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Identification of PWARX Hybrid Models with Unknown and Possibly Different Orders

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***Abstract*— We consider the problem of identifying the orders and the model parameters of PWARX hybrid models from noiseless input/output data. We cast the identification problem in an algebraic geometric framework in which the number of discrete states corresponds to the degree of a multivariate polynomial p and the orders and the model parameters are encoded on the factors of p. We derive a rank constraint on the input/output data from which one can estimate the coefficients of p. Given p, we show that one can estimate the orders and the parameters of each ARX model from the derivatives of p at a collection of regressors that minimize a certain objective function. Our solution does not require previous knowledge about the orders of the ARX models (only an upper bound is needed), nor does it constraint the orders to be equal. Also the switching mechanism can be arbitrary, hence the switches need not be separated by a minimum dwell time. We illustrate our approach with an algebraic example of a switching circuit and with simulation results in the presence of noisy data.**

I. Introduction

We consider the problem of identifying the orders and the model parameters of a class of discrete-time hybrid systems known as PieceWise Auto Regressive eXogenous (PWARX) systems, *i.e.* systems whose evolution is described as

$$y_t = \sum_{j=1}^{n_a(\lambda_{t-1})} a_j(\lambda_{t-1}) y_{t-j} + c_j(\lambda_{t-1}) u_{t-j}, \qquad (1)$$

where $u_t \in \mathbb{R}$ is the *input*, $y_t \in \mathbb{R}$ is the output, $\lambda_t \in \{1, 2, \dots, n\}$ is the *discrete state*, and $n_a(i)$, $\{a_\ell(i)\}_{\ell=1}^{n_a(i)}$ and $\{c_\ell(i)\}_{\ell=1}^{n_a(i)}$ are, respectively, the orders and the model parameters of the i^{th} ARX model for $i = 1, \dots, n$.

The evolution of the discrete state λ_t can be described in a variety of ways:

- In *Jump-linear systems (JLS)* λ_t is an unknown, deterministic and finite-valued input.
- In *Jump-Markov linear systems (JMLS)* λ_t is an irreducible Markov chain governed by the transition probabilities $\pi(i, i') \doteq P(\lambda_{t+1} = i' | \lambda_t = i)$.
- In *Piecewise affine systems* (PWAS) λ_t is a piecewise constant function of the continuous state that is defined by a polyhedral partition of the state space.

In this paper, we take the least restrictive model (JLS), so that our results also apply to other switching mechanisms. We therefore consider the following identification/filtering problem.

Problem 1: Let $\{u_t, y_t\}_{t=0}^T$ be input/output data generated by the PWARX model (1), with known number of discrete states n. Given an upper bound n_a for the orders of the ARX models, identify the order of each ARX model $\{n_a(i) \leq n_a\}_{i=1}^n$, the model parameters $\{a_\ell(i)\}_{\ell=1,\dots,n_a(i)}^{i=1,\dots,n}$ and $\{c_\ell(i)\}_{\ell=1,\dots,n_a(i)}^{i=1,\dots,n}$, and estimate the discrete state $\{\lambda_{t-1}\}_{t=n_a}^T$.

Work on identification/filtering of hybrid systems first appeared in the seventies (see [19] for a review). More recent works consider variations of Problem 1 in which the model parameters, the discrete state and/or the switching mechanism are *known*, and concentrate on the analysis of the observability of the hybrid state [2], [4], [9], [11], [18], [21], [22] and the design of hybrid observers [1], [3], [7], [8], [10], [12], [14], [16], [17], [20].

The more challenging case in which both the model parameters and the hybrid state are unknown has been recently addressed using mixed-integer quadratic programming [6] or iteratively by alternating between assigning data points to models and computing the model parameters starting from a random or ad-hoc initialization [5], [13]. The first algebraic approach to the identification of PWARX models appeared in [25], where it was shown that one can identify the model parameters in closed form when the ARX models are of known and equal order and the number of discrete states is less than or equal to four.

In this paper, we consider the case in which the orders of the ARX models are *unknown* and possibly *different* from each other. Following [25], we represent the number of discrete states as the degree of a polynomial p and the orders and model parameters as factors of p. We show that one can linearly solve for the coefficients of p, even in the case of unknown and different orders, thanks to a rank constraint on the data. Given p, the orders and the parameters of each ARX model are estimated from the derivatives of p evaluated at a collection of regressors that minimize a certain objective function. Our solution only requires an upper bound on the orders of each ARX model, which are not constrained to be equal. Also the switching mechanism can be arbitrary. In particular, the switching times need not be separated by a minimum dwell time. We illustrate our approach with an algebraic example of a switching circuit and with simulation results in the presence of noisy data.

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II. IDENTIFICATION OF LINEAR ARX HYBRID SYSTEMS

This section presents an algebraic geometric solution to Problem 1. Sections II-A and II-B show how to decouple the identification of the model parameters from the estimation of the discrete state via a suitable embedding into a higher-dimensional space, as proposed in [25]. Sections II-C and II-D show how to identify the orders and the model parameters from the derivatives of a polynomial whose coefficients are obtained from a rank constraint on the embedded data. Section II-E shows how to estimate the discrete state.

