Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Minimum-time sliding mode control for second-order systems

Boyko Iliev
Department of Technology
Örebro University
SE-701 82, Örebro, Sweden
boyko.iliev@tech.oru.se

Ivan Kalaykov
Department of Technology
Örebro University
SE-701 82, Örebro, Sweden
ivan.kalaykov@tech.oru.se

***Abstract*— Our approach for near time-optimal control is based on Takagi-Sugeno fuzzy model of the maximum slope SMC sliding surface as an adaptive technique for tuning the current slope of the sliding surface to the maximum feasible slope depending on the current state of system. The stability conditions of this method are proved and respective measures about the feasible maximum slope are presented. Experimental results demonstrate the system behaviour.**

I. INTRODUCTION

Sliding mode control (SMC) has proved to be a successful method for control of second-order nonlinear systems in the presence of uncertainty. In theory, while the system is in sliding mode it is completely insensitive to *matched* uncertainties and its dynamics is completely determined by the sliding surface. Therefore, it has been widely used for control of the motion of various mechanical systems, motor drives and robots (see [11], [8], [5]). However, this perfect performance comes at a price. As known from the literature [3], systems in sliding mode suffer from the so called *chattering effect*. Number of methods are proposed to deal with the problem - *boundary layer* [11], fuzzy SMC [9] and approximations of the relay function.

The control of mechanical systems often requires not only robustness with respect to disturbances and uncertainty, but also fast transient response. For that reason, many researchers have contributed with methods for improvement of the transient performance of SMC. The SMC design includes two stages: choice of a sliding surface and design of an appropriate control law. The most common sliding surface is the *hyperplane*. It's main advantage is that the original (possibly nonlinear) plant behaves as a linear system while in sliding mode. Consequently, the control design and analysis becomes much easier. This choice also has some drawbacks. For highly nonlinear plants the design procedure based on Lyapunov's direct method tends to produce rather conservative results, which results in poor transient performance. One way to deal with this problem is to find more accurate estimates of the upper bounds of uncertainties and nonlinearities in system's dynamics. In this way the stability conditions can be relaxed. Such approaches are reported in [12] and [10].

Alternatively, the form of the sliding surface can be nonlinear or adaptive. Newman [8] shows that for a class of mechanical systems, sliding mode can be achieved along the time-optimal switching curve of a second order-linear system provided that the control gain is sufficiently high. Bartoszewicz [2] and Furuta *et al* [4] propose time-varying sliding hyperplanes for uncertain second-order systems. Ha *et al* [5] introduces fuzzy moving SMC to improve the tracking performance of robot manipulators.

In Section 2 we present briefly the concept of maximum slope sliding lines (see [6]) applied to the double integrator, which is used as the most simple second order system that we consider as a reference system. As the time-optimal control for it is developed in well known explicit form, any other control algorithm aimed to obtain near TOC behavior is named minimum-time control. The conditions for global stability of the minimum-time SMC for the double integrator are proved, from which the maximum slope sliding line can be obtained. Further in Section 3 we expand the approach used for the double integrator for a general type of smooth nonlinear system of second order. The conditions for getting the sliding line slope that keeps the system stability are proved. Two basic control algorithms are presented and discussed. The first is based on a global maximum slope sliding line obtained by using the maximum bounds of the system state, nonlinearity and disturbance. Takagi-Sugeno fuzzy model of the sliding line established on a family of maximum slope sliding lines corresponding to a set of operating regimes is the core of the second algorithm. Conditions for global stability for this model are proved. Section 4 represents experimental tests of the proposed methodology on real one-link robot arm. The results show clearly the advantage of the second control algorithm.

II. SLIDING MODE CONTROL OF SECOND ORDER LINEAR SYSTEMS

In this section we briefly outline the most common design cases of sliding mode controllers. We take as a basic example the control of the simplest second order system - the double integrator.

