On Direct Adaptive Control of a Class of Nonlinear Scalar Systems

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Abstract— **In this paper, we develop an adaptive control for a scalar system with nonlinear uncertainty. Specifically, we consider scalar systems that are not stabilizable via a linear feedback and hence there does not exist a simple procedure to design stabilizing adaptive controllers for these systems. In this paper, we present an adaptive control framework for such scalar systems that guarantees convergence of the state to the origin. The overall objective of this note is to emphasize the inherent difficulties in designing adaptive controllers for systems with nonlinear uncertainties.**

I. Introduction

In light of the highly complex nature of modern engineering systems, accurate mathematical description of these systems is seldom possible. Hence, it is not surprising that adaptive and robust control theory plays a fundamental role in modern control design. While robust controllers are efficient in the case of bounded uncertainties and bounded disturbances, adaptive controllers have the ability to stabilize systems over a large range of uncertainties without sacrificing system performance [1], [2]. A key assumption in a typical adaptive control problem is that the system uncertainties are real constant parameters and the dynamics of the system are described in terms of an affine function of the uncertain parameters. To illustrate this point, consider the scalar dynamical system

$$\dot{x}(t) = f(x(t)) + u(t), \qquad x(0) = x_0, \qquad t \geq 0, \quad (1)$$

where $f(x) = \sum_{i=1}^{r} a_i f_i(x)$, $a_i \in \mathbb{R}$, $i = 1, \ldots, r$, are *unknown* and $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, r$, are known. In this case, depending on the function $f_i(x)$ and the knowledge of the sign of the corresponding uncertain parameter a_i the control input $u(t)$ is chosen as $u(t) = \sum_{i=1}^{r} k_i(t) g_i(x(t))$ where $g_i(x)$ is strongly related to $f_i(x)$ and the adaptation gain $k_i(t)$ is determined by an update law

$$\dot{k}_i(t) = \phi(x(t)), \qquad k_i(0) = k_{i0}, \qquad i = 1, \ldots, r, \quad (2)$$

and where $\phi_i(x)$ is typically derived based on an appropriate Lyapunov function. Hence, we need one adaptation gain for every uncertain parameter (unless it can be established *a priori* that $a_i f_i(x)$ for a particular i is *not* destabilizing).

A notable exception to this approach is given in [3] where the authors consider a special class of dynamic systems, namely, the second order systems. Specifically, the authors in [3] consider a second order system (in the scalar case) given by

$$m\ddot{x}(t) + c(x(t))\dot{x}(t) + k(x(t)) = u(t) \quad (3)$$
$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \quad t \geq 0, \quad (4)$$

where $m > 0$ is unknown, $c(x)$ and $k(x)$ are polynomials with unknown coefficients and with no restriction on the order of the polynomial. In this case, the adaptive control law was constructed using at most four adaptation gains. While this approach uses only a small number of adaptation gains compared to the arbitrarily large number of uncertain parameters, it is limited to second order systems given by (3).

In this paper, we consider a scalar system given by

$$\dot{x}(t) = a x^p(t) + u(t), \qquad x(0) = x_0, \qquad t \geq 0, \quad (5)$$

where $x_0 > 0$, $a > 0$, and $0 < p < 1$ are unknown. Note that if p is known then it is a trivial exercise using the standard approach to find an adaptive control law that stabilizes (5) (see Section II). Alternatively, if $p \geq 1$ and unknown one may use the approach similar to that of [3] to derive globally stabilizing control laws (see Section II). However, if $p \in (0, 1)$ and unknown, the scalar system is not stabilizable via a linear feedback and hence there does not exist a simple procedure to design a stabilizing adaptive controller. Specifically, the standard Lyapunov based methods given in the literature fail to provide any stabilizing adaptive controller. In this paper, we provide an adaptive control law that guarantees the convergence of the state $x(t)$ to the origin. Finally, the result is then extended to systems of the form

$$\dot{x}(t) = f(x) + u(t), \qquad x(0) = x_0, \qquad t \geq 0. \quad (6)$$

II. Main Result

In this section we consider the scalar dynamical system

$$\dot{x}(t) = a x^p(t) - u(t) \quad (7)$$

where $x_0 > 0$, $a > 0$, and $0 < p < 1$ are unknown. In this case the closed-loop system is given by

