Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Solutions of Optimal Feedback Control Problems with General Boundary Conditions using Hamiltonian Dynamics and Generating Functions

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***Abstract*— Given a nonlinear system and performance index to be minimized, we present a general approach to evaluating the optimal feedback control law for this system that can be easily modified to satisfy different types of boundary conditions. Formulated in the context of Hamiltonian systems theory, this work allows us to analytically construct optimal feedback control laws from generating functions. Given our feedback control law solution, our approach enables us to obtain the feedback control for a different set of boundary conditions only using a series of algebraic manipulations, partial differentiations, and solutions of implicit algebraic equations. Furthermore, the proposed approach reveals a fundamental insight: that the optimal cost function that satisfies the HJB equation can be expressed as a generating function for a class of canonical transformations of the Hamiltonian system defined by the necessary conditions for optimality. This result is formalized as a theorem, which relates the sufficient condition to the necessary conditions for optimality. The whole procedure provides an advantage over the conventional method based on dynamic programming, which requires one to solve the HJB PDE repetitively for each type of boundary condition.**

Key Words. Optimal Feedback Control, Hamilton-Jacobi-Bellman Equation, Hamiltonian System, Generating Function, Hamilton-Jacobi Equation, Legendre Transformation

I. INTRODUCTION

For a general nonlinear system with arbitrary performance criteria, optimal state feedback control laws can be derived from the solution to the Hamilton-Jacobi-Bellman (HJB) equation. The HJB equation does not have closed-form solutions in general, thus much research has been performed to find practical approaches for obtaining sub-optimal feedback controls. See Park and Scheeres [1][2] for a list of representative works on both infinite horizon regulator and finite horizon terminal controller problem.

Recently Park and Scheeres [1] and Scheeres et al [3] studied the optimal feedback control problem in the context of Hamiltonian dynamical systems. They observed that the optimal cost function is related to a generating function for a class of canonical transformations, which allowed them to devise a systematic methodology to evaluate the optimal feedback control and cost function satisfying the general boundary conditions. This approach was successfully applied to a nonlinear optimal rendezvous problem in a central gravity field.

The current work is a synthesis and extension of these results and considers a wider range of optimal feedback control problems with various types of boundary conditions. We show that our method can be applied to the optimal control of a given system with a number of different types of boundary conditions. Furthermore, by exploiting fundamental links between generating functions, we present an algorithm for evaluating optimal feedback controls for different types of boundary conditions without having to solve the HJB equation repetitively.

II. PROBLEM STATEMENT

Consider the minimization of a general performance index for an arbitrary initial point (x, t)

$$J(x,t) = \phi(x(t_f), t_f) + \int_t^{t_f} L(x(\tau), u(\tau), \tau) d\tau$$

subject to the following system with final time constraints

$$\dot{x}(t) = F(x(t), u(t), t) \quad , \quad \psi(x(t_f), t_f) = 0.$$

Here $x \in \Re^n$, $t \in \Re$, $u \in \Re^m$, and $\psi \in \Re^{p \leq n}$. We assume that there exist no constraints on state and control trajectories. Our objective is to evaluate the optimal trajectory satisfying the final time constraints and to find the optimal feedback control for an arbitrary initial point $(x, t) \in \Re^{n+1}$.

We consider two representative problem formulations, which are characterized by the types of terminal boundary conditions:[1]

- **Hard Constraint Problem (HCP)** Terminal boundary condition for states is pre-specified to a fixed point in $\Re^n$.
- **Soft Constraint Problem (SCP)** Terminal boundary condition for states is not pre-specified, but indirectly affected by the final time performance index $\phi(x, t)$.

Given the problem statement, the optimal trajectory and associated optimal control are determined by the following sufficient and necessary conditions.

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[1]This classification is just for simplicity of argument. It does not imply that the applicability of our approach is confined to these two kinds of boundary conditions. There may also exist some mixed boundary conditions, which are specified by a terminal hyper plane. Later we will briefly discuss how to manipulate such a situation.