A. Maximum slope sliding lines

The dynamical equations of double integrator are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u(t) \end{aligned} \quad (1)$$

where $|u(t)| \leq u_{max}$, $u_{max} = 1$. The SMC law is

$$u(t) = -K\ sgn(s), \ \ K > 0 \quad (2)$$

where the sliding surface is linear

$$s(\mathbf{x}, t) = \lambda x_1(t) + x_2(t). \tag{3}$$

Since the dynamics of the system in sliding mode is fully determined by the equation $s = 0$, it is obvious that higher value of λ implies faster response of the system. However, the choice of λ is not arbitrary. For global asympthotic stability the Lyapunov function $V(s) = \frac{1}{2}s^2$ has to have minimum under the condition

$$\dot{V}(s) < 0$$

Further we get

$$\begin{aligned} \dot{V}(s) = s(\lambda x_2 + \dot{x_2}) < 0 \\ s(\lambda x_2 - Ksgn(s)) < \\ |s|(\lambda x_2 sgn(s) - K) < 0 \end{aligned}$$

which means that the system will be asymptotically stable if

$$0 < \lambda < \frac{K}{|x_2|} \tag{4}$$

Then the highest feasible slope λ_{max} of the sliding surface within the entire range of x_2 for $K = u_{max}$ is

$$\lambda_{max} = \frac{u_{max}}{x_{2max}} \tag{5}$$

This means also that the entire control amplitude is used to drive the system to the origin. The switching line, which has the highest slope λ that does not violate the reaching condition, is called **maximum slope sliding line** [6]. In this case

$$s_{ms}(x, t) = \lambda_{max} x_1(t) + x_2(t). \tag{6}$$

However, the estimate given by (5) is rather conservative, because the highest value of x_2 is assumed. Therefore, for states x_2 much smaller than x_{2max} the value of λ_{max} is underestimated. Contrarily, we modify the expression of the sliding surface such that *the slope of the surface increases when x_2 decreases and vice versa*. This new sliding surface is a function of the system state and the time

$$s(\mathbf{x}, t) = \frac{\lambda}{|x_2|} x_1(t) + x_2(t) = 0. \tag{7}$$

Using a SMC law (2) with $K = u_{max}$, we expect that the new sliding surface will give faster response preserving system's global asymptotical stability. The question that arises naturally is if this is the true maximum slope or we can make further improvements? To answer, we have to compare with the time-optimal control (TOC) behavior of the double integrator. As TOC is the best control, any other control will give worse performance. Nevertheless, we will focus on SMC solutions giving as close as possible to the time-optimal peformance which we call *minimum-time SMC.*

B. Minimum-time SMC of the double integrator

The time-optimal control for the double integrator (see for example [1]) is

$$u_{opt}(t) = -sgn[s_{to}(\mathbf{x}, t)], \qquad |u(t)| \leq 1.$$

The respective switching line is

$$s_{to} = x_1(t) + \frac{1}{2}|x_2(t)|x_2(t). \tag{8}$$

Now we rewrite (7) in the form

$$s_1(\mathbf{x}, t) = x_1(t) + \frac{1}{\lambda}|x_2(t)|x_2(t). \tag{9}$$

Here we use index ”1” to distinguish between $s = 0$ from (3), and $s_1(\mathbf{x}, t)$ in (9). Comparing (9) with (8), the similarities and the differences become obvious. Equation (8) represents a nonlinear curve, which is fixed in system's phase domain, while (9) is a straight line through the origin with variable slope depending on $|x_2|$. However, the behavior of the system driven by (9) is close to the time-optimal. When $|x_2|$ is small, the slope of (9) can be large, but becomes smaller if $|x_2|$ is large. Therefore, our next step is to investigate the influence of the parameter λ on the transient time and the possibilities to adjust it in such way that the SMC performance will get as close as possible to the TOC.