A. The hybrid decoupling constraint

Notice from equation (1) that if we let $K \doteq 2n_a + 1$, $\boldsymbol{x}_t = [u_{t-n_a}, y_{t-n_a}, \cdots, u_{t-1}, y_{t-1}, -y_t]^T \in \mathbb{R}^K$, and $\boldsymbol{b}_i = [0, \cdots, 0, c_{n_a(i)}(i), a_{n_a(i)}(i), \cdots, c_1(i), a_1(i), 1]^T \in \mathbb{R}^K$, for $i = 1, \ldots, n$, then we have that for all $t \geq n_a$ there exists a discrete state $\lambda_{t-1} = i \in \{1, \ldots, n\}$ such that

$$\boldsymbol{b}_i^T \boldsymbol{x}_t = 0. \tag{2}$$

Therefore, the following *hybrid decoupling constraint* (HDC) [25] must be satisfied by the model parameters and the input/output data regardless of the value of the discrete state and regardless of the switching mechanism generating the evolution of the discrete state (JLS, JMLS, or PWAS)

$$\prod_{i=1}^{n} (\boldsymbol{b}_i^T \boldsymbol{x}_t) = 0. \tag{3}$$

B. The hybrid model parameters

The hybrid decoupling constraint is simply is a homogeneous polynomial of degree n in K variables

$$p_n(\boldsymbol{z}) \doteq \prod_{i=1}^{n} (\boldsymbol{b}_i^T \boldsymbol{z}) = 0, \tag{4}$$

which can be written as

$$p_n(\boldsymbol{z}) \doteq \sum h_{n_1,\ldots,n_K} z_1^{n_1} \cdots z_K^{n_K} = \boldsymbol{h}_n^T \nu_n(\boldsymbol{z}) = 0, \tag{5}$$

where $h_I \in \mathbb{R}$ represents the coefficient of the monomial $\boldsymbol{z}^I = z_1^{n_1} z_2^{n_2} \cdots z_K^{n_K}$ with $0 \leq n_j \leq n$ for $j = 1, \ldots, K$, and $n_1 + n_2 + \cdots + n_K = n$; $\nu_n : \mathbb{R}^K \to \mathbb{R}^{M_n}$ is the *Veronese map* of degree n which is defined as:

$$\nu_n : [z_1, \ldots, z_K]^T \mapsto [\ldots, \boldsymbol{z}^I, \ldots]^T, \tag{6}$$

with I chosen in the degree-lexicographic order; and

$$M_n = \binom{n + K - 1}{K - 1} = \binom{n + K - 1}{n} \tag{7}$$

is the total number of independent monomials. One can show [24] that the vector of coefficients $\boldsymbol{h}_n \in \mathbb{R}^{M_n}$ is simply a vector representation of the symmetric tensor product of the individual model parameters $\{\boldsymbol{b}_i\}_{i=1}^n$, *i.e.*

$$\sum_{\sigma \in \mathfrak{S}_n} \boldsymbol{b}_{\sigma(1)} \otimes \boldsymbol{b}_{\sigma(2)} \otimes \cdots \otimes \boldsymbol{b}_{\sigma(n)}, \tag{8}$$

where $\mathfrak{S}_n$ is the permutation group of n elements. Since $\boldsymbol{h}_n$ encodes the parameters of all the ARX models, we will refer to it as the *hybrid model parameters* from now on.

C. Identifying the hybrid model parameters

Since the HDC (3)-(5) is satisfied by all the data points $\{\boldsymbol{x}_t\}_{t=n_a}^T$, we can use it to derive the following linear system on the hybrid model parameters $\boldsymbol{h}_n$

$$L_n\, \boldsymbol{h}_n \doteq \begin{bmatrix} \nu_n(\boldsymbol{x}_{n_a})^T \\ \nu_n(\boldsymbol{x}_{n_a+1})^T \\ \vdots \\ \nu_n(\boldsymbol{x}_T)^T \end{bmatrix} \boldsymbol{h}_n = 0 \in \mathbb{R}^{T-n_a+1}, \tag{9}$$

where $L_n \in \mathbb{R}^{(T-n_a+1)\times M_n}$ is the matrix of embedded input/output data.

We are now interested in solving for $\boldsymbol{h}_n$ from (9). In our previous work [25] we considered ARX models of *equal* and *known* orders $n_a(1) = n_a(2) = \cdots = n_a(n) = n_a$ and showed that if the number of measurements is such that $T \geq M_n + n_a - 2$ and at least $2n_a$ measurements correspond to each discrete mode, then $\operatorname{rank}(L_n) = M_n - 1$. Therefore, one can uniquely solve for $\boldsymbol{h}_n$ from the nullspace of L_n, because the last entry of $\boldsymbol{h}_n$ is known to be equal to one.