Sufficient Conditions for Optimality

We first define the Hamiltonian H as

$$H(x, \lambda, u, t) = L(x, u, t) + \lambda^T F(x, u, t) \tag{1}$$

According to the classical derivation from dynamic programming [4][5], if

1) In the domain considered for (x, t), the Hamiltonian has a unique minimizer with respect to u such that
$$u = arg \min_{\bar{u}} H\left(x, \frac{\partial J}{\partial x}, \bar{u}, t\right)$$
2) $J(x, t)$ is sufficiently smooth (or analytic) and satisfies the Hamilton-Jacobi-Bellman (HJB) equation with the given boundary condition
$$\frac{\partial J}{\partial t}(x, t) + \min_{\bar{u}} H\left(x, \frac{\partial J}{\partial x}, \bar{u}, t\right) = 0 \tag{2}$$
$$J(x(t_f), t_f) = \phi(x(t_f), t_f) \text{ on } \psi(x(t_f), t_f) = 0$$

then J is the optimal cost and u is the corresponding optimal feedback control law.

For both of our formulations classified above, the HJB equation is the same nonlinear first order partial differential equation (PDE) for the spatial variables x and the time variable t. The mathematical expressions for the boundary conditions, however, become distinct from each other:

- HCP: $\phi(x(t_f), t_f) \equiv 0$, $\psi(x(t_f), t_f) = x(t_f) - x_f$. The pair $(x(t_f), t_f)$ is given a priori.
- SCP: $\psi(x(t_f), t_f)$ does not exist and $\phi(x(t_f), t_f) \in \Re$. The pair $(x(t_f), t_f)$, in contrast to HCP, is not given a priori but is indirectly constrained by minimizing $\phi(x(t_f), t_f)$.

Necessary Conditions for Optimality

Now re-consider the Hamiltonian (1) defined above. The standard derivation from the variational calculus and Pontryagin's principle provides the well-known 1st order necessary conditions [4]:

$$\begin{aligned} \dot{x} &= H_\lambda(x, \lambda, u, t) & (3)\\ \dot{\lambda} &= -H_x(x, \lambda, u, t) & (4)\\ u &= arg \min_{\bar{u}} H(x, \lambda, \bar{u}, t) & (5) \end{aligned}$$

where λ is the costate. Substituting (5) into (1), (3), and (4) yields

$$\begin{aligned} H(x, \lambda, t) &= L(x, t) + \lambda^T F(x, t) & (6)\\ \dot{x} &= H_\lambda(x, \lambda, t) & (7)\\ \dot{\lambda} &= -H_x(x, \lambda, t) & (8) \end{aligned}$$

which is a Hamiltonian canonical system for states and costates.

Evaluating the optimal trajectory corresponds to solving this system of ordinary differential equations (ODEs) satisfying the given boundary conditions. For HCP, the initial states x_0 and terminal states x_f are given and the initial costates λ_0 and terminal costates λ_f are to be determined. For SCP, the initial states are given whereas terminal states, initial costates, and terminal costates are to be determined. Note, however, that the well-known transversality condition applies for this case:

$$\lambda(t_f) = \frac{\partial \phi(x(t_f), t_f)}{\partial x(t_f)}, \tag{9}$$

which relates the terminal states and costates and provides an additional boundary condition for the SCP[2]. Since we need to solve this system of ODEs with split boundary conditions, the optimal control problem is again reduced to a two point boundary value problem (TPBVP).

III. Boundary Value Problems in Hamiltonian Systems

This section introduces the application of canonical transformation theory to solve boundary value problems in Hamiltonian systems. A more detailed description of this theory can be found in Guibout and Scheeres [6]. For a general review of Hamiltonian systems and canonical transformations, see Greenwood [7].