Theorem 1: Consider the double integrator system (2) with the control law

$$u(t) = -Ksgn(s_1(\mathbf{x}, t)), \quad K > 0,$$

where the switching line is given by (9). Assume that $u(t) \leq u_{max}$ and $K = u_{max} = 1$.
The closed-loop systems is asymptoticaly stable if λ is

$$\lambda < 2. \tag{10}$$

Proof. Consider the Lyapunov function

$$V(s_1) = \frac{1}{2} = s_1^2(\mathbf{x}, t)$$

The condition $\dot{V}(s_1) < 0$ is developed as follows

$$\begin{aligned} \dot{V}(s_1) = s_1\dot{s}_1 = \\ = s_1(\lambda\dot{x}_1 + \dot{x}_2|x_2| + x_2 sgn(x_2)\dot{x}_2) = \\ = s_1(\lambda x_2 + 2|x_2|\dot{x}_2). \end{aligned} \tag{11}$$

Subsituting $\dot{x}_2(t) = u(t)$ from (2) in (11) yields

$$\begin{aligned} \dot{V}(s_1) &= s_1(\lambda x_2 + 2|x_2|\dot{x}_2) \\ &= s_1(\lambda x_2 - 2K|x_2|sgn(s_1)) \\ &= |s_1|(\lambda x_2 sgn(s_1) - 2K|x_2|) \\ &= |s_1|(\lambda|x_2|sgn(x_2)sgn(s_1) - 2K|x_2|) \\ &= |s_1||x_2|(\lambda sgn(x_2)sgn(s_1) - 2K). \end{aligned} \tag{12}$$

To show that $\dot{V}(s_1) < 0$ for all $x \neq 0$ it is sufficient that

$$\lambda sgn(x_2)sgn(s_1) - 2K < 0.$$

This inequality holds since $K = 1$, that is

$$\lambda < 2$$

□

This result shows that if we increase the value of λ towards its limit $\lambda \to 2$, we achieve a faster transient response without violation of the stability condition. When $\lambda = 2$, the SMC and TOC control laws become identical. If $\lambda > 2$, the reaching condition no longer holds, which results in an overshoot in the transient response.

III. Minimum-time control of second order nonlinear systems

In this section we expand the method to a more general class of systems showing its limitations and possible remedies. Consider the following second-order nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(\mathbf{x},t) + u(t) + \eta(\mathbf{x},t) \end{aligned} \quad (13)$$

where $\eta(\mathbf{x},t)$ represents uncertainty and external disturbances, f is a smooth, (possibly) nonlinear function of the state and the time. We assume that $|u| \le u_{max}$, $|x_1| \le x_{1max}$, $|x_2| \le x_{2max}$ and $|\eta| \le \eta_{max}$. The sliding mode control law and surface are

$$u(t) = -K\, sgn(s), \qquad s(\mathbf{x},t) = \lambda x_1 + x_2 = 0. \quad (14)$$

Similarly, we find the maximum slope of this surface that minimizes the Lyapunov function for the system. In other words, the system motion in sliding mode should be as fast as possible without loss of stability. To prove that, we take again the Lyapunov function

$$V(s) = \frac{1}{2}s^2(\mathbf{x},t)$$

Its derivative is

$$\begin{aligned} \dot{V}(s) &= s\,\dot{s} = s(\lambda\dot{x}_1 + \dot{x}_2) \\ &= s(\lambda x_2(t) + f(\mathbf{x},t) + \eta(\mathbf{x},t) + u(t)) \quad (15) \\ &= s(\lambda x_2(t) + f(\mathbf{x},t) + \eta(\mathbf{x},t) - K\, sgn(s)) \\ &= |s|\{sgn(s)[\lambda x_2 + f(\mathbf{x},t) + \eta(\mathbf{x},t)] - K\}. \end{aligned}$$

To show that $\dot{V} < 0$, it is sufficient that

$$sgn(s)[\lambda x_2 + f(\mathbf{x},t) + \eta(\mathbf{x},t)] < K.$$

Then the slope λ can be determined by

$$\lambda < \frac{K - f(\mathbf{x},t) - \eta(\mathbf{x},t)}{|x_2|}. \quad (16)$$

Comparing (16) with (4), it is clear that the slope (16) cannot be higher than (4) as its value depends on $f(\mathbf{x},t)$ and $\eta(\mathbf{x},t)$. Therefore, the maximum slope given by (16) is always smaller than (4).

Since the solution of the time-optimal control problem for the general nonlinear system (13) is not available in closed form, it is not possible to make a comparison with the current SMC law.