In the case of ARX models of *unknown* and possibly *different* orders, one may not be able to uniquely recover the hybrid model parameters from the equation $L_n \boldsymbol{h}_n = 0$. This is because in this case some of the entries of the vector of model parameters $\boldsymbol{b}_i$ associated with the i^{th} ARX model may be zero. Therefore, depending on the model orders and on the switching sequence (see Example 1), there could be a vector $\boldsymbol{b} \neq \boldsymbol{b}_i$ such that $\boldsymbol{b}^T \boldsymbol{x}_t = 0$ for all data points $\boldsymbol{x}_t$ generated by the i^{th} ARX model. If such a $\boldsymbol{b}$ exists, then there is a vector $\boldsymbol{h} \neq \boldsymbol{h}_n$, defined as in (8) but with $\boldsymbol{b}_i$ replaced by $\boldsymbol{b}$, which is also in the nullspace of L_n, thus the solution of $L_n \boldsymbol{h}_n = 0$ is no longer unique.

Example 1 (Effect of n_a and λ_t on the nullspace of L_n): Consider a PWARX model with orders $n_a(1) = n_a(2) = 1$ and assume that $n_a = 2$ so that the vectors of model parameters are $\boldsymbol{b}_1 = [0, 0, c_1(1), a_1(1), 1]^T$ and $\boldsymbol{b}_2 = [0, 0, c_1(2), a_1(2), 1]^T$, respectively. If $\lambda_{t-2} = \lambda_{t-1} = 1$, then the vector $\boldsymbol{b} = [c_1(1), a_1(1), -1, 0, 0]^T \neq \boldsymbol{b}_1$ satisfies $\boldsymbol{b}_1^T \boldsymbol{x}_t = \boldsymbol{b}^T \boldsymbol{x}_t = 0$. Therefore, if the switching sequence is such that one of the discrete states is always visited for at least two time instances, then the nullspace of L_n is at least two-dimensional. On the other hand, if there is at least one time instance t such that $\lambda_{t-3} \neq \lambda_{t-2} \neq \lambda_{t-1} = i$, then there is no $\boldsymbol{b} \neq \boldsymbol{b}_i$ such that $\boldsymbol{b}^T \boldsymbol{x}_t = 0$ for all $\boldsymbol{x}_t$ generated by the i^{th} ARX model. If this is true for all $i = 1, \ldots, n$, $\boldsymbol{h}_n$ can be uniquely obtained from the nullspace of L_n.

In practice, we would like to estimate $\boldsymbol{h}_n$ uniquely, regardless of what the model orders are and regardless of the evolution of discrete state. To this end, recall from (4) and (5) that $\boldsymbol{h}_n$ is uniquely defined as the symmetric tensor product of the vectors $\{\boldsymbol{b}_i\}_{i=1}^n$. Therefore, some of the entries of $\boldsymbol{h}_n$ must be zero, because they involve products of entries of $\boldsymbol{b}_i$ which are also zero. Since we have chosen the entries of $\boldsymbol{h}_n$ in the degree-lexicographic order and the zero entries of each $\boldsymbol{b}_i$ as the first $2(n_a - n_a(i))$ entries, the zero entries of $\boldsymbol{h}_n$ must be the first m entries, where

m is a function of n_a and $\{n_a(i)\}_{i=1}^n$. If m was known, we could readily remove the first m entries of $\boldsymbol{h}_n$ in (9) and solve uniquely for the remaining (nonzero) entries from the nullspace of the submatrix of L_n consisting of its last $M_n - m$ columns. The following theorem shows that, although m is unknown, one can still estimate it from a rank constraint on L_n and then uniquely solve for $\boldsymbol{h}_n$.

Theorem 1 (Identifying the hybrid parameters): Given input/output data $\{u_t, y_t\}_{t=0}^T$ generated by the PWARX model (1) with $a_{n_a(i)}(i) \neq 0$ and $c_{n_a(i)}(i) \neq 0$ for $i = 1, \ldots, n$, let $L_n^j \in \mathbb{R}^{(T-n_a+1)\times(M_n-j)}$ be a matrix whose columns are the last $M_n - j$ columns of L_n as defined in (9). If $T \geq M_n + \max_{i=1,\ldots,n}(n_a(i)) - 2$ and at least $2n_a(i)$ measurements come from the i^{th} ARX model, for $i = 1, \ldots, n$, then

$$\operatorname{rank}(L_n^j) \begin{cases} \leq M_n - j - 1, & j < m, \\ = M_n - j - 1, & j = m, \\ = M_n - j, & j > m. \end{cases} \tag{10}$$

Therefore, the number of leading zero entries of $\boldsymbol{h}_n$ is

$$m = \max\{j : \operatorname{rank}(L_n^j) = M_n - j - 1\}, \tag{11}$$

and the hybrid model parameters are given by

$$\boxed{\boldsymbol{h}_n = \frac{\boldsymbol{g}_n}{e_K^T \boldsymbol{g}_n},} \tag{12}$$

where $\boldsymbol{g}_n = [0, \ldots, 0, \boldsymbol{h}_n^{mT}]^T \in \mathbb{R}^{M_n}$, $\boldsymbol{h}_n^m \in \mathbb{R}^{M_n - m}$ is the unique vector in the nullspace of L_n^m, *i.e.*

$$\boxed{L_n^m \boldsymbol{h}_n^m = 0,} \tag{13}$$

and $e_K = [0, \cdots, 0, 1]^T \in \mathbb{R}^K$.

