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On the Domain and Error Characterization in the Singular Perturbation Modeling of Closed Kinematic Chains

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Abstract— **This paper addresses two important issues in modeling Closed Kinematic Chains (CKC) as singularly perturbed systems, namely, the validity domain and the error characterization. The Singular Perturbation Formulation (SPF) is obtained by replacing the algebraic constraint equation in an index-1 Differential Algebraic Equations (DAE) model with an artificial fast dynamics. We first show that the SPF model has a larger validity domain than the DAE model, and boundaries of the domain are easy to determine. We then characterize the error between the SPF model and the DAE model by deriving explicit error bounds. Sufficient conditions that guarantee exponential convergence of the model error are established. We verify the analysis by simulating the dynamics of a CKC mechanism, the Rice Planar Delta Robot, and validating the simulation results with experimental data obtained on the real robot.**

I. Introduction

The dynamics of Closed Kinematic Chains (CKC) are conventionally described by differential-algebraic equations (DAE). Dynamic equations in DAE form pose difficulties for simulation and control design. The DAE that characterize CKC are of index-3 and they are difficult to simulate due to numerical ill-conditioning [1]. On the other hand, the difficulty with control of the DAE formulation of CKC lies in the fact that most control design techniques are devised for explicit state space models.

Efforts have been made to extend the wealth of results from the control of Open Kinematic Chains (OKC) to the control of CKC. A recent result is the “reduced model” proposed in [2], which is in terms of independent coordinates and enables model-based control design and implementation [3]. This model also presents two challenges. First, it is based on an implicit transformation from independent coordinates to dependent coordinates, which is valid only (locally) in a compact domain. As a prerequisite for stability analysis, the boundaries of the compact domain need to be explicitly characterized. This is no easy task for general closed chains. Nevertheless, conservative estimates have been developed in [2] and [3]. The second challenge is that, since the transformation is implicit, effective numerical schemes must be devised for real-time control implementation. The above difficulties suggest the need for an alternative to directly considering the DAE formulation as the basis for control design.

In [4], an index-1 DAE formulation of the dynamics of CKC, from which the reduced model is derived, was combined with a realization method from [5] to construct a singularly perturbed system as an ODE approximation to the DAE system. The idea consists of replacing the algebraic constraints in the DAE system with artificially introduced fast dynamics characterized by a small perturbation parameter ϵ. The transformed system is thus called Singular Perturbation Formulation (SPF). The motivations are first the well-known asymptotic connection between singular perturbation systems and corresponding reduced DAE systems [6] [7], and second the reality that more established control design and stability analysis techniques are available for singularly perturbed ODE systems than for DAE systems [8]. The proposed SPF model was shown to have the following properties that can facilitate control design [4]:

- *P1* The DAE characterizing a CKC are approximated using a singularly perturbed formulation in which the slow second-order differential equations are equal in number to the degrees of freedom (DOF) of the system.
- *P2* The validity domain of this singularly perturbed system contains the entire singularity-free workspace of the CKC. This property is investigated in this paper.

For the SPF model to be a valid approximation to the index-1 DAE model, the error between these two systems needs to be characterized. We will give explicit upper bound for the model error and compare our results to those from the standard Tikhonov’s theorem. In this paper, we make three main contributions:

- *C1* Explicit bounds for the error between the SPF model and the original DAE model are derived. Sufficient conditions for the exponential convergence of the error are established.
- *C2* Numerical simulations as well as experiments on a closed chain mechanism, the Rice Planar Delta Robot (RPDR), are performed to support the analytical results.
- *C3* The validity domain of the SPF model is characterized based on the work in [4]. The SPF model is shown to have a larger domain than the index-1 DAE model, hence much larger than the reduced model.

The remainder of the paper is organized as follows.

Section II describes the singular perturbation approach and characterizes the validity domain of the SPF model. Section III gives explicit error bounds and establishes stability properties which guarantee exponential convergence of the error. In section IV we illustrate the analysis by simulating the dynamics of the RPDR and validating the simulation results with experimental data. Finally, section V concludes this paper and outlines future work.

II. A Singular Perturbation Model for Closed Kinematic Chains

In [4], an index-1 DAE model originated in [2] was used to develop the singular perturbation approach. In this section we first review the SPF model developed in [4], and then characterize its validity domain.