Let us for a while set $f(\mathbf{x},t) = 0$ in (13). The system (13) becomes a double integrator, subject to a disturbance. The respective maximum slope for $K = u_{max}$ is given by

$$\lambda_{max} < \frac{u_{max} - \eta_{max}}{x_{2max}}. \quad (17)$$

This means a part of the available control resource u_{max} is reduced to compensate the disturbance. Therefore, the maximum slope sliding line provides worse performance if the slope (17) is used for the entire range of x_2.

In the genarl case of (13) the maximum feasible slope is

$$\lambda_{max} < \frac{u_{max} - max\{f(\mathbf{x},t)\} - \eta_{max}}{x_{2max}}, \quad (18)$$

which is even worse than (17).

Having this in mind, we suggest two approaches to find minimum-time SMC control laws for the nonlinear system (13). The first one is a direct extention of the approach in Section II-B used for the double integrator. The second is to apply a method for constructing the sliding surface by means of a fuzzy model, as introduced in [6].

A. Fixed sliding lines

The design of the sliding surface is based on the property of the systems in sliding mode to reject matched disturbances. Certain (nonlinear) terms in the system dynamics can also be *treated as a bounded disturbance*. Hence, we simply add the upper bounds of these terms to $\eta(t)$ and calculate the respective slope of the sliding surface.

Generally, this approach is conservative and might not give a result if the control resources are not sufficient. However, for plants with linear overall dynamics it can be a very good option. Below we describe two typical cases:

(1) Linear sliding surface $s = 0$ together with the control law given by (14) with the maximum slope (18). Such choice is conservative, but can be relaxed exploiting the properties of $f(\mathbf{x},t)$. See for example [6] and [7].

(2) Sliding surface similar to the time-optimal switching line given by (9)

$$s(\mathbf{x},t) = \alpha x_1 + x_2|x_2| = 0. \quad (19)$$

To obtain α_{max}, we search for the highest slope α that preserves the stability of the system using a Lyapunov function. The respective sufficient condition is

$$\alpha_{max} < 2(K - max\{f(\mathbf{x},t)\} - \eta_{max}). \quad (20)$$

The stability proof of this control law is very similar to Theorem 1, but it will not be presented here due to the lack of space. It is difficult to say which of the two cases gives better performance because it depends on the particular system dynamics. Therefore, the best option is to evaluate both designs in order to choose the right one. If both solutions are too conservative, the method given in the next section can relax the stability condition at the expense of more complex design procedure and a higher computational load.

B. Takagi-Sugeno fuzzy model of the sliding surface

The conservatism of the methods with fixed sliding surface results from the fact that the slopes in (18) and (20) are calculated for the *worst-case values* of the nonlinear function $f(\mathbf{x},t)$. The main idea of the *fuzzy minimum-time sliding mode control*, introduced in [6], is to overcome this drawback by means of constructing the sliding surface as a convex combination of maximum slope sliding lines obtained for different operating points. In this way, the slope of the line can be increased in points where the respective value is higher than the worst case value given in (18).

Assume that the state variables vary in certain ranges:

$$x_{1l} \leq x_1 \leq x_{1h}$$
$$x_{2l} \leq x_2 \leq x_{2h}$$

We choose a set of values of x_1^i and x_2^j to cover the whole operating range

$$x_1^i = \{x_1^1,\ x_1^2,\ x_1^3\ \ldots\ x_1^r\},\quad i = 1\ldots r$$
$$x_2^i = \{x_2^1,\ x_2^2,\ x_2^3\ \ldots\ x_2^r\},\quad i = 1\ldots r$$

and for all these values, we obtain the maximum feasible slopes

$$\lambda_{max} = \{\lambda_{max}^1,\ \lambda_{max}^2,\ \lambda_{max}^3,\ \ldots,\ \lambda_{max}^r\}. \tag{21}$$

