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Sliding Mode Extremum Seeking Control for Linear Quadratic Dynamic Game

Yaodong Pan* and Ümit Özgüner**

* ITS Research Group, AIST Tsukuba East
Namiki 1-2-2, Tsukuba-shi,Ibaraki-ken 305-8564, Japan
e-mail: pan.yaodong@aist.go.jp
** Department of Electrical Engineering, The Ohio State University
2015 Neil Avenue, Columbus, OH 43210
e-mail: umit@ee.eng.ohio-state.edu

***Abstract*— The extremum seeking control has been proposed to find a set point and/or track a time-varying set point so that a performance index of the system reaches its extremum point. In this paper, the extremum seeking control with sliding mode is extended to solve the Nash equilibrium solution for an n-person linear quadratic dynamic game. For each player, a sliding mode extremum seeking controller is designed to let the player's linear quadratic performance index track a decreasing signal so that the Nash equilibrium point is reached.**

Keyword: Noncooperative Dynamic Game, Linear Quadratic Performance Index, Nash Equilibrium Solution, Extremum Seeking, Sliding Model.

I. INTRODUCTION

Extremum seeking control approaches have been studied since 50's [1] and have been successfully implemented in the control of a axial-flow compressor [2], the optimization of the operation of biological reactors [3], and the optimization of spark ignition automotive engines [4]. With the extremum seeking control, a setpoint and/or a time-variable setpoint is tracked to minimize or maximize a performance index of the system [5][6][7][8].

As one of the extremum seeking control approaches, the sliding mode extremum seeking controller shown in Figure 1 ensures ([9][10]) that the system converges to a predesigned sliding mode in a finite time, enters a vicinity of the extremum point on the sliding mode, and stays there with oscillation [11]. Although it is a tradeoff problem to determine the control accuracy and convergence speed by choosing controller parameters, it is possible to obtain both high control accuracy and fast convergence speed after introducing time-variable parameters [11].

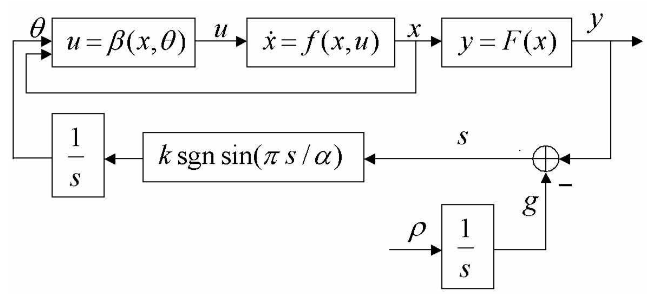

Fig. 1. Extremum Seeking Control Using Sliding Mode

Recently, it is found that the sliding mode extremum seeking control can also be implemented to solve the Nash Solution [12]. For an n-person noncooperative dynamic game, each player defines a performance index and adjusts some of the control parameters to minimize his own performance index [13][14] to find a Nash equilibrium solution. In [12], it is assumed that the performance index for each player is a function on the state, the input and the output variables of the system, which may be in an unknown form but is measurable or can be calculated from the measurable variables.

In this paper, we will consider the n-person noncooperative linear quadratic dynamic game, where the control inputs to be adjusted by each player are the feedback control gains K_i $(i = 1, 2, ..., n)$ and the performance index J_i $(i = 1, 2, ..., n)$ is the integration of a linear quadratic function on the system state and input. In this case, the calculation of the performance indices needs the future information of the system state and input. The performance indices can not be calculated on-line and thus can not be directly used in the extremum seeking controller. As the first step to design an extremum seeking controller for the n-person noncooperative linear quadratic dynamic game, the linear quadratic index is transferred to an equivalent performance index, which can be calculated on-line based on the system parameters and the feedback gain.

The arrangement of the paper is as follows. Section 2 describes the problem formulation and the transformation of the linear quadratic index; Section 3 proposes the sliding mode extremum seeking approach to find the Nash equilibrium solution for a linear quadratic dynamic game; and Section 4 gives simulation results.

