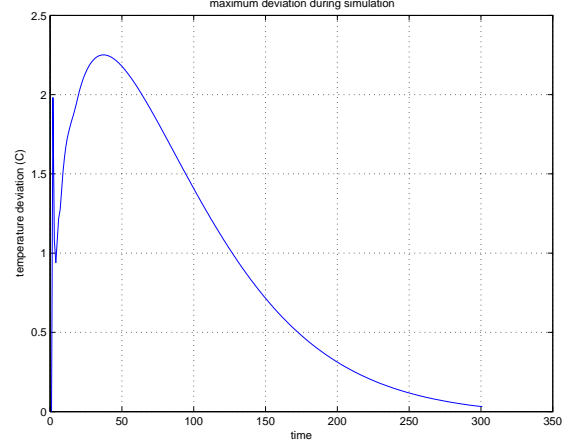


Fig. 5. Maximum deviation of every time step of reduced order model by MPE

V. CONTROLLER DESIGN

With the available fast model, a model based controller can be designed. The control objective is to achieve a desired temperature distribution optimally. Since we use reduced order model as the base of the control design, the control objective has to be translated in terms of reduced order model.

Definition 1: Given the temperature $T_{ref}(x, t)$, with $x \in \mathbb{X}$ and $t \in \mathbb{T}$, find a control input $u(t) \in \mathcal{U}$ such that the squared L2-norm of the tracking error $e(t) = r_{ref} - a(t)$ where $r(t) = \Phi^T T_{ref}$ is minimized with $u(t)$ defined as:

$$u(t) = \arg \min_{u \in \mathcal{U}} \sum_{t=0}^{T_{opt}} \|r_{ref}(t) - a(t)\|^2$$

As the controller is going to be applied to a discretised model, the time notation t in the next discussion is replaced by the time step notation $k = \frac{t}{\Delta t}$.

Consider again the MPE-based model equation (25). After decomposing the term \tilde{S} into the control input $u \in \mathbb{R}^2$ (there are west heat flux and northern temperature as the control inputs) and $B(k)$, we obtain a discrete, LTV state space equation for $\tilde{a}(k+1)$. For convenience, we refer $\tilde{a}(k+1)$ in the following discussion as the state variables $\mathbf{x}(k+1)$.

The state space equation is:

$$\begin{aligned} \mathbf{x}(k+1) &= A(k)\mathbf{x}(k) + B(k)u(k) \\ \mathbf{y}(k+1) &= C\mathbf{x}(k) \end{aligned}$$

The designed controller for this system is an LQR with a reference signal. The modified classic LQR objective functions to be minimized then reads:

$$\begin{aligned} J(x_o, u) &= \sum_{k=0}^{N_s-1} [(r(k) - \bar{x}(k))^T Q (r(k) - \bar{x}(k))] \\ &+ u^T(k) R u(k) \\ &+ x(N_e)^T E [r(N_e) - x(N_e)] \end{aligned} \quad (29)$$

The optimal control input u^* has to be found such that $J(x_0, u^*) \leq J(x_0, u) \forall u \in \mathcal{U}$. Such input, is found to be [3]

$$u^* = -F(k)x(k) - Gv(k+1) \quad (30)$$

where

$$F(k) = (R + B(k)^T P(k+1)B(k))^{-1} B^T(k)P(k+1)A(k)$$

$$G(k) = (R + B^T(k)P(k+1)B(k))^{-1} B^T(k)$$

where $P(k+1)$ is the solution of symmetric Riccati equation.

The steady state closed loop response with this LQR controller of prediction horizon equals 100 time steps can be seen in Figure 6. The deviation from the desired temperature is still reasonable, considering that the model used as the reference for the controller is based on 200 data points only. This shows the capability of MPE-based reduced model, which is computed very fast, to control a full order model with reasonable deviation.

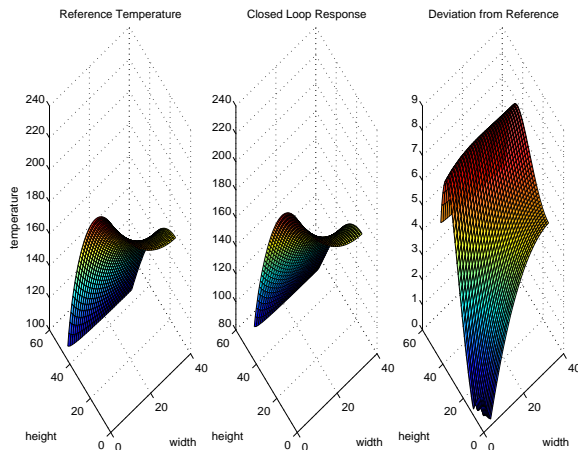


Fig. 6. Closed Loop Response

VI. CONCLUSION AND OUTLOOK

We have proposed a methodology to modify classic POD procedure which enables fast prediction of the states of the original model. With Missing Point Estimation (MPE), we are able to do fast prediction with limited available data. This is advantageous for control design, as well as other possible needs such as fault detection. This technique can also be used as a means to estimate variables which cannot be measured or missing from experiments. There are many areas still challenging to be explored, such as the suitable input choices for wide-operating range coverage, stability issues, and more advanced optimisation techniques to choose the number of pre-selected states.

VII. ACKNOWLEDGMENTS

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