Then we define a number of fuzzy sets over the state variables x_1^i and x_2^i and denote them by LX_1^i and LX_2^i correspondingly. If the plant is operating in a certain range $\mathbf{x}^j \in [\mathbf{x}^j - \Delta_j,\quad \mathbf{x}^j + \Delta_j]$ the switching line with a maximum feasible slope λ_{max}^i in this range is

$$s^i(t) = \lambda_{max}^i x_1(t) + x_1(t) \tag{22}$$

In this way, we build Takagi-Sugeno model

$$\begin{array}{l}\text{Rule } i:\ \text{IF } x_1 \text{ IS } LX_1^i \text{ AND } x_2 \text{ IS } LX_2^i \quad \text{THEN}\\ \quad s^i(t) = \lambda_{max}^i x_1(t) + x_2(t) \qquad i = 1\ldots r\end{array} \tag{23}$$

A combination of all active rules leads to the final output of the fuzzy model

$$s_f(\mathbf{x},t) = \frac{\sum_{i=1}^r w^i(\mathbf{x},t)\cdot s^i(t)}{\sum_{i=1}^r w^i(\mathbf{x},t)} \tag{24}$$

$w^i(\mathbf{x},t)$ is the i:th rule strength, where $\sum_{i=1}^r w^i(\mathbf{x},t) > 0$ and $w^i(\mathbf{x},t) \geq 0$ for $i = 1,\ldots r$. Sliding line (24) can be rewritten in the form:

$$s_f(\mathbf{x},t) = \sum_{i=1}^r \tilde{w}^i(\mathbf{x},t)\lambda_{max}^i\, x_1(t) + x_2(t), \tag{25}$$

where $\tilde{w}^i(\mathbf{x},t) = \frac{w^i(\mathbf{x},t)}{\sum_{i=1}^r w^i(\mathbf{x},t)}$, $w^i(\mathbf{x},t) = \mu_1^i(x_1(t))\cdot\mu_2^i(x_2(t))$, and $\mu_j^i(x_j)$ is the degree of membership of the $x_j(t)$ to the fuzzy set TX_j^i for the i^{th} rule.

The control law is

$$u(t) = -K_{max} sgn(s_f(\mathbf{x},t)). \tag{26}$$

To justify the stability of the system we state the following theorem.

Theorem 2: Consider the dynamical system (13) with the control law (26). Let the sliding surface be given by the Takagi-Sugeno fuzzy system

$$\begin{array}{l}\text{Rule } i:\ \text{IF } x_1 \text{ IS } LX_1^i \quad \text{AND } x_2 \text{ IS } LX_2^i \quad \text{THEN}\\ \qquad s_1^i(\mathbf{x},t) = \lambda_{max}^i x_1(t) + \dot{x}_1(t),\quad i = 1\ldots r\end{array} \tag{27}$$

where $s^i(\mathbf{x},t) = 0$ is a maximum slope sliding line for $x_1 = x_1^i$ and $x_2 = x_2^i$.
The sliding mode on the surface given by $s_f(\mathbf{x},t) = 0$ is asymptotically stable.
Proof. Consider the following Lyapunov function: $V(e) = \frac{1}{2}s_f^2(\mathbf{x},t)$. It is sufficient to show that $\dot{V} = s_f\dot{s}_f < 0$. From (25) we obtain

$$\begin{aligned}\dot{s}_f(\mathbf{x},t) = \sum_{i=1}^r \tilde{w}^i(\mathbf{x})\cdot\lambda_{max}^i\dot{x}_1(t) + \ddot{x}_1(t) +\\ + \sum_{i=1}^r \dot{\tilde{w}}^i(\mathbf{x})\cdot\lambda_{max}^i x_1(t)\end{aligned} \tag{28}$$

Furthermore we assume

$$\left|\sum_{i=1}^r \dot{\tilde{w}}^i(x_1,x_2)\cdot\lambda_{max}^i x_1(t)\right| < M_{1max} \tag{29}$$