II. Problem Formulation

Consider an n-person noncooperative linear quadratic dynamic game described by a linear dynamic system

$$\frac{d}{dt}x(t) = Ax(t) + \sum_{i=1}^{n} B_i u_i(t) \tag{1}$$

with a linear quadratic performance index for i-th player

$$J_i = \int_{t_0}^{\infty} (x^T(t)Q_i x(t) + u^T(t)R_i u(t))dt, (i \in N) \tag{2}$$

where A and B_i ($i \in N$) are known constant matrices of appropriate dimensions, (A, B_i) ($i \in N$) are controllable pairs, N is the index set of player defined as

$$N = \{1, 2, \cdots, n\},$$

$x(t) \in R^m$ is the system state variable, $u_i(t) \in R$ ($i \in N$) is the control variable for each player, t_0 is the system initial time, $Q_i \in R^{m\times m}$ and $R_i \in R$ ($i \in N$) are semi-positive and positive definite symmetric matrices, respectively, and (Q_i, A) ($i \in N$) are observable pairs.

To simply the notation and the discussion, the 2-player case is considered from now on. Extension to the n-player case is straightforward. The optimal control of the above system is a game problem. The Nash Solution with the Nash feedback strategy is given by [13]

$$u_i^*(t) = -R_i^{-1}B_i^T P_i x(t), \quad (i = 1, 2)$$

which ensures for any system initial state $x(t_0)$ ($x(t_0) \in R^m$) that

$$\begin{aligned} J_1^*|_{u_1(t)=u_1^*(t),u_2(t)=u_2^*(t)} &\leq J_1|_{u_1(t)\in\mathcal{U}_1,u_2(t)=u_2^*(t)} \\ J_1^*|_{u_1(t)=u_1^*(t),u_2(t)=u_2^*(t)} &\leq J_2|_{u_1(t)=u_1^*(t),u_2(t)\in\mathcal{U}_2} \end{aligned}$$

where $\mathcal{U}_1$ and $\mathcal{U}_2$ are sets of possible control input for player 1 and 2, respectively, P_1 and P_2 are the positive definite solutions of the following coupled algebraic Riccati equations:

$$\begin{aligned} &P_1A + A^TP_1 - P_1B_1R_1^{-1}B_1^TP_1 - \\ &\quad P_1B_2R_2^{-1}B_2^TP_2 - P_2B_2R_2^{-1}B_2^TP_1 = -Q_1 \\ &P_2A + A^TP_2 - P_2B_2R_2^{-1}B_2^TP_2 - \\ &\quad P_2B_1R_1^{-1}B_1^TP_1 - P_1B_1R_1^{-1}B_1^TP_2 = -Q_2. \end{aligned}$$

Instead of solving the above Riccati equations, in this paper the optimal feedback gains K_i^* ($K_i^* = -R_1^{-1}B_1^TP_1, i = 1, 2$) as the Nash solution to minimize the performance index J_i ($i = 1, 2$) are calculated by the sliding mode extremum seeking control approach.

Denote the linear feedback control input for each player as

$$u_i(t) = K_i x(t), \quad (i = 1, 2). \tag{3}$$

Then the closed-loop system of (1) with the control input (3) is determined by

$$\frac{d}{dt}x(t) = \bar{A}x(t), \tag{4}$$

and the linear quadratic performance index given in (2) can be rewritten as

$$J_i = \int_{t_0}^{\infty} x^T(t)\bar{Q}_i x(t)dt, \quad (i = 1, 2), \tag{5}$$

where

$$\begin{aligned} \bar{A} &= A - B_1K_1 - B_2K_2 \\ \bar{Q}_i &= \bar{A}^T(Q_i + K_i^TR_iK_i)\bar{A}. \quad (i = 1, 2) \end{aligned}$$

As (A, B_i) and (Q_i, A) ($i = 1, 2$) are controllable and observable pairs, respectively, there exist two parameter sets $\mathcal{K}_1$ and $\mathcal{K}_2$ so that for any feedback gain K_i ($K_i \in \mathcal{K}_i, i = 1, 2$), the Lyapunov functions

$$M_i\bar{A} + \bar{A}^TM_i = -\bar{Q}_i \quad (i = 1, 2) \tag{6}$$

have positive definite symmetric solutions M_i ($i = 1, 2$). It is clear that M_i ($i = 1, 2$) is a function on K_1 and K_2 and thus may be denoted as

$$M_i = M_i(K_1, K_2). \quad (i = 1, 2) \tag{7}$$

Then the linear quadratic performance index (2) can be rewritten as

$$\begin{aligned} J_i &= -x^T(t)M_ix(t)|_{t_0}^{\infty} = x^T(t_0)M_ix(t_0) \\ &= tr(x(t_0)x^T(t_0)M_i) \quad \forall K_i \in \mathcal{K}_i, (i = 1, 2) \end{aligned}$$

where $tr(.)$ is the trace operation.