As shown in [2], an n DOF closed chain is considered to be an n' DOF holonomic system (free system containing only open chains) to which $p = n' - n$ independent holonomic constraints are imposed. The dynamics of the constrained system is completely described by the following index-3 DAE

$$\begin{cases} \mathbf{D}'(\mathbf{q}')\ddot{\mathbf{q}}' + \mathbf{C}'(\mathbf{q}', \dot{\mathbf{q}}')\dot{\mathbf{q}}' + \mathbf{g}'(\mathbf{q}') = \phi_{q'}^T(\mathbf{q}')\lambda \\ \phi(\mathbf{q}') = 0 \end{cases} \tag{1}$$

where [1] $\mathbf{q}' \in \mathbf{V}' \subset \Re^{n'}$ is the vector of dependent generalized coordinates, typically representing all the joint positions, and $\mathbf{V}'$ denotes the singularity-free workspace in $\Re^{n'}$ (as defined below). $\phi(\mathbf{q}') = 0$ denotes the p constraints, where $\phi(\mathbf{q}')$ is at least twice continuously differentiable. λ is the p-vector of Lagrange multipliers. $\mathbf{D}'(\mathbf{q}')$ represents the $n \times n$ inertia matrix, $\mathbf{C}'(\mathbf{q}', \dot{\mathbf{q}}')\dot{\mathbf{q}}'$ represents the Coriolis and centrifugal terms, $\mathbf{g}'(\mathbf{q}')$ represents the gravity terms.

Due to the constraints, the generalized coordinates $\mathbf{q}'$ are confined to the reachable workspace

$$\mathbf{U}' = \{\mathbf{q}' \in \Re^{n'} : \phi(\mathbf{q}') = 0\} \subset \Re^{n'}, \tag{2}$$

and the constrained system has n degrees of freedom. Hence there exists a minimum set of $n-$independent generalized coordinates $\mathbf{q} \in \Re^n$ which can describe the constrained dynamics. The independent coordinates $\mathbf{q}$ can be chosen to satisfy the twice continuously differentiable parameterization $\mathbf{q} = \alpha(\mathbf{q}')$. We further define $\psi(\mathbf{q}') \triangleq \begin{bmatrix} \phi(\mathbf{q}') \\ \alpha(\mathbf{q}') \end{bmatrix}$, $\psi_{q'}(\mathbf{q}') \triangleq \frac{\partial \psi}{\partial \mathbf{q}'}$, $\overline{\psi}(\mathbf{q}', \mathbf{q}) \triangleq \begin{bmatrix} \phi(\mathbf{q}') \\ \alpha(\mathbf{q}') \end{bmatrix} - \begin{bmatrix} 0 \\ \mathbf{q} \end{bmatrix}$, and $\overline{\psi}_{q'}(\mathbf{q}') \triangleq \frac{\partial \overline{\psi}}{\partial \mathbf{q}'}$. It can be shown [2] that $\dot{\mathbf{q}}' = \rho(\mathbf{q}')\dot{\mathbf{q}}$ with $\rho(\mathbf{q}') = \psi_{q'}^{-1}(\mathbf{q}') \begin{bmatrix} 0_{p \times n} \\ I_{n \times n} \end{bmatrix}$. This leads to the definition of the singularity-free workspace

$$\mathbf{V}' = \{\mathbf{q}' \in \mathbf{U}' : \det[\overline{\psi}_{q'}(\mathbf{q}')] \neq 0\} \subset \mathbf{U}'. \tag{3}$$

The principle of virtual work was used in [2] to eliminate the Lagrange multipliers in the index-3 DAE (1). The resulting equations of motion is in the form of an index-1 DAE

$$\begin{cases} \mathbf{D}(\mathbf{q}')\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}', \dot{\mathbf{q}}')\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}') = 0 \\ \mathbf{D}(\mathbf{q}') = \rho(\mathbf{q}')^{\mathrm{T}}\mathbf{D}'(\mathbf{q}')\rho(\mathbf{q}') \\ \mathbf{C}(\mathbf{q}', \dot{\mathbf{q}}') = \rho(\mathbf{q}')^{\mathrm{T}}\mathbf{C}'(\mathbf{q}', \dot{\mathbf{q}}')\rho(\mathbf{q}') \\ \qquad\qquad + \rho(\mathbf{q}')^{\mathrm{T}}\mathbf{D}'(\mathbf{q}')\dot{\rho}(\mathbf{q}', \dot{\mathbf{q}}') \\ \mathbf{g}(\mathbf{q}') = \rho(\mathbf{q}')^{\mathrm{T}}\mathbf{g}'(\mathbf{q}') \\ \dot{\mathbf{q}}' = \rho(\mathbf{q}')\dot{\mathbf{q}} \\ \phi(\mathbf{q}') = 0 \end{cases} \tag{4}$$