$$\dot{x}(t) = a x^p(t) - c x(t), \quad x(0) = x_0, \quad t \geq 0. \quad (8)$$

Note that for every $c > 0$ the origin is an unstable equilibrium. However, with $V(x) = \frac{1}{2}x^2$ it follows from Theorem 3.2 of [4] that (8) is ultimately bounded with the ultimate bound $\varepsilon = (\frac{a}{c})^{(\frac{1}{1-p})}$. Since, $\varepsilon \to 0$ as $c \to \infty$, in the sequel we provide an adaptive control law that guarantees the convergence of $x(t)$ to the origin. Specifically, consider the adaptive feedback control law $u(t) = -k(t)x(t)$, $t \geq 0$, where $k(t)$, $t \geq 0$, satisfies

$$\dot{k} = \frac{1}{b}x^2(t), \quad k(0) = 0, \quad t \geq 0 \quad (9)$$

and where $b > 0$. In this case, the closed-loop system is given by

$$\dot{x} = a x^p(t) - k(t)x(t), \quad x(0) = x_0, \quad t \geq 0, \quad (10)$$
$$\dot{k} = \frac{1}{b}x^2(t), \quad k(0) = 0 \quad (11)$$

Next, with the adaptive control law $u(t) = -k(t)x(t)$, where $k(t)$ satisfies (11), we show that for every $x_0 > 0$,

This research was supported in part by the National Science Foundation under Grant ECS-0133038.

$x(t) \to 0$ as $t \to \infty$. First however, we need the following lemma. For this result, let $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ be defined by $\alpha(\theta) \triangleq a\theta^{p-1}$ and $\beta(\theta) = (\theta/a)^{1/1-p}$ so that $\beta(\alpha(\theta)) = \alpha(\beta(\theta)) = \theta$. Furthermore, let

$$\mathcal{X} \triangleq \{(x, k) \in \mathbb{R} \times \mathbb{R} : x \geq 0,\ k \geq \alpha(x)\}. \tag{12}$$

Lemma 2.1: Consider the closed-loop dynamical system (10),(11). Then the following statements hold:

i) If there exists $\hat{t} > 0$ such that $k(\hat{t}) < \alpha(x(\hat{t}))$ then there exists $T > \hat{t}$ such that $k(T) \geq \alpha(x(T))$.

ii) $\mathcal{X}$ is a positive invariant set with respect to (10),(11).

iii) If there exists $\hat{t} > 0$ such that $\alpha(x(\hat{t})) \leq k(\hat{t})$ then $\lim_{t\to\infty} x(t) = 0$.

Proof. i) Suppose, *ad absurdum*, $k(t) < \alpha(x(t))$, $t \geq \hat{t}$. Now note that

$$\dot{x}(t) \;=\; x(t)(\alpha(x(t)) - k(t)) > 0 \qquad t \geq \hat{t}, \tag{13}$$

which implies that $x(t) > x(\hat{t}) > 0$, $t \geq \hat{t}$. Next since $\dot{k}(t) = \frac{1}{b}x^2(t) \geq \frac{1}{b}x^2(\hat{t}) > 0$, $t \geq \hat{t}, x \neq 0$, it follows that $k(t) \to \infty$ as $t \to \infty$. Next, since $0 \leq \alpha(x(t)) \leq \alpha(x(\hat{t}))$ it follows that $\alpha(x(t)) \to c$ as $t \to \infty$. Since $\alpha(x(t)) > k(t), t \geq \hat{t}$ it follows that $c = \lim_{t\to\infty} \alpha(x(t)) > \lim_{t\to\infty} k(t) = \infty$ which is a contradiction.

ii) Let $(x(0), k(0)) \in \mathcal{X}$ and suppose, *ad absurdum*, there exists $T > 0$ such that $(x(T), k(T)) \notin \mathcal{X}$; that is, $\alpha(x(T)) > k(T)$. Now, it follows from the continuity of $x(\cdot)$, $k(\cdot)$, and $\alpha(\cdot)$ that there exists $\hat{t} \geq 0$ such that $\alpha(x(t)) > k(t)$, $t \in (\hat{t}, T]$ and $\alpha(x(\hat{t})) = k(\hat{t})$. Next, note that $\dot{x}(t) = x(t)(\alpha(x(t)) - k(t)) \geq 0$, $t \in (\hat{t}, T]$ which implies that

$$x(T) = x(\hat{t}) + \int_{\hat{t}}^{T} \dot{x}(t)\mathrm{d}t \geq x(\hat{t}).$$