According to the linear quadratic optimal control theory, the optimal solution of the above performance index is independent to the initial time t_0 and the system initial state $x(t_0)$, i.e. the optimal feedback gain K_i^* $(i = 1,2)$ is unique no matter the system initial state $x(t_0)$ is. Denote the column vector of the identity matrix I_m as e_j $(j = 1,2,...,m)$, i.e.

$$I_m = diag\{1,1,...,1\} = \begin{bmatrix} e_1 & e_2 & \cdots & e_m \end{bmatrix}.$$

Then the optimal feedback gain K_i^* $(i = 1,2)$ can be described as

$$\begin{aligned}\{K_1^*, K_2^*\} &= Arg \min_{K_i \in \mathcal{K}_i, i=1,2} tr(x(t_0)x^T(t_0)M_i) \\ &= Arg \min_{K_i \in \mathcal{K}_i, i=1,2} tr(e_j e_j^T M_i), (j = 1,2,...,m) \\ &= Arg \min_{K_i \in \mathcal{K}_i, i=1,2} \sum_{j=1}^{m} tr(e_j e_j^T M_i) \\ &= Arg \min_{K_i \in \mathcal{K}_i, i=1,2} tr(\sum_{j=1}^{m} e_j e_j^T M_i) \\ &= Arg \min_{K_i \in \mathcal{K}_i, i=1,2} tr(I_m M_i) \\ &= Arg \min_{K_i \in \mathcal{K}_i, i=1,2} tr(M_i). \end{aligned} \quad (8)$$

Thus the control objective to solve the Nash equilibrium solution is turned to minimize the performance index

$$J_i(K_1,K_2) = tr(M_i) = tr(M_i(K_1,K_2)) \quad (i = 1,2) \quad (9)$$

by adjusting the feedback gain K_i by each player $(i = 1,2)$ independently.

It follows from the linear quadratic optimal control theory [13] that a unique Nash equilibrium solution $(K_1^*,\ K_2^*)$ exists such that

$$\begin{aligned} J_1^*(K_1^*,K_2^*) &\leq J_i(K_1,K_2^*), \quad \forall K_1 \in \mathcal{K}_1 \\ J_2^*(K_1^*,K_2^*) &\leq J_i(K_1^*,K_2), \quad \forall K_2 \in \mathcal{K}_2. \end{aligned}$$

During searching the Nash solution, the feedback gain K_i $(i = 1,2)$ is adjusted on-line by the extremum seeking controller. Therefore K_i $(i = 1,2)$ is a function on time. The performance index given in (9) thus can be described as a function on time, too, i.e.

$$J_i(t) = J_i(K_1(t),K_2(t)) = tr(M_i(K_1(t),K_2(t))) \quad (i = 1,2) \quad (10)$$

To simplify the notation, the same symbol J_i is used to denote the performance indices in (2), (9), and (10).

III. Extremum Seeking with Sliding Mode

To design an extremum seeking controller with sliding mode for the i-th player $(i = 1,2)$, a switching function is defined as

$$s_i(t) = J_i(t) - g_i(t) \quad (11)$$

where the reference signal $g_i(t) \in R$ is determined by

$$\dot{g}_i(t) = -\rho_i, \quad (12)$$

where ρ_i $(i = 1,2)$ are positive constants.