Remark 1: One of the assumptions of Theorem 1 is that $a_{n_a(i)}(i) \neq 0$ and $c_{n_a(i)}(i) \neq 0$ for $i = 1, \ldots, n$. This assumption guarantees that the nullspace of L_n^j is not further increased by having additional zeros on some $\boldsymbol{b}_i$.

Remark 2 (Identifying $\boldsymbol{h}_n$ and m from noisy data): In the presence of noise, we can still solve for $\boldsymbol{h}_n^m$ in (13), hence for the hybrid model parameters $\boldsymbol{h}_n$ in (12), in a least-squares sense: we let $\boldsymbol{h}_n^m$ be the eigenvector of $L_n^{mT} L_n^m$ associated with its smallest eigenvalue. However, we cannot directly estimate m from (11), because the matrix L_n^j may be full rank for all j. Instead, we compute m from a noisy matrix L_n as

$$m = \underset{j=0,\ldots,M_n-1}{\arg\min} \frac{\sigma^2_{M_n-j}(L_n^j)}{\sum_{k=1}^{M_n-j-1} \sigma_k^2(L_n^j)} + \mu(M_n - j), \tag{14}$$

where $\sigma_k(L_n^j)$ is the k^{th} singular value of L_n^j and μ is a parameter. The above formula for estimating m is motivated by model selection techniques [15] in which one minimizes a cost function that consists of a data fitting term and a model complexity term. The data fitting term measures how well the data is approximated by the model – in this case how close the matrix L_n^j is to dropping rank. The model complexity term penalizes choosing models of high complexity – in this case choosing a large rank. This model selection technique has worked well in our experiments.

D. Identifying the model parameters

Theorem 1 allow us to determine the hybrid model parameters $\boldsymbol{h}_n$ from input/output data $\{u_t, y_t\}_{t=0}^T$. The rest of the problem is to recover the model parameters $\{\boldsymbol{b}_i\}_{i=1}^n$ and the orders of the ARX models $\{n_a(i)\}_{i=1}^n$ from $\boldsymbol{h}_n$.

In our previous work [25], which deals with the case of ARX models of equal and known orders, we showed that one can identify the model parameters directly from the derivatives of $p_n(\boldsymbol{z})$ at a collection of n regressors $\{\boldsymbol{z}_i\}_{i=1}^n$ corresponding to each one of the n ARX models, *i.e.*

$$\boxed{\boldsymbol{b}_i = \frac{Dp_n(\boldsymbol{z}_i)}{e_K^T Dp_n(\boldsymbol{z}_i)}, \quad i = 1, \ldots, n.} \tag{15}$$

However, since the value of the discrete state λ_{t-1} is unknown, we do not know which data points in $\{\boldsymbol{x}_t\}_{t=n_a}^T$ correspond to which ARX model, hence we do not know how to choose the regressors $\{\boldsymbol{z}_i\}_{i=1}^n$. In [25] we proposed a simple algebraic algorithm that obtains the regressors $\{\boldsymbol{z}_i\}_{i=1}^n$ from the roots of a univariate polynomial as follows. One first chooses a line, $\mathcal{L} = \{\boldsymbol{z}_0 + \alpha \boldsymbol{v}, \alpha \in \mathbb{R}\}$, in $\mathbb{R}^K$ with base point $\boldsymbol{z}_0$ and direction $\boldsymbol{v}$ and uses it to build a univariate polynomial $q_n(\alpha) = p_n(\boldsymbol{z}_0 + \alpha \boldsymbol{v})$. Then one chooses the regressors as $\boldsymbol{z}_i = \boldsymbol{z}_0 + \alpha_i \boldsymbol{v}$, for $i = 1, \ldots, n$, where $\{\alpha_i\}_{i=1}^n$ are the n roots of $q_n(\alpha)$. By construction these points satisfy $p_n(\boldsymbol{z}_i) = 0$ and $\boldsymbol{b}_i^T \boldsymbol{z}_i = 0$, as needed.

As it turns out, the above solution does not depend on whether some entries of $\boldsymbol{b}_i$ or $\boldsymbol{h}_n$ are zero or not. Therefore, in the case of ARX models with unknown and possibly different orders, one may obtain the model parameters as in (15). Furthermore, in the case of perfect data the first m_i entries of each $\boldsymbol{b}_i$ will automatically be zero, hence we can obtain the orders of each ARX model as

$$\boxed{n_a(i) = n_a - \frac{m_i}{2}, \quad i = 1, \ldots, n.} \tag{16}$$

In the case of noisy data, however, the performance of this purely algebraic scheme will depend on the choice of the parameters $\boldsymbol{z}_0$ and $\boldsymbol{v}$ that define the randomly chosen line. Therefore, we now propose a new algorithm that chooses the regressors $\{\boldsymbol{z}_i\}_{i=1}^n$ in a more robust fashion. The main idea is to choose points from the data set $\{\boldsymbol{x}_t\}_{t=n_a}^T$ that are "as close as possible" to one of the ARX models in the sense of minimizing the distance $|\boldsymbol{b}_i^T \boldsymbol{x}_t|$. Notice that in the case of zero-mean white Gaussian noise added to the PWARX model in (1), minimizing the distance $|\boldsymbol{b}_i^T \boldsymbol{x}_t|$ corresponds to choosing a measurement that results in small standard deviation. However, since we do not yet know the model parameters $\boldsymbol{b}_i$, we do not know how to evaluate such a distance in the first place. The following lemma allows us to compute a first order approximation to such a distance, *without* having to know the model parameters in advance.