where $\mathbf{q}' \in \mathbf{V}' \subset \Re^{n'}$, $\mathbf{q} \in \alpha(\mathbf{V}') \subset \Re^n$. Note that the above index-1 DAE model is the common basis from which the reduced model of [2] and the SPF model of [4] are derived.

Next we briefly review the reduced model of [2]. If we solve the constraint equation in (4), equivalently $\bar{\psi}(\mathbf{q}', \mathbf{q}) = 0$ for $\mathbf{q}'$ in terms of $\mathbf{q}$, we end up with an ODE system. For any given point $\mathbf{q}'_\star \in \mathbf{V}'$, let $\bar{\psi}(\mathbf{q}'_\star, \mathbf{q}_\star) = 0$. In [3] an explicit estimate of a compact set Ω centered at $\mathbf{q}_\star$ was characterized such that for each $\mathbf{q} \in \Omega$, there exists a unique $\mathbf{q}' \in \mathbf{W}' \subset \mathbf{V}'$ satisfying $\mathbf{q}' = \sigma(\mathbf{q})$, where $\mathbf{W}'$ denotes the corresponding set of $\mathbf{q}'$ centered at $\mathbf{q}'_\star$. The reduced model in terms of independent coordinates $\mathbf{q}$ are given by [2]

$$\begin{cases} \mathbf{D}(\mathbf{q}')\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}', \dot{\mathbf{q}}')\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}') = 0 \\ \dot{\mathbf{q}}' = \rho(\mathbf{q}')\dot{\mathbf{q}} \\ \mathbf{q}' = \sigma(\mathbf{q}) \end{cases} \tag{5}$$

where $\mathbf{q}' \in \mathbf{W}' \subset \Re^{n'}$, $\mathbf{q} \in \Omega \subset \Re^n$. Note that the reduced model (5) is only valid in Ω where the transformation $\mathbf{q}' = \sigma(\mathbf{q})$ exists. This makes the reduced model different from explicit models of open chain mechanical systems in two aspects. First, as a prerequisite for control design and stability analysis, the boundaries of the compact domain Ω need to be explicitly characterized. This is not easy for general closed chains. Explicit estimates of the domain have been reported in [2] and [3]. Second, an effective numerical algorithm for solving the nonlinear algebraic constraints must be sought for implementing model-based control. This problem is addressed in [3] where guaranteed convergence to prescribed precision within a fixed number of iterations is achieved using a modified Newton iteration.

An alternative to directly considering the index-1 DAE (4) as a basis for control design is the singular perturbation approach proposed in [4], where the index-1 DAE of (4) was transformed into a singularly perturbed ODE system (the SPF model). In practice, the independent coordinates $\mathbf{q}$ in (4) are often chosen as components of $\mathbf{q}'$ corresponding to actuated joints. We denote the remaining components as $\mathbf{z}$ and rewrite the index-1 DAE model in (4) as

$$\begin{cases} \mathbf{D}(\mathbf{q}, \mathbf{z})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{z})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}, \mathbf{z}) = \mathbf{u} \\ \phi(\mathbf{q}, \mathbf{z}) = 0. \end{cases} \tag{6}$$

[1] In this paper, we use the following standard notation and terminology: $\Re$ denotes the set of real numbers, and $\Re^n$ denotes the usual n-dimensional vector space over $\Re$ endowed with the Euclidean norm $\|\mathbf{x}\| = \left\{\sum_{i=1}^n {x_i}^2\right\}^{\frac{1}{2}}$. $\Re^{n \times m}$ denotes the set of all $n \times m$ matrices with real elements. Unless otherwise specified, for $M \in \Re^{n \times n}$, $\|M\|$ is the induced–2 matrix norm of M corresponding to the Euclidean vector norm on $\Re^n$.