Let the variable structure control law be

$$v_i(t) = -k_i \begin{bmatrix} \mathrm{sgn}(sin(\pi s_i(t)/\alpha_i)) \\ \mathrm{sgn}(sin(2\pi s_i(t)/\alpha_i)) \\ \cdots \\ \mathrm{sgn}(sin(2^{m-1}\pi s_i(t)/\alpha_i)) \end{bmatrix}, \quad (i = 1,2) \quad (13)$$

and the feedback gain K_i $(i = 1,2)$ satisfy

$$\dot{K}_i(t) = v_i(t), \quad (i = 1,2) \quad (14)$$

where α_i and k_i $(i = 1,2)$ are positive constants.

Assumption 1: The partial derivative of the performance index $J_i(t)$ $(i = 1,2)$ satisfies

$$|\frac{\partial}{\partial K_i} J_i(K_1,K_2)| >> |\frac{\partial}{\partial K_j} J_i(K_1,K_2)|, \forall j \neq i (i,j = 1,2)$$

which means that each player can adjust his performance index most effectively

Assumption 2: The Nash equilibrium point $(K_1^*,\ K_2^*)$ is in the vicinity of the initial 2-tuple of $K_i(0)$ $(i = 1,2)$. Thus the partial derivative of the performance index $J_i(t)$ on K_i is bounded by a positive constant γ_i, i.e.,

$$|\frac{\partial}{\partial K_i} J_i(K_1,K_2)| \leq \gamma_i. \quad (i = 1,2) \quad (15)$$

Theorem 1: Consider the dynamic noncooperative game described by the state equation in (1) with the linear quadratic performance index (2), the extremum seeking controller with sliding mode for the i-th player $(i = 1,2)$ designed by Equations (11), (12), (13), and (14) ensures that the performance index $J_i(t)$ $(i = 1,2)$ are minimized to get the Nash equilibrium solution $J_1^*(K_1^*,K_2^*)$ and $J^*(K_1^*,K_2^*)$ if positive constants ρ_i, k_i, and α_i $(i = 1,2)$ as the controller parameters are chosen suitable.

Proof: The completed proof of this theorem needs much space [11]. In this paper, only the steps of the proof are described as follows, which are similar to the results in [12] and [11].

Based on the above assumptions, the derivative of the switching function $s_i(t)$ is given by

$$\begin{aligned}\frac{d}{dt}s_i(t) &= \sum_{j=1}^{2}\frac{\partial}{\partial K_j}J_i(K_1,K_2)\dot{K}_j(t)-\dot{g}_i(t)\\ &\approx \frac{\partial}{\partial K_i}J_i(K_1,K_2)\dot{K}_i(t)-\dot{g}_i(t). \qquad (16)\end{aligned}$$

From which, it can be shown that

- For each player there exists a vicinity of the minimum point, which is determined by $\frac{\rho_i}{k_i}$ $(i = 1, 2)$. Outside this vicinity, a sliding mode

$$s_i(t) = l\alpha_i \quad \text{or} \quad s_i(t) = -l\alpha_i$$

  will happen for some number l determined by the initial condition of the system.
- On the sliding mode, the system converges to the vicinity.
- After entering the vicinity, it is possible that either the system stays inside the vicinity or go through the vicinity. In the later case, another sliding mode will happen and the system will enter the vicinity again on the sliding mode.
- In both cases, i.e. the case staying inside the vicinity and the case moving out of the vicinity, the performance index $J_1(t)$ oscillates and decreases in each oscillation period as shown in Figures 2, 3 and 4.

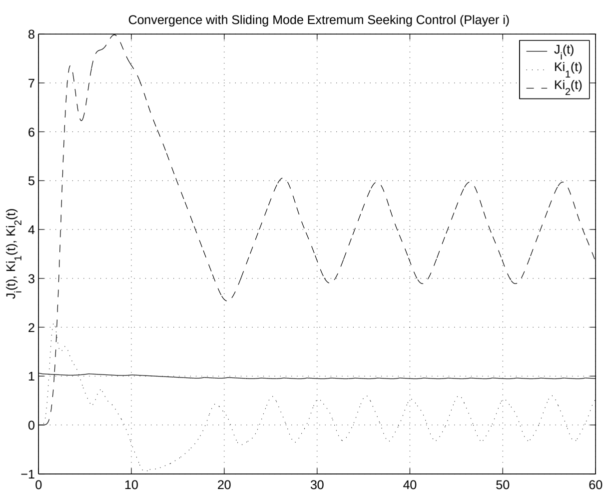

Fig. 2. Convergence with Sliding Mode Extremum Seeking Controller

- Finally the system oscillates around the Nash equilibrium point.