Lemma 1: Let $\tilde{\boldsymbol{x}} \in \mathbb{R}^K$ be the projection of a point $\boldsymbol{x} \in \mathbb{R}^K$ onto the algebraic variety $\mathcal{V} = \{\boldsymbol{z} : p_n(\boldsymbol{z}) = 0\}$. Then the Euclidean distance from $\boldsymbol{x}$ to $\mathcal{V}$ is given by

$$\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\| = \frac{|p_n(\boldsymbol{x})|}{\|(I - e_K e_K^T) Dp_n(\boldsymbol{x})\|} + O\big(\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|^2\big). \tag{17}$$

Thanks to Lemma 1, we can now choose a point corresponding to say the n^{th} ARX model as

$$\boldsymbol{z}_n = \underset{\boldsymbol{x}\in\{\boldsymbol{x}_t\}_{t=n_a}^T}{\arg\min} \frac{|p_n(\boldsymbol{x})|}{\|(I-e_K e_K^T)Dp_n(\boldsymbol{x})\|}. \tag{18}$$

Given $\boldsymbol{z}_n$, we can immediately compute the vector of parameters $\boldsymbol{b}_n$ as in (15). Given $\boldsymbol{b}_n$, we can divide the polynomial of degree n, $p_n(\boldsymbol{z})$, by the polynomial of degree 1, $\boldsymbol{b}_n^T\boldsymbol{z}$, to obtain a polynomial of degree $n-1$

$$p_{n-1}(\boldsymbol{z}) \doteq \frac{p_n(\boldsymbol{z})}{\boldsymbol{b}_n^T\boldsymbol{z}} = \frac{\boldsymbol{h}_n^T\nu_n(\boldsymbol{z})}{\boldsymbol{b}_n^T\boldsymbol{z}} = \boldsymbol{h}_{n-1}^T\nu_{n-1}(\boldsymbol{z}). \tag{19}$$

Notice that given $\boldsymbol{h}_n \in \mathbb{R}^{M_n}$ and $\boldsymbol{b}_n \in \mathbb{R}^K$, solving for $\boldsymbol{h}_{n-1} \in \mathbb{R}^{M_{n-1}}$ is simply a linear problem of the form $\mathcal{D}_n(\boldsymbol{b}_n)\boldsymbol{h}_{n-1} = \boldsymbol{h}_n$, where $\mathcal{D}_n(\boldsymbol{b}_n) \in \mathbb{R}^{M_n\times M_{n-1}}$. Now, by definition of p_{n-1}, points $\{\boldsymbol{x}_t\}$ such that $p_{n-1}(\boldsymbol{x}_t)=0$ must correspond to one of the remaining $(n-1)$ ARX models. Thus we can choose a new point $\boldsymbol{z}_{n-1}$ from the data set that minimizes $|p_{n-1}(\boldsymbol{x})|/\|(I-e_Ke_K^T)Dp_{n-1}(\boldsymbol{x})\|$. By repeating this procedure for the remaining ARX models, we obtain the following algorithm for computing the points $\{\boldsymbol{z}_i\}_{i=1}^n$ and the corresponding model parameters $\{\boldsymbol{b}_i\}_{i=1}^n$:

for $i = n:1$,

$$\boldsymbol{z}_i = \underset{\boldsymbol{x}\in\{\boldsymbol{x}_t\}}{\arg\min} \frac{|p_i(\boldsymbol{x})|}{\|(I-e_Ke_K^T)Dp_i(\boldsymbol{x})\|},$$
$$\boldsymbol{b}_i = \frac{Dp_i(\boldsymbol{z}_i)}{e_K^T Dp_i(\boldsymbol{z}_i)},$$
$$\boldsymbol{h}_{i-1} = \mathcal{D}_i(\boldsymbol{b}_i)^\dagger\boldsymbol{h}_i, \quad p_{i-1}(\boldsymbol{x}) = \boldsymbol{h}_{i-1}^T\nu_{i-1}(\boldsymbol{x})$$

end

where $A^\dagger$ is the pseudo-inverse of A.

Given the model parameters $\{\boldsymbol{b}_i\}_{i=1}^n$, the order of each ARX model can be determined as in (16). However, in the presence of noise, the first m_i entries of each $\boldsymbol{b}_i$ will not be exactly zero. In this case, one may determine the number of zero entries of $\boldsymbol{b}_i$ as the first m_i entries whose absolute value is below a threshold $\epsilon > 0$. Alternatively (see Remark 2), we can use model selection techniques to determine m_i as

$$m_i = \underset{j=0,2,\ldots,K-3}{\arg\min} \frac{b_{ij}^2}{\sum_{k=j+1}^K b_{ik}^2} + \mu(K-j),$$

where b_{ik} is the k^{th} entry if $\boldsymbol{b}_i$, $b_{i0}=0$ by convention, and μ is a parameter.