Next we replace the algebraic constraint in (6) with a fast dynamics ODE in terms of violation of the constraints, $\dot{\mathbf{w}} = -\frac{1}{\epsilon}\mathbf{w}$, with $\mathbf{w} \stackrel{\triangle}{=} \phi(\mathbf{q},\mathbf{z})$. Note that ϵ is a small positive parameter. We obtain a singularly perturbed system,

$$\begin{cases} \mathbf{D}(\mathbf{q},\mathbf{z})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}},\mathbf{z})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q},\mathbf{z}) = \mathbf{u} \\ \epsilon\phi_z(\mathbf{q},\mathbf{z})\dot{\mathbf{z}} = -\phi(\mathbf{q},\mathbf{z}) - \epsilon\phi_q(\mathbf{q},\mathbf{z})\dot{\mathbf{q}}. \end{cases} \tag{7}$$

where $\phi_z = \frac{\partial \phi}{\partial \mathbf{z}}$ and $\phi_q = \frac{\partial \phi}{\partial \mathbf{q}}$.

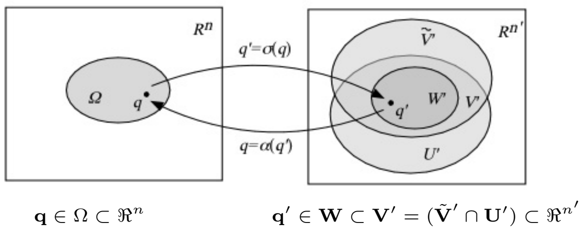


Fig. 1. Relations among Domains $\Omega, \mathbf{W}', \mathbf{V}', \tilde{\mathbf{V}}'$ and $\mathbf{U}'$

The proposed Singular Perturbation Formulation (SPF) in (7) has two attractive properties. *C1*) The fast dynamics of the constraint error always die out rapidly making the overall SPF system converge to a slow subsystem with the dimension equal in number to the DOF of the system. Since a minimal-order dynamics model is preferred by most existing control design techniques, once an appropriate controller is devised based on the augmented SPF system (7), it would be a promising candidate for the control of the index-1 DAE system. *C2*) It can be shown that the validity domain of the proposed SPF model, namely $\tilde{\mathbf{V}}'$, contains the entire singularity-free workspace $\mathbf{V}'$. Thus the SPF model is capable of approximating the index-1 DAE model over the entire scope of its domain. Furthermore, this property implies that the SPF system may be exploited as a better basis for control design than the index-1 DAE model because its domain of validity is much larger and much more useful for performing stability analysis. Next we summarize property *C2* as the following Lemma.

Lemma 2.1: The validity domain of the augmented SPF system (7), namely $\tilde{\mathbf{V}}'$, is given by

$$\tilde{\mathbf{V}}' = \{(\mathbf{q},\mathbf{z}) \in \Re^{n'} : \det[\phi_z(\mathbf{q},\mathbf{z})] \neq 0\} \subset \Re^{n'}.$$

Furthermore, $\tilde{\mathbf{V}}'$ contains the entire singularity free workspace $\mathbf{V}'$, i.e. $\mathbf{V}' \subset \tilde{\mathbf{V}}'$.

Proof: Details are in [9].

Remark 2.1: A by-product of Lemma 2.1 is that the boundary of $\tilde{\mathbf{V}}'$ can be found by solving the nonlinear equation $\det[\phi_z(\mathbf{q},\mathbf{z})] = 0$. This can be achieved by applying Newton iterations and in general requires less effort compared to characterizing the boundary of the validity domain of the reduced model Ω, where a one-to-one mapping $\mathbf{q}' = \sigma(\mathbf{q})$ needs to be guaranteed.

Figure 1 gives a conceptual representation of the relations among domains $\Omega, \mathbf{W}', \mathbf{U}', \mathbf{V}'$, and $\tilde{\mathbf{V}}'$. Note that the singularity-free workspace $\mathbf{V}'$ of a parallel robot may be physically not connected (see, e.g., [10]). We next consider the Rice Planar Delta Robot (RPDR) as an illustrative example for Lemma 2.1. Pictures of the RPDR are shown in Figure 4 in section IV. We choose $\mathbf{q}' = [q_1, q_2, q_3, q_4]^T$ (therefore $\mathbf{V}' \subset \Re^4$ and $\tilde{\mathbf{V}}' \subset \Re^4$), $\mathbf{q} = [q_1, q_2]^T$ and $\mathbf{z} = [q_3, q_4]^T$. Here q_i is defined as the angle made by link-i with respect to the horizontal line. It can be shown [4] that Direct Kinematics (DK) singularities occur when $\det[\phi_z(\mathbf{q},\mathbf{z})] = 0$, i.e. $q_3 - q_4 = n\pi$ for $n = 0, \pm 1, \pm 2, \ldots$. Figure 2 shows type-1 and type-2 singularities for odd and even n respectively. For the RPDR, the direct kinematics has two solutions for each non-singular $\mathbf{q}$, one of which is depicted by region-2 and region-4 (darkly shaded) in Figure 2, and the other is depicted by region-1 and region-3 (lightly shaded).