Hamiltonian Systems and Canonical Transformations

Suppose we have a system whose equations of motion can be represented by Hamilton's canonical form

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} \frac{\partial H(x(t), \lambda(t), t)}{\partial \lambda} \\ -\frac{\partial H(x(t), \lambda(t), t)}{\partial x} \end{bmatrix}$$

where

- $H = H(x(t), \lambda(t), t)$ is the Hamiltonian of the system,
- $x(t) = [x_1(t) \quad x_2(t) \quad \cdots \quad x_n(t)]^T$ is the generalized coordinate vector,
- $\lambda(t) = [\lambda_1(t) \quad \lambda_2(t) \quad \cdots \quad \lambda_n(t)]^T$ is the generalized momentum vector conjugate to $x(t)$.

In our application we restrict ourselves to canonical transformations with time as the independent parameter, i.e., solutions to the given dynamical system.

Suppose that the following transformation between (x, λ, t) and (x_0, λ_0, T) is canonical:

$$\begin{aligned} x_0(T) &= x_0(x(t), \lambda(t), t, T) & (10)\\ \lambda_0(T) &= \lambda_0(x(t), \lambda(t), t, T). & (11) \end{aligned}$$

Then there exist generating functions for these transformations that can have one of the four classical forms:

$$F_1(x, x_0, t, T), \quad F_2(x, \lambda_0, t, T)$$
$$F_3(\lambda, x_0, t, T), \quad F_4(\lambda, \lambda_0, t, T)$$

Note that each of them are a function of n old coordinates and n new coordinates. If we choose (x_0, λ_0) to be the

[2]Other than the HCP and SCP, if there exists a terminal hyper plane $\psi(x(t_f), t_f) \in \Re^{p<n}$, then we have a more general type of terminal boundary conditions, where the terminal states are partially determined and the transversality condition relates the undetermined terminal states with terminal costates [4].

initial conditions at time T, they also satisfy a boundary value problem and a partial differential equation [7]:

$$\lambda = \frac{\partial F_1(x, x_0, t, T)}{\partial x} \tag{12}$$
$$\lambda_0 = -\frac{\partial F_1(x, x_0, t, T)}{\partial x_0} \tag{13}$$
$$0 = H(x, \lambda, t) + \frac{\partial F_1(x, x_0, t, T)}{\partial t} \tag{14}$$

$$\lambda = \frac{\partial F_2(x, \lambda_0, t, T)}{\partial x} \tag{15}$$
$$x_0 = \frac{\partial F_2(x, \lambda_0, t, T)}{\partial \lambda_0} \tag{16}$$
$$0 = H(x, \lambda, t) + \frac{\partial F_2(x, \lambda_0, t, T)}{\partial t}. \tag{17}$$

$$x = -\frac{\partial F_3(\lambda, x_0, t, T)}{\partial \lambda} \tag{18}$$
$$\lambda_0 = -\frac{\partial F_3(\lambda, x_0, t, T)}{\partial x_0} \tag{19}$$
$$0 = H(x, \lambda, t) + \frac{\partial F_3(\lambda, x_0, t, T)}{\partial t} \tag{20}$$

$$x = \frac{\partial F_4(\lambda, \lambda_0, t, T)}{\partial \lambda} \tag{21}$$
$$x_0 = -\frac{\partial F_4(\lambda, \lambda_0, t, T)}{\partial \lambda_0} \tag{22}$$
$$0 = H(x, \lambda, t) + \frac{\partial F_4(\lambda, \lambda_0, t, T)}{\partial t}. \tag{23}$$

As noted, the generating functions satisfy a partial differential equation found by substituting for λ in (14) and (17), and for x in (20) and (23), which are usually referred to as the Hamilton-Jacobi (HJ) equation.