E. Estimation of the discrete state

Given the model parameters $\{\boldsymbol{b}_i\}_{i=1}^n$, the discrete state can be estimated as [25]

$$\boxed{\lambda_{t-1} = \arg\min_{i=1,\ldots,n} (\boldsymbol{b}_i^T\boldsymbol{x}_t)^2,} \tag{20}$$

because for each time $t \geq n_a$ there exists a generally unique[1] i such that $\boldsymbol{b}_i^T\boldsymbol{x}_t = 0$.

[1]In principle, it is possible that a data point $\boldsymbol{x}_t$ belongs to more than one hyperplane $\boldsymbol{b}_i^T\boldsymbol{z}=0$. However, the set of all such points is a zero measure set on the variety $\{\boldsymbol{z} : p_n(\boldsymbol{z}) = 0\}$.

III. EXAMPLE: A SWITCHING CIRCUIT

In this section we present a numerical example that illustrates the proposed algorithm in the absence of noise, as well as simulation results with noisy data.

A. Algebraic example

We consider the switching circuit shown in Fig. 1 where the input u is the voltage in the source and the output y is the current in the inductance. The circuit can be modeled as a continuous-time hybrid system with $n=2$ discrete states, corresponding to whether the switch is connected to the capacitor or to the resistor. The dynamics in each state are described by the linear differential equations

$$\ddot{y} + 2\dot{y} + 4y = \frac{1}{2}\dot{u} \quad \text{and} \quad \dot{y} + 3y = \frac{1}{2}u, \tag{21}$$

respectively. By discretizing the above differential equations with a sampling time τ we obtain a PWARX model with $n_a(1)=2$ and $n_a(2)=1$ consisting of the two ARX models

$$y_t = -2(\tau-1)y_{t-1} - (1-2\tau+4\tau^2)y_{t-2} + \frac{\tau}{2}u_{t-1} - \frac{\tau}{2}u_{t-2}$$

and

$$y_t = (1-3\tau)y_{t-1} + \frac{\tau}{2}u_{t-1},$$

respectively. If we assume $n_a=2$ and $\tau=0.2$, then we have $K=5$, $M_n=15$ and

$$\begin{aligned}
\boldsymbol{x}_t &= [u_{t-2}, y_{t-2}, u_{t-1}, y_{t-1}, -y_t] \in \mathbb{R}^5\\
\boldsymbol{z} &= [z_1, z_2, z_3, z_4, z_5]^T \in \mathbb{R}^5\\
\boldsymbol{b}_1 &= [-0.1, -0.76, 0.1, 1.6, 1]^T \in \mathbb{R}^5\\
\boldsymbol{b}_2 &= [0, 0, 0.1, 0.4, 1]^T \in \mathbb{R}^5.
\end{aligned}$$

By generating data in discrete time for the switching circuit, we obtain $\text{rank}(L_2)=14$, $\text{rank}(L_2^1)=13$, $\text{rank}(L_2^2)=12$ and $\text{rank}(L_2^j)=15-j$ for $j\geq 3$. Therefore the number of zeros of $\boldsymbol{h}_2$ is $m=2$, and we can obtain $\boldsymbol{h}_2$ from the nullspace of L_2^2. Notice that in this particular example one can also obtain $\boldsymbol{h}_2$ from the nullspace of L_n or L_n^1, which are also one-dimensional. This is predicted by Theorem 1, since $\text{rank}(L_n^j) \leq M_n - j$ when $j \leq m$ (See also Example 1). However, in general we will have $\text{rank}(L_n^j) < M_n - j$, hence it is preferable to remove the zeros before computing $\boldsymbol{h}_2$.

Given $\boldsymbol{h}_2$, we can build the polynomial $p_n(\boldsymbol{z})$, as

$$\begin{aligned}
p_n(\boldsymbol{z}) &= (-0.1z_1 - 0.76z_2 + 0.1z_3 + 1.6z_4 + z_5)(0.1z_3 + 0.4z_4 + z_5)\\
&= -0.01z_1z_3 - 0.04z_1z_4 - 0.1z_1z_5 - 0.076z_2z_3 - 0.304z_2z_4\\
&\quad - 0.76z_2z_5 + 0.01z_3^2 + 0.2z_3z_4 + 0.2z_3z_5 + 0.64z_4^2 + 2z_4z_5 + z_5^2\\
&= [0, 0, -0.01, -0.04, -0.1, 0, -0.076, -0.304, -0.76,\\
&\qquad 0.01, 0.2, 0.2, 0.64, 2, 1]^T\nu_2(\boldsymbol{z})\\
&= \boldsymbol{h}_2^T\nu_2(\boldsymbol{z}),
\end{aligned}$$

and obtain its partial derivatives as

$$Dp_2(\boldsymbol{z}) = \begin{bmatrix} -0.01z_3 - 0.04z_4 - 0.1z_5\\ -0.076z_3 - 0.304z_4 - 0.76z_5\\ -0.01z_1 - 0.076z_2 + 0.02z_3 + 0.2z_4 + 0.2z_5\\ -0.04z_1 - 0.304z_2 + 0.2z_3 + 1.28z_4 + 2z_5\\ -0.1z_1 - 0.76z_2 + 0.2z_3 + 2z_4 + 2z_5 \end{bmatrix}.$$