Denote the one-to-one mapping from $\mathbf{q}$ to $\mathbf{z}$ by $\mathbf{z} = \mathbf{h}(\mathbf{q})$. Then the singularity-free workspace $\mathbf{V}'$ in (3) is given by $\mathbf{V}' = \{(\mathbf{h}^{-1}(\mathbf{z}), \mathbf{z}) \in \Re^4,\ \mathbf{z} \in \bigcup_{i=1}^4 \text{Region-}i\}$, and the definition domain of the SPF model, $\tilde{\mathbf{V}}'$, is given by $\tilde{\mathbf{V}}' = \{(\mathbf{q},\mathbf{z}) \in \Re^4,\ \mathbf{q} \in \Re^2,\ \mathbf{z} \in \bigcup_{i=1}^4 \text{Region-}i\prime\}$. Notice that the unreachable area-2 for $\mathbf{V}'$ is "reachable" for $\tilde{\mathbf{V}}'$ since the kinematic constraint as in (4) may or may not be satisfied for the SPF model. Thus Figure 2 and Figure 3 clearly show that for the RPDR $\mathbf{V}' \subset \tilde{\mathbf{V}}'$.

III. Model Error Analysis

In this section, we derive explicit upper bounds for the error between the SPF model (7) and the index-1 DAE model (4), or equivalently(6) in the context of trajectory tracking. In the analysis of singularly perturbed systems, the Implicit Function Theorem (IFT) is traditionally invoked to insure solvability of the algebraic equation obtained by setting the perturbation parameter $\epsilon = 0$ [7] [11] [12]. This results in the fact that the solution of the algebraic equation, and hence the entire singularly perturbed analysis, is valid in a neighborhood set (open set, ϵ-neighborhood) of the solution. While the existence of the ϵ-neighborhood is insured by the IFT, its boundary is not specified. In our analysis, we will utilize a recent result from [3], where a compact subset $\mathbf{B}_q$[2] of the solvability domain (i.e. the validity domain Ω), is characterized with well defined boundaries. By definition the domain Ω guarantees the existence of function $\mathbf{q}' = \sigma(\mathbf{q})$ as introduced in Section II. Also $\mathbf{z}$ is a component of $\mathbf{q}'$, therefore there exists a function $\mathbf{h} : \mathbf{B}_q \to \mathbf{B}_z$, such that $\phi(\mathbf{q}, \mathbf{h}(\mathbf{q})) = 0, \forall \mathbf{q} \in \mathbf{B}_q$. It follows that our analysis is valid for any $\mathbf{q} \in \mathbf{B}_q \subset \Omega$, $\mathbf{z} \in \mathbf{B}_z \subset \Re^p$.

We consider the error of the SPF approach in the context of trajectory tracking. Assume that, for the DAE system (6), equivalently (4), the controller $\mathbf{u}$ is chosen such that the state $\bar{\mathbf{x}} = [\bar{\mathbf{q}}, \dot{\bar{\mathbf{q}}}]^T$ exponentially follows a desired trajectory $\mathbf{x}_d = [\mathbf{q}_d, \dot{\mathbf{q}}_d]^T$. Examples of such controllers include e.g. the inverse dynamics control law and the non-adaptive scheme of [13].