Substituting (13) in (28) and further into $\dot{V}$ yields

$$\begin{aligned}\dot{V} &= s_f\dot{s}_f\\ &= s_f[\sum_{i=1}^r \tilde{w}^i\lambda_{max}^i\dot{x}_1(t) + \ddot{x}_1(t) + \sum_{i=1}^r \dot{\tilde{w}}^i\lambda_{max}^i x_1(t)] =\\ &= s_f[(\sum_{i=1}^r \tilde{w}^i\lambda_{max}^i\dot{x}_1(t) + f(\mathbf{x},t) + \eta_1 - K_{max}sgn(s_f)\\ &\quad + \sum_{i=1}^r \dot{\tilde{w}}^i\lambda_{max}^i x_1(t)]\\ &= |s_f|\,[\sum_{i=1}^r \tilde{w}^i\lambda_{max}^i\dot{x}_1(t) + f(\mathbf{x},t) + \eta_1\\ &\quad + \sum_{i=1}^r \dot{\tilde{w}}^i\lambda_{max}^i x_1(t)]sgn(s) - K_{max}sgn(s_f)]\end{aligned}$$

The expression for $\dot{V}$ is negative if

$$\begin{aligned}K_{max} > |\sum_{i=1}^r \tilde{w}^i\lambda_{max}^i\dot{x}_1(t) + f(\mathbf{x},t) +\\ + \eta_1(\mathbf{x},t) + \sum_{i=1}^r \dot{\tilde{w}}^i\lambda_{max}^i x_1(t)|.\end{aligned} \tag{30}$$

As $s^i(\mathbf{x},t)$ represents the maximum slope sliding line for the point $x_1 = x_1^i$ and $x_2 = x_2^i$, from (18) it follows that

$$K_{max} > \lambda^i_{max} x_{2max} + max\{f(\mathbf{x},t)\} + \eta_{1max} + M_{1max}$$

Then inequality (30) remains valid, since

$$K_{max} > \lambda^i_{max} x_{2max} + max\{f(\mathbf{x},t)\} + \eta_{1max} + M_{1max} \geq \left| \sum_{i=1}^{r} \tilde{w}^i \lambda^i_{max} \dot{x}_1 + max\{f(\mathbf{x},t)\} + \eta_1(\mathbf{x},t) + \sum_{i=1}^{r} \dot{\tilde{w}}^i \lambda^i_{max} x_1 \right|$$

Obviously, $\sum_{i=1}^{r} \tilde{w}^i \lambda^i_{max} \leq \lambda_{max}$ is true as $\lambda^i_{max} \leq \lambda_{max}$ and $\tilde{w}^i \leq 1$.

This concludes the proof.

□

IV. Real experiments - one degree-of-freedom robot arm

In this section we demonstrate the design of a fuzzy minimum-time sliding mode of a one degree-of-freedom robot arm. The robot link is driven by a Maxon DC servomotor with a gearbox. The motor is controlled via voltage amplifier. The control system is implemented on a dSpace 1103 6-axis control card, while the control design is performed using MATLAB and Real-Time Workshop.
The equation of motion is:

$$J\,\ddot{\theta} + F\,\dot{\theta} + g\,sin\,\theta = B\,u(t) \tag{31}$$

the parameters are: J - load and rotor inertia; F - viscous friction coefficient; g - gravity force coefficient, B - input gain. The joint angle is denoted by $\theta,\ deg$, the joint angular velocity is $\dot{\theta},\ deg/s$, while u is the normalized control voltage applied to the input of the power amplifier. The identified parameters are: $J = 0.0514$, $F = 1.097$, $g = 9.09$, $B = 414$.

The upper bounds of the state variables are: $|\theta| \leq 180\,deg$, $|\dot{\theta}| \leq 250\,deg/s$.

The maximum slope is obtained from (16)

$$\lambda_{max} = \frac{K - |B\dot{\theta}| - |g\,sin\,\theta| - \eta_{max}}{J\,|\dot{\theta}|} \tag{32}$$

In our case η_{max} stands for the upper bound of the system uncertainty and unmodelled dynamics. To find an estimate of it, we performed an experiment with a linear sliding surface where the slope is obtained from (32) for $\eta_{max} = 0$. The value we got is $\lambda_{max} = 10.09$. After few experimental runs we found out that the highest λ that guarantees global stability is $\lambda \approx 8$. This suggests that η is bounded by $\eta_{max} \approx 60$.