Alternatively we can also derive a similar result between a set of *fixed* final time conditions (x_f, λ_f, t_f) and the *moving* initial variable (x, λ, t) with $t \le t_f$:

$$\begin{aligned} 0 &= -\frac{\partial F_1(x_f, x, t_f, t)}{\partial t} + H\left(x, -\frac{\partial F_1(x_f, x, t_f, t)}{\partial x}, t\right) \\ \lambda &= -\frac{\partial F_1(x_f, x, t_f, t)}{\partial x} \\ \lambda_f &= \frac{\partial F_1(x_f, x, t_f, t)}{\partial x_f} \end{aligned} \tag{24}$$

A crucial property of the generating functions related to a given transformation is that they are linked to each other via Legendre transformations, which can be represented by the following identities:

$$F_2(x, \lambda_0, t, T) = F_1(x, x_0, t, T) + \lambda_0^T x_0 \tag{25}$$
$$F_3(\lambda, x_0, t, T) = F_1(x, x_0, t, T) - \lambda^T x \tag{26}$$
$$F_4(\lambda, \lambda_0, t, T) = F_2(x, \lambda_0, t, T) - \lambda^T x \tag{27}$$

Given an analytical solution to any generating function, it is then possible to evaluate the analytical form of any other generating function as long as some uniqueness conditions are satisfied [6].

Solving Boundary Value Problems with Generating Functions

The choice of the appropriate generating function depends on the type of boundary condition of the TPBVP. For the hard constraint problem (HCP), $F_1(x, x_0, t, t_0)$ is the most appropriate choice as we know the initial and terminal states. Indeed if we can find F_1, we can directly evaluate the initial and final costates from (12)- (13)[3]

$$\lambda_f = \left.\frac{\partial F_1}{\partial x}\right|_{t=t_f, x=x_f} = \frac{\partial F_1(x_f, x_0, t_f, t_0)}{\partial x_f} \tag{28}$$
$$\lambda_0 = -\left.\frac{\partial F_1}{\partial x_0}\right|_{t=t_f, x=x_f} = -\frac{\partial F_1(x_f, x_0, t_f, t_0)}{\partial x_0} \tag{29}$$

Furthermore since any time $t \le t_f$ can be the initial time, the equation (29) should hold for arbitrary initial conditions $x = x(t)$ and $\lambda = \lambda(t)$:

$$\lambda = -\frac{\partial F_1(x_f, x, t_f, t)}{\partial x}. \tag{30}$$

Substitution of (30) into (5) yields the optimal feedback control for the hard constraint problem:

$$u = arg \min_{\bar{u}} H\left(x, -\frac{\partial F_1(x_f, x, t_f, t)}{\partial x}, \bar{u}, t\right) \tag{31}$$

For the soft constraint problem (SCP), however, it is not apparent in general which generating function is the most appropriate since we have $3n$ unknown boundary conditions $(\lambda_0, x_f, \lambda_f)$. Whatever generating function we may choose, we cannot evaluate the unknown boundary conditions only by a series of partial differentiations without solving a set of implicit equations. However since we are interested in both HCP and SCP, we choose F_1 to avoid evaluating an additional generating function.

Consider the $2n$ corresponding necessary conditions (28)-(29) for F_1 along with the n transversality conditions (9). Taking the initial states x_0 as independent parameters and solving these $3n$ implicit equations for $(x_f, \lambda_0, \lambda_f)$ results in

$$x_f = x_f(x_0, t_f, t_0). \tag{32}$$
$$\lambda_0 = \lambda_0(x_0, t_f, t_0) \tag{33}$$
$$\lambda_f = \lambda_f(x_0, t_f, t_0) \tag{34}$$

Finally, a similar procedure leads to the optimal feedback control for the SCP[4]:

$$u = arg \min_{\bar{u}} H\left(x, -\left.\frac{\partial F_1(x_f, x, t_f, t)}{\partial x}\right|_{x_f(x, t_f, t)}, \bar{u}, t\right) \tag{35}$$

[3]In fact, any other generating functions can be adopted to evaluate the initial and terminal costates. For that purpose, however, using F_2, F_3, or F_4 requires one to solve a set of implicit equations as well as to take partial differentiations, whereas employing F_1 only necessitate taking partial differentiations. This observation suggests some computational advantages to using F_1.