Hence, $k(T) < \alpha(x(T)) \leq \alpha(x(\hat{t})) = k(\hat{t})$ which is a contradiction since $k(\cdot)$ is monotonically increasing.

iii) It follows from ii) that if there exists $\hat{t} > 0$ such that $\alpha(x(\hat{t})) \leq k(\hat{t})$ then $\alpha(x(t)) \leq k(t)$, $t \geq \hat{t}$. Hence, it follows that $\dot{x}(t) = x(t)(\alpha(x(t)) - k(t)) \leq 0$, $t \geq \hat{t}$, which implies that $x(t)$, $t \geq \hat{t}$ is monotonically decreasing and hence $\lim_{t\to\infty} x(t)$ exists. Now suppose, *ad absurdum*, $\lim_{t\to\infty} x(t) = c > 0$ which implies that $x(t) \geq c > 0$ or equivalently, $\alpha(x(t)) \leq \alpha(c)$, $t \geq \hat{t}$. Next note that

$$\begin{aligned} k(t) &= k(\hat{t}) + \int_{\hat{t}}^{t} \dot{k}(s)ds = k(\hat{t}) + \int_{\hat{t}}^{t} \frac{1}{b}x^2(s)ds \\ &\geq k(\hat{t}) + c^2(t - \hat{t}), \qquad t \geq \hat{t}, \end{aligned}$$

and

$$\begin{aligned} x(t) &= x(\hat{t}) + \int_{\hat{t}}^{t} \dot{x}(s)ds \\ &= x(\hat{t}) + \int_{\hat{t}}^{t} x(s)(\alpha(x(s)) - k(s))ds \\ &\leq x(\hat{t}) + \int_{\hat{t}}^{t} x(\hat{t})[\alpha(c) - k(\hat{t}) - c^2(s - \hat{t})]ds \\ &= x(\hat{t})[1 + \theta(t - \hat{t}) + \frac{c^2\hat{t}^2}{2} - \frac{c^2t^2}{2}], \qquad t \geq \hat{t}, \end{aligned}$$

where $\theta \triangleq \alpha(c) - k(\hat{t}) + c^2\hat{t}$. Now it can be shown that there exists $t > \hat{t}$ such that $x(t) < c$, which is a contradiction. □

Theorem 2.1: Consider the scalar system given by (7) where $a > 0$, and $p \in (0, 1)$ are unknown with the feedback control law $u(t) = -k(t)x(t)$, $t \geq 0$, where $k(\cdot)$ is given by the update law (9). Then for every $x_0 > 0$, $x(t) \to 0$ as $t \to \infty$.

Finally, we extend Theorem 2.1 to scalar systems of the form

$$\dot{x}(t) = f(x(t)) + u(t), \quad x(0) = x_0, \quad t \geq 0 \tag{14}$$

where $f(\cdot) : [0, \infty) \to [0, \infty)$, such that $f(0) = 0$ and $\frac{f(x)}{x}$ is a non-increasing function of $x \in (0, \infty)$

Theorem 2.2: Consider the scalar system given by (14) with the adaptive feedback control law $u(t) = -k(t)x(t)$, $t \geq 0$, where $k(\cdot)$ is given by the update law (9). Then for every $x_0 > 0$, $x(t) \to 0$ as $t \to \infty$.

III. Conclusion

In this paper, we developed an adaptive control for a scalar system with nonlinear uncertainty. Specifically, we considered scalar systems that are not stabilizable via a linear feedback. Hence, there does not exist a simple procedure to design stabilizing adaptive controllers for these systems. In this paper, we presented an adaptive control framework for such scalar systems that guarantees convergence of the state to the origin. The overall objective of this note is to emphasize the inherent difficulties in designing adaptive controllers for systems with nonlinear uncertainties.

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