We simplified the Takagi-Sugeno fuzzy model as the variation of λ with respect to $\dot{\theta}$ is more significant than the variation with respect to θ.

$$\begin{aligned} &\text{Rule } i: \text{ IF } \dot{\theta} \text{ IS } TH_d^i \quad \text{THEN} \\ &s^i(\dot{\theta},t) = \lambda^i_{1max} e(t) + \dot{e}(t), \quad i = 1 \ldots r \end{aligned} \tag{33}$$

The gaussian type input fuzzy sets TH_d^i cover the entire velocity range from 0 to $300\,deg/s$ with 9 supporting points at { 0, 37.5, 75, 112.5, 150, 187.5, 225, 262.5, 300 }. The slopes λ^i_{1max} in the antecedent part of the rules are obtained by evaluation of expression (32) at each supporting point are: { ∞, 162.3, 70.4, 39.8, 24.5, 15.4, 9.3, 4.9, 1.6 }. From practical viewpoint, values over 25 does not make sense because the mechanical time constant of the DC motor is about $0.045s$. Moreover, the fuzzy sets TH_d^8 and TH_d^9 correspond to velocities that are not reached during normal operation. Therefore, we modify the original values and the slopes used in the experiments are: { 25, 25, 25, 20, 20, 15, 9, 6, 6 }

The control law is in the form given in Theorem 2, the gain is $K_{1max} = 0.6$, which is the normalized maximum value allowed in real experiments. To limit the chattering effect we implemented a *boundary layer*, i.e. the function $sgn(s)$ in the control law is replaced by $sat(s/\phi)$, where the thickness of the layer ϕ = 12.5. The positioning error was less that 0.01 deg, which lies within the backlash of the gearbox.

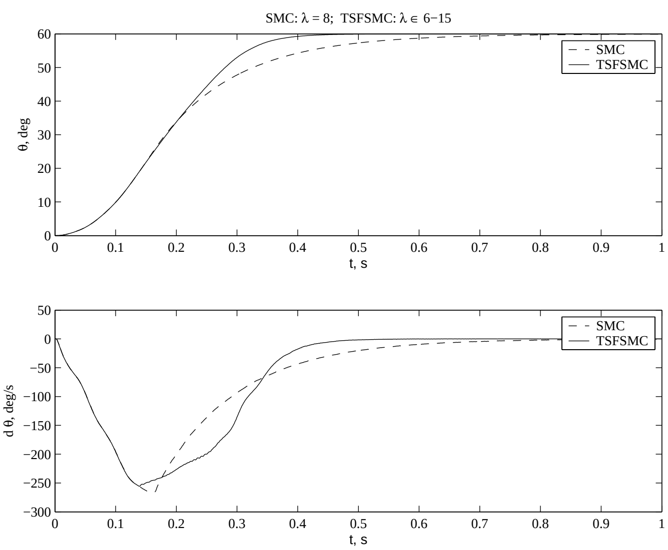

Fig. 1. Joint position and velocity for SMC and TSFSMC

To evaluate the performance of the new algorithm we compared it with a SMC with a linear sliding surface with λ = 8. The test run was a move from zero initial conditions to a new position θ_d = 60 deg. Fig. 1 shows the time history of the joint positions and angular velocities. It can be clearly seen that the TSFSMC controller reaches the desired position for about 0.4 s, while the SMC - for about 0.7 s. The motion in reaching phase is identical for both cases while the difference comes in the sliding phase where the fuzzy surface guides the state towards the origin much faster. The same result can be verified if we look at the phase plane plot (Fig. 2). The joint velocity achieved by TSFSMC is higher for almost all positions while in sliding mode. The only exception is in