[2] The notation $\mathbf{B}_v$ is used to denote a compact ball centered at $\mathbf{v} = 0, \mathbf{v} \in \Re^p$, p interger, that is, $\mathbf{B}_v = \{\mathbf{v} \in \Re^p : \|\mathbf{v}\| \leq \rho_v, \rho_v$ positive real $\} \subset \Re^p$

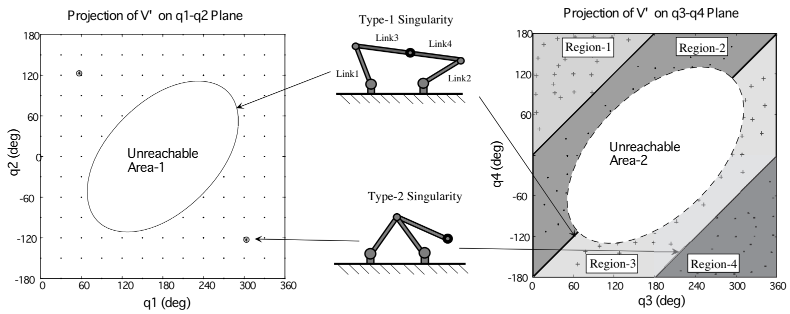


Fig. 2. Projections of $\mathbf{V}'$ and Singularities of the RPDR

Fig. 3. Projections of $\tilde{\mathbf{V}}'$

We first reformulate the original problem into the standard singular perturbation forms. Define $\mathbf{x} \triangleq [\mathbf{q}, \dot{\mathbf{q}}]^T$, $\mathbf{y} \triangleq \mathbf{z} - \mathbf{h}(\mathbf{q})$ and $\mathbf{e} \triangleq \mathbf{x} - \mathbf{x}_d$. Then the original *full system* in (7) can be rewritten in terms of $\mathbf{e}$ and $\mathbf{y}$,

$$\begin{aligned} \dot{\mathbf{e}} &= \mathbf{F}(t, \mathbf{e} + \mathbf{x}_d, \mathbf{y}) - \dot{\mathbf{x}}_d = \mathbf{F}'(t, \mathbf{e}, \mathbf{y}), \quad (8)\\ &\qquad \mathbf{e}(0) = \mathbf{x}(0) - \mathbf{x}_d(0),\\ \epsilon\dot{\mathbf{y}} &= \mathbf{G}(\mathbf{e} + \mathbf{x}_d, \mathbf{y}, \epsilon) = \mathbf{G}'(t, \mathbf{e}, \mathbf{y}, \epsilon) \quad (9)\\ &\qquad \mathbf{y}(0) = \zeta_0 - \mathbf{h}(\theta_0), \end{aligned}$$

where $\mathbf{F}(t, \mathbf{x}, \mathbf{y})$ is given by

$$\mathbf{F}(t, \mathbf{x}, \mathbf{y}) \triangleq \begin{bmatrix} \dot{\mathbf{q}} \\ -D^{-1}(C\dot{\mathbf{q}} + g) \end{bmatrix} + \begin{bmatrix} 0 \\ D^{-1}\mathbf{u} \end{bmatrix}$$

and $\mathbf{G}(\mathbf{x}, \mathbf{y}, \epsilon)$ is given by

$$\mathbf{G}(\mathbf{x}, \mathbf{y}, \epsilon) \triangleq w(\mathbf{q}, \mathbf{y}) + \epsilon\left[\mathbf{v}(\mathbf{q}, 0) - \mathbf{v}(\mathbf{q}, \mathbf{y})\right]\dot{\mathbf{q}}$$

with

$$\begin{aligned} w(\mathbf{q}, \mathbf{y}) &\triangleq -\phi_z^{-1}(\mathbf{q}, \mathbf{y} + \mathbf{h})\phi(\mathbf{q}, \mathbf{y} + \mathbf{h}),\\ \mathbf{v}(\mathbf{q}, \mathbf{y}) &\triangleq \phi_z^{-1}(\mathbf{q}, \mathbf{y} + \mathbf{h})\phi_q(\mathbf{q}, \mathbf{y} + \mathbf{h}). \end{aligned}$$

The corresponding *boundary layer system* is

$$\epsilon\dot{\mathbf{y}} = \mathbf{G}(\mathbf{x}, \mathbf{y}, 0) = w(\mathbf{q}, \mathbf{y}), \qquad \mathbf{y}(0) = \zeta_0 - \mathbf{h}(\theta_0). \quad (10)$$