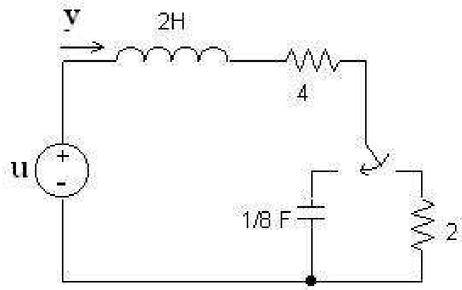

Fig. 1. A switching first and second order circuit

Assuming $y_0 = y_1 = 0$, $u_0 = 10$, $u_1 = 5$, $u_1 = 0$, $\lambda_1 = 1$ and $\lambda_2 = 2$ we obtain $y_2 = -0.5$ and $y_3 = -0.2$. Thus we obtain the data points $\boldsymbol{x}_2 = [10, 0, 5, 0, 0.5]^T$ and $\boldsymbol{x}_3 = [5, 0, 0, -0.5, 0.2]^T$. Evaluating Dp_2 at $\boldsymbol{z}_1 = \boldsymbol{x}_2$ and $\boldsymbol{z}_2 = \boldsymbol{x}_3$ we obtain

$$[Dp_2(\boldsymbol{z}_1) \qquad Dp_2(\boldsymbol{z}_2)] = [\boldsymbol{b}_1 \quad -1.1\boldsymbol{b}_2], \tag{22}$$

which shows how we can effectively recover the model parameters $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ and the dimensions $n_a(1)$ and $n_a(2)$ from the derivatives of p_2.

B. Simulation results

We now present simulation results showing the performance of the proposed algorithm when the output measurements are corrupted with zero-mean white Gaussian noise with standard deviation $\sigma \in [0, 0.01]$. The input u is chosen as a zero-mean white Gaussian noise with unit standard deviation. The discrete state starts at $\lambda_1 = 1$ and then switches periodically between the two discrete states a) every 10 seconds, b) every 5 seconds, c) every second, or d) every 0.4 seconds. Recall that the sampling time is $\tau = 0.2$ seconds.

Table I shows the mean error over 10,000 trials for the estimation of the number of zeros of the hybrid model parameters, m, the model orders $\{n_a(i)\}_{i=1}^n$, the model parameters $\{\boldsymbol{b}_i\}_{i=1}^n$ and the discrete state $\{\lambda_{t-1}\}_{t=n_a}^T$ as a function of the level of noise σ. The error E_m between the estimated number of zeros[2] $\hat{m}$ and the true number of zeros m was computed as the percentage of trials for which $\hat{m} \neq m$. The error $E_{n_a(i)}$ between the estimated order $\hat{n}_a(i)$ and the true order $n_a(i)$ was estimated as the percentage of trials for which $\hat{n}_a(i) \neq n_a(i)$. The error $E_{\boldsymbol{b}}$ between the estimated model parameters $(\hat{c}_2, \hat{a}_2, \hat{c}_1, \hat{a}_1)$ and the true model parameters (c_2, a_2, c_1, a_1) was computed as $\|(\hat{c}_2, \hat{a}_2, \hat{c}_1, \hat{a}_1) - (c_2, a_2, c_1, a_1)\|$, averaged over the number of models and the number of trials. The error E_λ between the estimated discrete state $\hat{\lambda}_t$ and the true discrete state λ_t was computed as the percentage of times in which $\hat{\lambda}_t \neq \lambda_t$, averaged over the number of trials.

[2]The number of zeros was estimated as described in Remark 2, with the weight of the complexity term chosen as $\mu = 2 \times 10^{-8}$.

Notice that the number of zeros, the model orders, the model parameters and the discrete state are perfectly estimated when $\sigma = 0$. For $\sigma > 0$, the model selection based algorithm for estimating the rank of noisy matrices gives an estimate of the number of zeros m which is correct over 97% of the times. The model orders are correctly estimated over 90% of the times for system 1 and over 70% of the times for system 2. This suggests that, as expected, it is harder to estimate the orders $n_a(i)$ for which $n_a - n_a(i)$ is larger. On the other hand, the estimation errors $E_{\boldsymbol{b}}$ and E_λ increase approximately linearly with the amount of noise, as shown in Fig. 2. Notice also that the errors tend to increase when the switching times are either too close or too far. Indeed, the best performance is obtained when the switches are separated by 1 second. Intuitively, one would expect that it is easier to identify a slowly switching PWARX model. However, as shown by Example 1, data points at the switching times provide independent equations that typically increase the rank of L_n. In the presence of noise, this makes the linear system $L_n^m \boldsymbol{h}_n^m = 0$ better conditioned, especially when m is incorrectly estimated. A deeper theoretical understanding of the effect of fast versus slow switching in the algorithm's performance is part of our future work.