[4]For the terminal constraint given by a hyper plane $\psi(x(t_f), t_f) = 0$ in $\Re^{p<n}$, we will have mixed terminal conditions for both states and costates in general. In this case, a more generalized kind of generating function is required, which would mix all 4 kinds of variables (initial and terminal states and costates).

As is shown, once the appropriate generating function has been found, the unknown boundary conditions are simply evaluated by a series of partial differentiations and algebraic manipulations without solving a differential equation. Furthermore, the evaluation of the initial costates λ_0 enables us to develop the optimal trajectory by simple forward integration.

Note that we need to solve the HJ PDE only once for any one kind of a generating function. The Legendre transformations (25)-(27) enable us to evaluate the rest of the generating functions simply by algebraic manipulations. This observation is at the heart of our application and provides a substantial advantage over the classical dynamic programming approach. Unlike the dynamic programming approach, we can initially choose any one kind of generating function which may be easier to solve than others. Then without solving the HJ PDE repetitively, a series of partial differentiations and solutions of a set of implicit functions provide the other generating functions, which ultimately yields the optimal feedback control for the relevant optimal control formulations. For example, if we have computed $F_2(x, \lambda_0, t, t_0)$, we can directly find $F_1(x, x_0, t, t_0)$ from (16) and (25). Indeed, we have

$$\begin{aligned} x_0 &= \frac{\partial F_2(x, \lambda_0, t, t_0)}{\partial \lambda_0} && (36) \\ F_1(x, x_0, t, t_0) &= F_2(x, \lambda_0, t, t_0) - \lambda_0^T x_0 && (37) \end{aligned}$$

Assuming the uniqueness of inversion for the initial costates λ_0 in (36), we can express λ_0 as a function of the initial and terminal states (x_0, x):

$$\lambda_0 = \lambda_0(x_0, x, t, t_0)$$

Then, introducing this to (37) yields F_1 as a function of the desired variables:

$$\begin{aligned} F_1(x, x_0, t, t_0) = F_2(x, \lambda_0(x_0, x, t, t_0), t, t_0) \\ - x_0^T \lambda_0(x_0, x, t, t_0) \end{aligned}$$

IV. The Optimal Cost Function and the Generating Function

So far we have shown that the generating functions can be used to find optimal feedback control laws. This strongly suggests that there should be connections between the optimal cost function (solution to the HJB equation) and generating functions (solutions to the HJ equation). The following theorem confirms that this is indeed the case.

Theorem 4.1 (Sufficient Condition for Optimality): For both hard constraint problem and soft constraint problem, the function

$$V(x, t) = -F_1(x_f, x, t_f, t) + \phi(x_f, t_f)$$

satisfies the HJB equation and the corresponding boundary condition (2), thus it is the optimal cost function. Furthermore, the optimal feedback control can be expressed as

$$u = arg \min_{\bar{u}} H\left(x, \frac{\partial V(x, t)}{\partial x}, \bar{u}, t\right)$$

Proof We first consider the HCP (Note $\phi(x_f, t_f) = 0$ by definition of HCP). Since $-V = F_1$ satisfies the modified HJ equation and the associated relations (24), we have

$$0 = \frac{\partial V}{\partial t}(x, t) + H\left(x, \frac{\partial V}{\partial x}, u, t\right), \qquad (38)$$

which is indeed the HJB equation. Now consider the boundary condition for V. For our dynamical system, at the terminal condition we must find the identity transformation:

$$x = x_f \quad \lambda = \lambda_f$$

Here lies the fundamental difficulty; the function F_1 cannot generate identity transformations. Instead, we consider F_2, which can generate such identity transformation [7]. For F_2 we see that at the terminal time the function $F_2 = x_f^T \lambda$ generates the

$$x = \frac{\partial F_2}{\partial \lambda} = x_f \quad , \quad \lambda_f = \frac{\partial F_2}{\partial x_f} = \lambda$$