In this way, the Nash solution can be found by the proposed extremum seeking controller with sliding mode. And it

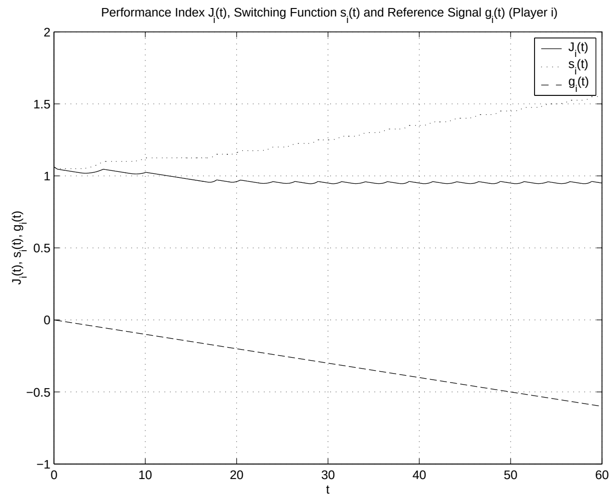

Fig. 3. Switching Function with Sliding Mode Extremum Seeking Controller

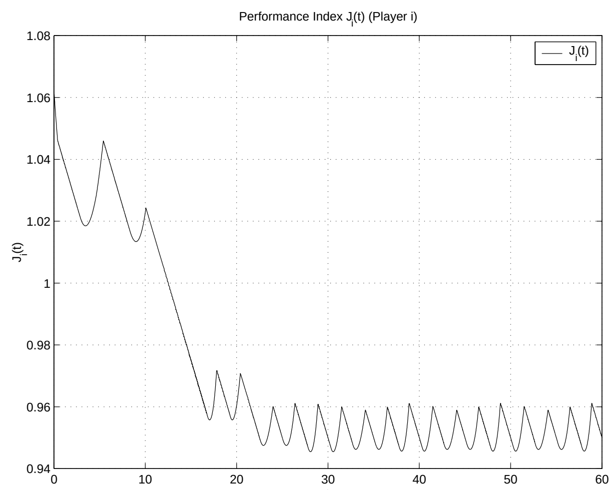

Fig. 4. Performance Index with Sliding Mode Extremum Seeking Controller

has been shown in [11] that fast convergent speed and high control accuracy may be obtained if the controller parameters ρ_i, k_i, and α_i are chosen suitably and adjusted online. ■

IV. EXAMPLES

Consider a two-person noncooperative linear quadratic dynamic game described by a second-order linear system.

$$\dot{x}(t) = \begin{bmatrix} -0.7 & 0.2 \\ 0.1 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2(t).$$

The parameter matrices of the performance index (2) are respectively given by

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0.2 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0.1 \end{bmatrix}.$$

The proposed extremum seeking control algorithm is implemented to the above system with sampling interval as $T = 0.01$ second and other controller parameters as

$$k_i = 0.5, \quad \rho_i = 0.05. \quad (i = 1, 2)$$

The simulation results with $\alpha_1 = \alpha_2 = 0.05$ are given in Figures 5 and 6, which shows that the system reaches a sliding mode in a finite time, converges to the vicinity of the Nash equilibrium point and then oscillates while the performance index keeps decreasing in each oscillation period until the Nash equilibrium point is reached.