TABLE I

Error in the estimation of the number of zeros E_m, the model orders $E_{n_a(1)}$, $E_{n_a(2)}$, the model parameters $E_{\boldsymbol{b}}$ and the discrete state E_λ for different levels of noise with standard deviation σ and for different dwell times: a) 10 seconds, b) 5 seconds, c) 1 second and d) 0.4 seconds.

σ		0.000	0.002	0.004	0.006	0.008	0.010
	a)	0.00	0.00	0.00	0.00	0.05	2.05
E_m	b)	0.00	0.00	0.00	0.00	0.04	2.46
	c)	0.00	0.00	0.00	0.00	0.00	1.09
	d)	0.00	0.00	0.00	0.06	0.19	0.47
	a)	0.00	0.00	0.13	0.16	0.08	0.09
$E_{n_a(1)}$	b)	0.00	0.00	0.00	0.00	0.00	0.01
	c)	0.00	0.00	0.00	0.00	0.05	0.55
	d)	0.00	0.00	0.00	0.38	3.10	8.23
	a)	0.00	0.00	0.01	0.22	1.93	7.38
$E_{n_a(2)}$	b)	0.00	0.00	0.00	0.20	2.33	9.02
	c)	0.00	0.00	0.00	0.00	0.14	1.36
	d)	0.00	0.00	0.04	3.84	14.96	29.41
	a)	0.000	0.0176	0.0442	0.0772	0.1092	0.1422
$E_{\boldsymbol{b}}$	b)	0.000	0.0193	0.0491	0.0809	0.1123	0.1444
	c)	0.000	0.0191	0.0382	0.0619	0.0837	0.1162
	d)	0.000	0.0266	0.0544	0.0814	0.1077	0.1408
	a)	0.000	3.744	8.187	13.464	18.777	23.174
E_λ	b)	0.000	5.710	12.404	17.951	22.531	25.983
	c)	0.000	4.499	7.739	10.893	14.159	17.243
	d)	0.000	4.747	10.050	15.358	19.299	23.120

IV. Conclusions and Open Issues

We have proposed an algebraic geometric solution to the identification ARX hybrid models of unknown and possibly different orders. By representing the number of discrete states n as the degree of a polynomial p and encoding the orders and the model parameters as factors of p, we showed that one can solve the identification problem using

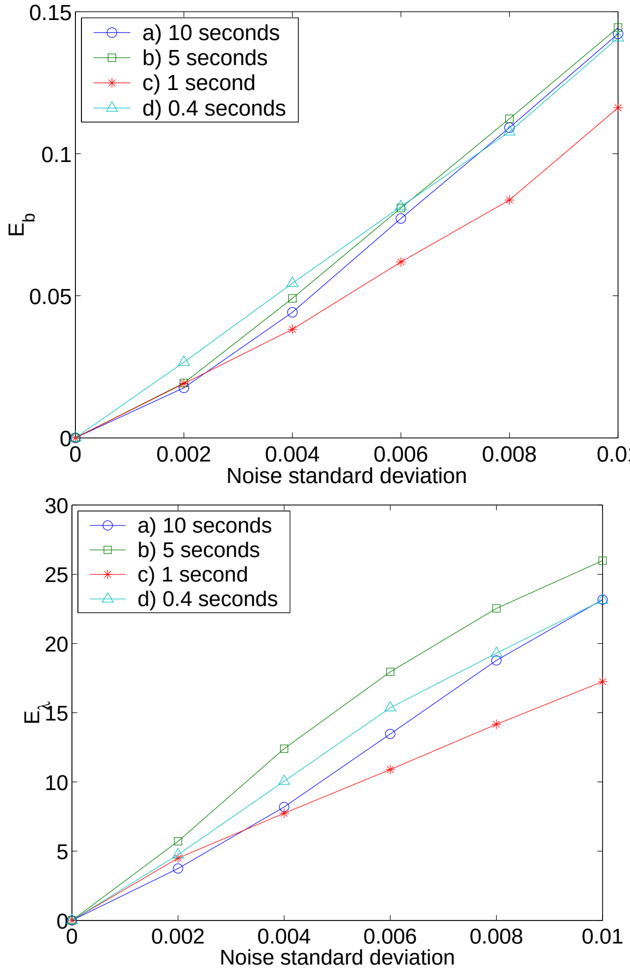

Fig. 2. Error in the estimation of the model parameters E_b and the discrete state E_λ for different levels of noise with standard deviation σ and for different dwell times: a) 10 seconds, b) 5 seconds, c) 1 second and d) 0.4 seconds.

simple linear-algebraic techniques: the coefficients of p can be obtained from a linear system, and the orders and parameters of each ARX model from the derivatives of p. We presented simulation results evaluating the performance of the algorithm on a switching circuit with noisy data.

Open issues include a detailed analysis of the robustness of the algorithm with noisy data, relaxing the assumptions that some of the parameters may not be zero (see Remark 1), and a deeper theoretical understanding of the effect of fast versus slow switching in the algorithm's performance. Extensions include imposing stability constraints on the estimation of the model parameters, developing recursive algorithms for the on-line identification of the model parameters, and dealing with PWARX systems with multiple inputs and multiple outputs (MIMO) by exploring connections with recent developments on subspace clustering [23].

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