With the aid of the Legendre transformation (25), we can solve for F_1 at the terminal condition:[5]

$$F_1(x_f, x_f, t_f, t_f) = \left[F_2(x_f, \lambda, t_f, t) - \lambda^T x\right]\Big|_{t=t_f} \equiv 0,$$

which finally yields

$$V(x_f, t_f) = -F_1(x_f, x_f, t_f, t_f) \equiv 0 \qquad (39)$$

Combining (38) and (39), we see that $V = -F_1$ satisfies the HJB equation and the associated boundary condition. The optimal control law has been obtained from (31), which completes the proof for the HCP.

Now consider the soft constraint problem. First from the HJ-equation and associated relations (24) along with the transversality conditions (9), we have

$$\lambda_f = \frac{\partial F_1}{\partial x_f} = \frac{\partial \phi}{\partial x_f} \qquad (40)$$

Also consider (32):

$$x_f = x_f(x, t_f, t)$$

Taking partial derivatives of V with respect to t and x yields,

$$\begin{aligned} \frac{\partial V}{\partial t} &= -\frac{\partial F_1}{\partial t} - \frac{\partial F_1}{\partial x_f}\left(\frac{\partial x_f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial x_f}{\partial t}\right) \\ &\quad + \frac{\partial \phi}{\partial x_f}\left(\frac{\partial x_f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial x_f}{\partial t}\right) = -\frac{\partial F_1}{\partial t} \\ \frac{\partial V}{\partial x} &= -\frac{\partial F_1}{\partial x} - \frac{\partial F_1}{\partial x_f}\frac{\partial x_f}{\partial x} + \frac{\partial \phi}{\partial x_f}\frac{\partial x_f}{\partial x} = -\frac{\partial F_1}{\partial x} \end{aligned}$$

These expressions satisfy the HJ-equation (24) for F_1, which is also equivalent to the HJB equation for V. Then similarly as in the HCP, consideration of the Legendre transformation yields

$$V(x_f(x, t_f, t), t_f) = \phi(x_f(x, t_f, t), t_f)$$

[5]We see that F_1 is singular as it loses independence of its arguments. In fact, this is an equivalent statement that the optimal cost function becomes singular at the terminal time for the hard constraint problem.

which indeed satisfies the boundary condition for the SCP. Hence $V = -F_1 + \phi$ is the optimal cost function for the SCP. The optimal control law has been determined from (35), which completes the proof for the SCP. (Q.E.D.)

Also note that for each problem F_1 satisfies the necessary conditions for optimality by definition, and hence we have both necessary and sufficient conditions for an optimal feedback control law.

These results imply that the optimal cost function is related to a special kind of generating function, and that the optimal feedback control problem can be considered as part of a more comprehensive field of canonical transformations for Hamiltonian dynamical systems.

V. Application to Nonlinear Optimal Maneuvers in a Central Gravity Field

In order to solve optimal feedback control problems for different types of boundary conditions, we need to solve the HJ PDE at least and at most once for one kind of generating function. Recently, Guibout and Scheeres [6] have devised a solution algorithm for a class of problems where:

1) the system $\dot{x} = F(x(t), u(t), t)$ is analytic and has a zero equilibrium, i.e., $F(x = 0, u = 0, t) = 0$
2) the integrand of the cost function $L(x(t), u(t), t)$ is analytic

We apply this specific algorithm to a nonlinear optimal control problem. Consider minimizing

$$J = \frac{1}{2}\int_{t_0}^{t_f} u^T(t)u(t)dt$$

subject to the system dynamics

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ 2x_4 - (1 + x_1)\left(\frac{1}{r^3} - 1\right) + u_1 \\ -2x_3 - x_2\left(\frac{1}{r^3} - 1\right) + u_2 \end{bmatrix}$$

where $r = \sqrt{(x_1 + 1)^2 + x_2^2}$. This system represents the planar motion of a particle in a central gravity field, expressed in a rotating coordinate frame. The origin of this frame corresponds to a circular orbit, the coordinates (x_1, x_2, x_3, x_4) represent radial displacement, tangential displacement, radial velocity, and tangential velocity deviations from the circular orbit, and (u_1, u_2) represent the radial and tangential control input, respectively.