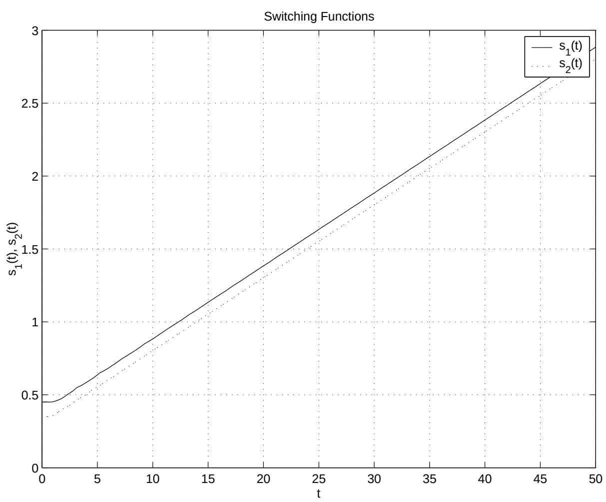

Fig. 5. Switching Functions with Sliding Mode Extremum Seeking Control

The amplitude of the oscillation can be reduced by increasing the positive constant ρ_i $(i = 1, 2)$ as shown in Figure 7 with $\rho_1 = \rho_2 = 0.2$ but in this case, the convergent speed is slow.

In this paper, a kind of on-line adjusting method is proposed. The parameter ρ is determined by a time-variable function as

$$\rho_i = \begin{cases} 0.05 + 0.0075 * t & t \le 20 \\ 0.2 & t > 20 \end{cases}. \quad (i = 1, 2)$$

The simulation results in Figure 8 with the above time-variable parameter ρ_i $(i = 1, 2)$ show that the Nash solution is obtained quickly with higher control accuracy.

To confirm the results obtained by the proposed sliding mode extremum seeking control approach, the optimal feedback gains for the Nash solution are obtained as

$$K_1 = \begin{bmatrix} 1.643 & 0.0265 \end{bmatrix} \quad (17)$$

$$K_2 = \begin{bmatrix} 0.095 & 3.6658 \end{bmatrix} \quad (18)$$

by the ϵ-coupling approach [14], which are the same to those results shown in Figures 6, 7, and 8.

V. Conclusion

The sliding mode extremum seeking control approach is successfully extended to the Nash equilibrium solution for an n-person noncooperative linear quadratic dynamic game. With the designed extremum seeking controller for each player in the game, the system reaches a sliding mode, enters a vicinity of the Nash equilibrium point, and stays there with oscillating behavior. The simulation results show the effectiveness.

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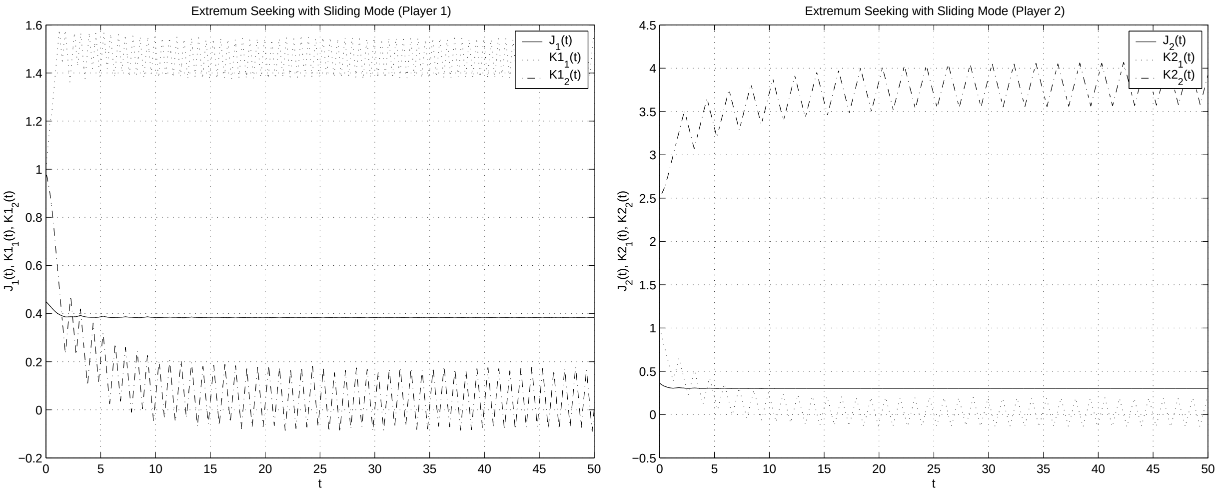

Fig. 6. Nash Solution by Extremum Seeking Control($\alpha_1 = \alpha_2 = 0.05$)