The *reduced system* in the sense of singular perturbations is obtained by setting $\epsilon = 0$ in (8)-(9), which turns out to be the same as the index-1 DAE model introduced in Section II. We define $\bar{\mathbf{x}} \triangleq \left[\bar{\mathbf{q}}, \dot{\bar{\mathbf{q}}}\right]^T$ and $\bar{\mathbf{e}} = \bar{\mathbf{x}} - \mathbf{x}_d$. The DAE system in (6) can be rewritten in terms of $\bar{\mathbf{e}}$

$$\begin{aligned} \dot{\bar{\mathbf{e}}} &= \mathbf{F}(t, \bar{\mathbf{e}} + \mathbf{x}_d, 0) - \dot{\mathbf{x}}_d = \mathbf{F}'(t, \bar{\mathbf{e}}, 0), \quad (11)\\ &\qquad \bar{\mathbf{e}}(0) = \bar{\mathbf{x}}(0) - \mathbf{x}_d(0),\\ 0 &= \mathbf{G}'(t, \bar{\mathbf{e}}, 0, 0). \quad (12) \end{aligned}$$

We have the following properties.

Remark 3.1: The constraint given by $\phi(\mathbf{q}, \mathbf{z})$ in (6) is at least twice continuously differentiable. The control $\mathbf{u}$ and the desired trajectory $\mathbf{x}_d$ are assumed to be smooth. Thus the right hand sides of (8)-(9) contains only compositions of smooth functions. Furthermore we assume that the desired trajectory $\mathbf{x}_d$ and the control $\mathbf{u}$ are selected such that the functions $\mathbf{F}', \mathbf{G}', \mathbf{h}$ and their partial derivatives up to order 2 are bounded for $(t, \mathbf{e}, \mathbf{y}, \epsilon) \in [0, \infty) \times \mathbf{B}_e \times \mathbf{B}_\rho \times [0, \epsilon_0]$, where $\mathbf{B}_e$ and $\mathbf{B}_\rho$ are compact domains of $\mathbf{e}$ and $\mathbf{y}$ respectively.

Property 3.1: The exponential stability of the origin of the boundary layer system (10) can be concluded by considering the Jacobian matrix

$$\begin{aligned} &\left\{\frac{\partial \mathbf{G}(\mathbf{x}, \mathbf{y}, 0)}{\partial y}\right\}_{y=0}\\ &= -\left\{\frac{\partial}{\partial y}\left[\phi_z^{-1}(\mathbf{q}, \mathbf{y} + \mathbf{h})\phi(\mathbf{q}, \mathbf{y} + \mathbf{h})\right]\right\}_{y=0} = -\mathbf{I}_m, \end{aligned}$$

where we used the fact that $\mathbf{h}(\mathbf{q})$ satisfies the constraint equation $\phi(\mathbf{q}, \mathbf{h}(\mathbf{q})) = 0$.

Property 3.2: The origin of the error dynamics of the reduced system (11) is exponentially stable due to the assumption of an appropriate controller $\mathbf{u}$.

Under the above conditions and Properties, the infinite-time version of Tikhonov's theorem [7] applies to the problem and gives the following error estimates

$$\begin{aligned} \mathbf{q}(t, \epsilon) - \bar{\mathbf{q}}(t) &= \mathbf{O}(\epsilon),\\ \dot{\mathbf{q}}(t, \epsilon) - \dot{\bar{\mathbf{q}}}(t) &= \mathbf{O}(\epsilon),\\ \mathbf{z}(t, \epsilon) - \bar{\mathbf{z}}(t) &= \mathbf{O}(\epsilon), \qquad \forall t > t_b, \forall \epsilon < \epsilon^0. \end{aligned}$$

Note that Tikhonov's theorem is not a stability result and hence the model error between the SPF solution and the

DAE solution is not guaranteed to converge to zero. In general, one can only get the above error order statement by using Tikhonov's theorem. See, for example, an application in [14].

The above result may be refined by taking into account the specific structure of the full system (8)-(9). These are given as follows. *A*) The initial conditions are independent of the singular perturbation parameter ϵ. *B*) The fast dynamics in (7) guarantees exponential convergence of $\mathbf{y}(t, \epsilon)$ [9]. *C*) The slow subsystem (8) is independent of ϵ. We now present the following result which guarantees exponential convergence of the approximation error,

Theorem 3.1: Under conditions of Remark 3.1, Property 3.1 and Property 3.2, the error between the solution to the full system (8)-(9), or equivalently (7), and the solution to the reduced system (11)-(12), or equivalently (6), has the following explicit upper bound

$$\begin{aligned} \|\mathbf{q}(t,\epsilon) - \bar{\mathbf{q}}(t)\| &< \epsilon K_d e^{-K_a t} \\ \|\dot{\mathbf{q}}(t,\epsilon) - \dot{\bar{\mathbf{q}}}(t)\| &< \epsilon K_d e^{-K_a t} \\ \|\mathbf{z}(t,\epsilon) - \bar{\mathbf{z}}(t)\| &< (C_1 + \epsilon C_2 K_d) e^{-K_a t}, \\ & \forall t > 0, \forall \epsilon \leq \epsilon^{**} \triangleq \min\{\epsilon^*, \frac{\alpha}{2K_a}\} \end{aligned}$$

where $C_1, C_2, K_d, K_a, \epsilon^*$ and ϵ^{**} are defined in [9]. Therefore the error exponentially converges to zero.