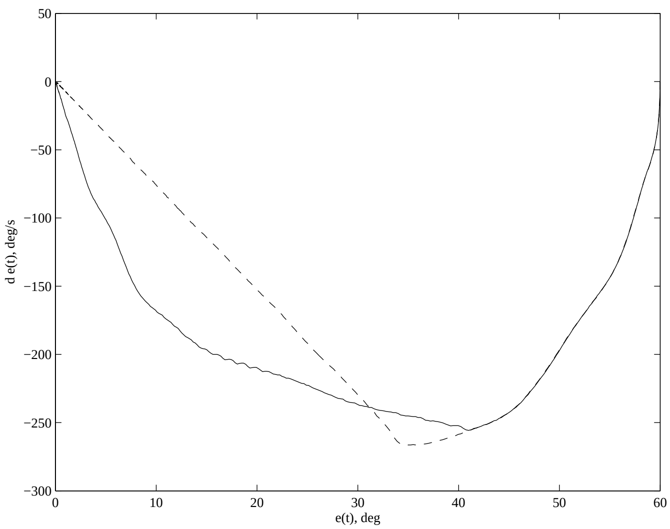

Fig. 2. Phase plane trajectories for SMC and TSFSMC

the end of the reaching of the sliding phase where TSFSMC starts sliding slightly earlier at $\lambda \approx 6$ (Rule 8 is most active), while the SMC continues to reach the line with $\lambda = 8$. For all other velocities the slope used by TSFSMC is higher since it only increases as the state moves towards the origin.

V. Conclusion

We presented an approach for near time-optimal control of second order smooth (possibly) nonlinear systems using Takagi-Sugeno fuzzy model of the maximum slope SMC sliding line. Actually, this is an adaptive technique as the current slope of the sliding line is permanently tuned to the maximum feasible slope depending on the current state of system. It was shown that the maximum slope sliding lines are rather conservative and do not provide the best time response. On contrary, the proposed method contributes performance close to the theoretical time-optimal control.

The stability conditions of this method are proved and respective measures about the feasible maximum slope are presented. Experimental results demonstrate the system behaviour in comparison to the maximum slope SMC algorithm. As shown in it, the Takagi-Sugeno model can be simplified depending on the particular properties of the controlled system.

Indicated for second order nonlinear systems with smooth nonlinearity, this approach can be applied for higher order systems using the same way for proving the global stability of the entire system.

VI. Acknowledgements

This work forms part of the research programs sponsored by the KK-foundation, Sweden, whose support is gratefully acknowledged.

VII. References

[1] M. Athans and P. Falb. Optimal Control: an introduction to the theory and its applications. *McGraw Hill*, 1966

[2] A. Bartoszewicz. Time-varying sliding modes for second-order systems *IEE Proc., Part D, Control Theory and Applications*, 143, pp.455-462, 1996

[3] C. Edwards and S. Spurgeon. Sliding Mode Control: Theory and Applications. *Taylor & Francis Ltd*, 1998

[4] T. Furuta and K. Tomiyama. Sliding mode control with time-varying hyperplane. *Proceedings of the International Conference on Robotics and Intelligent Systems*, pp. 576-581, 1996

[5] Q. P. Ha, D. Rye and H. Durrant-Whyte. Robust sliding mode control with robotic application *Int. Journal of Control*, Vol. 72, No. 12, pp. 1087-1096, 1999

[6] B. Iliev, I. Hristozov. Variable structure control using Takagi-Sugeno fuzzy system as a sliding surface *in Proc. IEEE International Conference on Fuzzy Systems FUZZ-IEEE02* May 12-17, 2002, Honolulu, USA

[7] B. Iliev, I. Kalaykov. Improved Sliding Mode Robot Control - a Fuzzy Approach. *in Proc. 3rd International Workshop on Robot Motion and Control RoMoCo'02*, pp. 393-298, 2002, Bukowy Dworek, Poland

[8] W. S. Newman. Robust Near Time-Optimal Control *IEEE Transactions on Automatic Control, Vol. 35, No 7, Juli 1990*

[9] R. Palm, D. Driankov, H. Hellendoorn. Model based fuzzy control: fuzzy gain schedulers and Sliding mode fuzzy controllers. *Springer-Verlag Ltd.*, 1996

[10] K.-K. Shyu and S.-R. Chen. Estimation of asymptotic stability region and sliding domain of uncertain variable structure systems with bounded controllers. *Automatica*, vol. 32, No. 5, 1996

[11] J.-J. Slotine, W. Li. Applied nonlinear control. *Prentice-Hall Inc.*, 1997

[12] G. Wheeler, C.-Y Su and Y. Stepanenko. A Sliding Mode Controller with Improved Adaptation Laws for the Upper Bounds on the Norm of Uncertainties *Automatica*, Vol. 34. No. 12. pp. 1657-1661, 1998