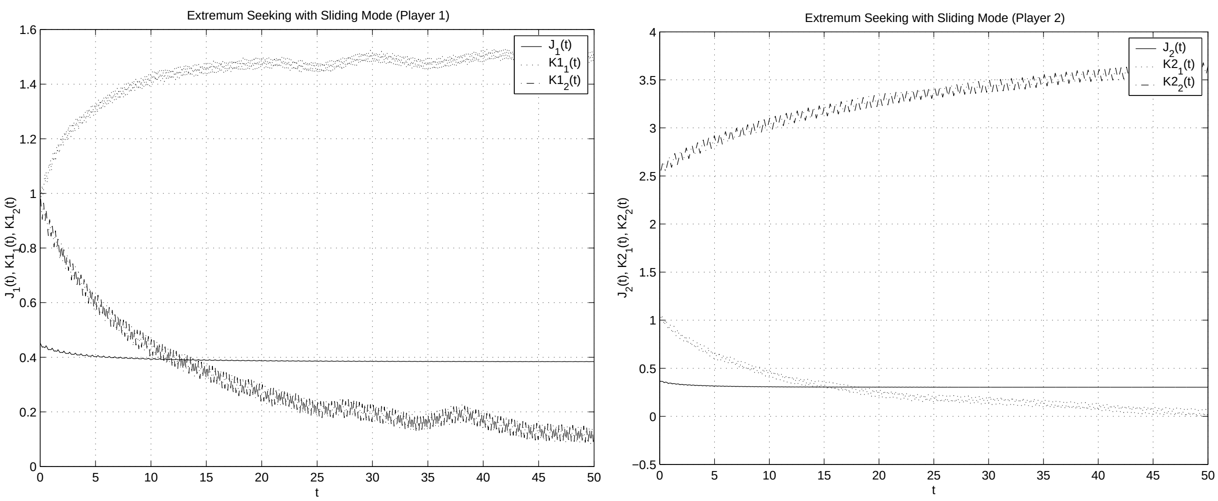

Fig. 7. Nash Solution by Extremum Seeking Control($\alpha_1 = \alpha_2 = 0.2$)

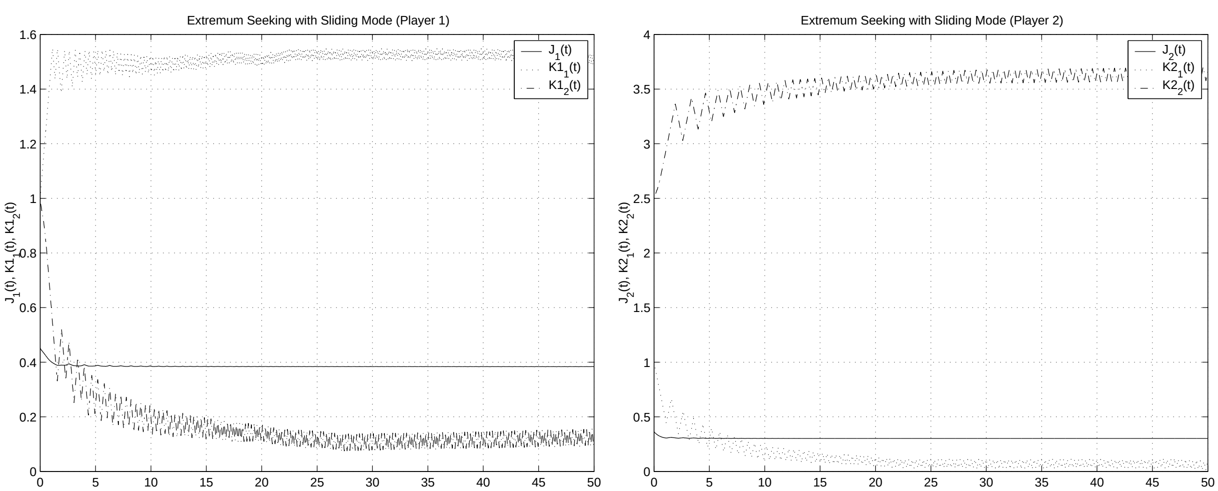

Fig. 8. Nash Solution by Extremum Seeking Control($\alpha_1 = \alpha_2 = 0.05 \rightarrow 0.2$)