We first expand the given system as a polynomial series about the zero equilibrium point. Using the necessary condition for optimality, we derive the Hamiltonian as a function of states and costates. As is suggested in [6], it is then introduced into the HJ equation for F_2. Finally with the aid of Legendre transformation, we can obtain F_1 to evaluate the optimal feedback control law. Refer to Scheeres et al [3] for more detailed review and derivation of this problem.

In our implementation of the method we expand H, F_1, and F_2 to the third order using Matlab. Figures 1–3 show the state and control trajectories of the HCP for an arbitrarily chosen boundary condition and time interval chosen as

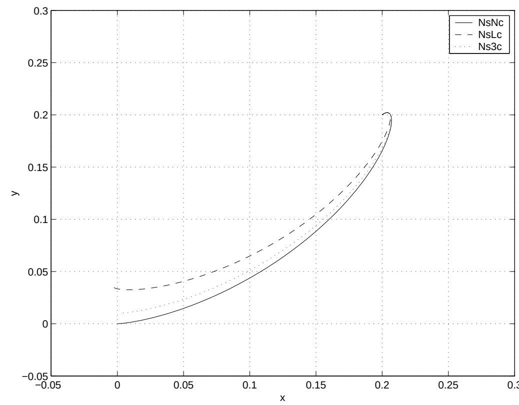

Fig. 1. Radial Position x_1 vs. Tangential Position x_2 (HCP)

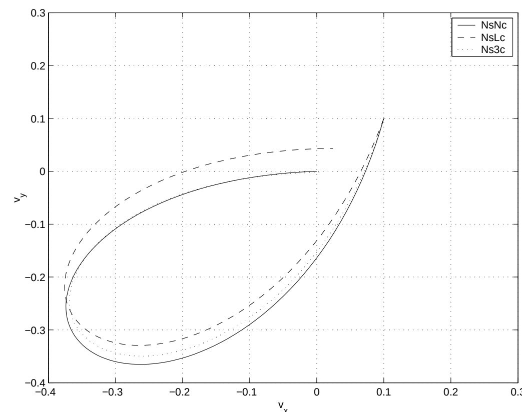

Fig. 2. Radial Velocity x_3 vs. Tangential Velocity x_4 (HCP)

$$\begin{aligned} x(t_0 = 0) &= [0.2\ \ 0.2\ \ 0.1\ \ 0.1]^T \\ x(t_f = 1) &= [0\ \ 0\ \ 0\ \ 0]^T \end{aligned}$$

For the control histories, the solid line, dashed line, and dotted line indicate the solution of the nonlinear TPBVP using a shooting method (which is our reference "true" solution), a linear systems solution, and the 3rd order analytical solution described here, respectively. For the state trajectories, each line represents the application of each control history to the original nonlinear system. It is clear that the 3rd order control is a better approximation than the linear control and is close to the true solution. By introducing additional higher order terms in the system dynamics, we can approximate the original system to as high a degree as desired.

Finally Figures 4–6 show the state and control trajectories of the SCP for the same numerical data except that x_f is not given in advance and that the terminal performance weight $Q_f = diag(10, 0, 10, 10)$. The difference between trajectories by application of the linear controller and the 3rd order controller is even more apparent.

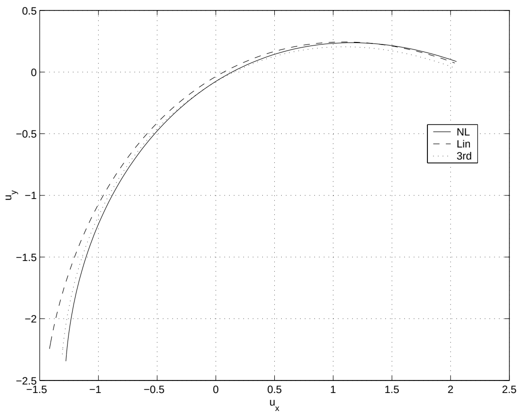

Fig. 3. Radial Control u_1 vs. Tangential Control u_2 (HCP)

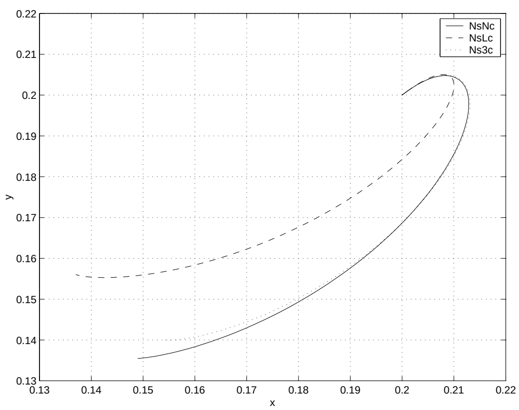

Fig. 4. Radial Position x_1 vs. Tangential Position x_2 (SCP)

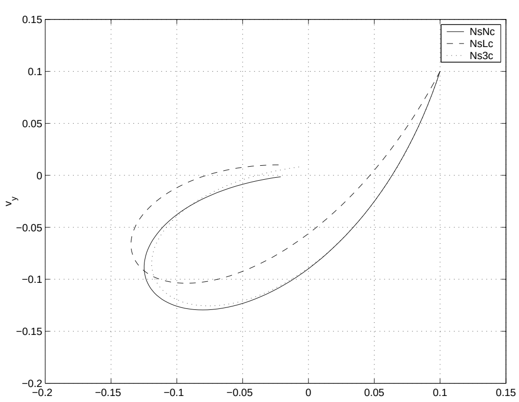

Fig. 5. Radial Velocity x_3 vs. Tangential Velocity x_4 (SCP)

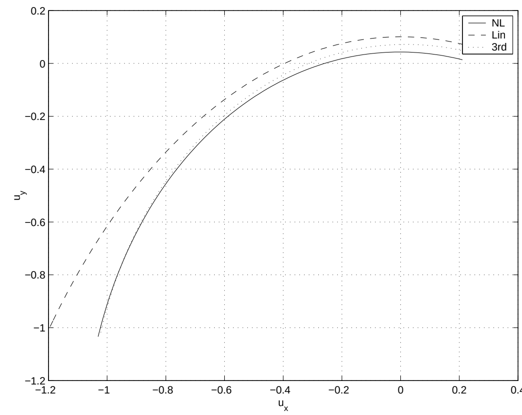

Fig. 6. Radial Control u_1 vs. Tangential Control u_2 (SCP)

VI. Conclusion

We have introduced a new approach for finding the optimal feedback control for different types of boundary conditions. Then we have applied our approach to two extreme cases of boundary conditions (hard constraint problem and soft constraint problem). It has been formally proven that our method satisfies the sufficient condition. Furthermore it relates the HJB sufficient condition to a solution of necessary conditions for optimality through generating functions. Our method is advantageous over the classical dynamical programming approach in that we do not need to solve the HJ-type PDE repetitively for different types of boundary conditions. All this implies that the optimal feedback control problem can be included within the more fundamental problem of canonical transformations for Hamiltonian systems.

Our future research will be directed toward the application of this approach to more general types of boundary conditions and a more systematic and efficient numerical implementation of our approach. We will also explore the group properties of Hamiltonian systems to search for additional advantages of our approach.

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