Proof: Details are in [9].

Roughly speaking, this theorem ensures that as long as the original DAE system in (6) is stabilized by an appropriate controller, the SPF model in (7) will exponentially approach the DAE system in the infinite-time interval. The exponential convergence of the error is a direct result of combining the comparison principle [7] with the structure features A), B) and C) of our case. This result is stronger than the error order statement from Tikhonov's theorem, which analysis is based on more general assumptions (e.g. all initial conditions are dependent on ϵ [7]) and thus only guarantees that under certain conditions, $\mathbf{x}(t, \epsilon)$ and $\mathbf{z}(t, \epsilon)$ will stay in an $\mathbf{O}(\epsilon)$ neighborhood of $\bar{\mathbf{x}}(t)$ and $\bar{\mathbf{z}}(t)$ respectively.

IV. Simulations and Experimental Study

The Rice Planar Delta Robot (RPDR) is a two degree-of-freedom parallel robot (See Figure 4 (a)). It was designed and constructed at Rice University as a test-bed to perform experiments on closed chain mechanisms. In previous work we implemented on the RPDR a PD plus gravity compensation set point control [2] and an inverse dynamics control based on the reduced model [3]. A detailed derivation of the singular perturbation model for the RPDR can be found in [4]. Corresponding to foregoing analysis, we consider the trajectory tracking of the RPDR. The inverse dynamics control is used to force the end-effector to follow a circle of diameter 6.0 inches that is centered 10 inches above the axes of joints one and two (see Figure 4 (b)). Due to the usage of the inverse dynamics control, the assumption on the exponential stability of the origin of (11) is satisfied. In simulation, we also included an initial configuration error of $\mathbf{e}(0) = [2°, 2°]^T$ (with respect to the exact starting point on the circle). The error in the initial $\mathbf{z}(0)$ is $\delta\mathbf{z}_0 = [-15°, -15°]^T$. For the RPDR, the constraint equation in the DAE model can be explicitly solved and the DAE model is transformed into an explicit ODE model. The resulted ODE was solved using ODE45 in MATLAB. The SPF model was solved using the stiff ODE solver ODE15S. The tolerance settings are $RelTol = 10^{-8}$ and $AbsTol = 10^{-12}$.

We first study the effect of changing the singular perturbation parameter ϵ on the transient behavior of the SPF model. Figure 5 shows that with a smaller ϵ, the initial error in z_1 vanishes faster resulting in a better approximation to the DAE model. On the other hand, decreasing ϵ will increase the stiffness of the SPF model and requires more computational effort. Thus the choice of ϵ is a tradeoff between convergence rate and computational burden. We then compare our simulation results for $\epsilon = 0.01$ to experimental data obtained on the real robot. Figure 6 shows that motion profiles from simulation are very close to experimental data. Thus, the validity of the simulations is demonstrated. Figure 6 also shows that with an exponentially stabilized reduced system, the solution of the SPF model approaches its DAE counterpart rapidly and stay very close to it afterward, which agrees quite well with the analysis in Section III.

V. Conclusion

Two important issues associated with the Singular Perturbation Formulation (SPF) model for Closed Kinematic Chain (CKC) are investigated in this paper. The first is the domain of validity. We showed that the SPF model has a larger domain than the index-1 DAE model, hence much larger than the reduced model, and the boundary of its domain is easier to characterize. The second issue is the characterization of the closeness between the SPF model and the DAE model. Our analysis takes into account the special structure of the SPF model and produces stronger results than the statement by the Tikhonov's Theorem in the sense that it guarantees exponential convergence of the model error. The analysis is supported by numerical simulation and experimental data, indicating that the SPF approach is an effective tool for modeling closed chain mechanisms. The SPF model will be used to develop new control schemes for closed kinematic chains.

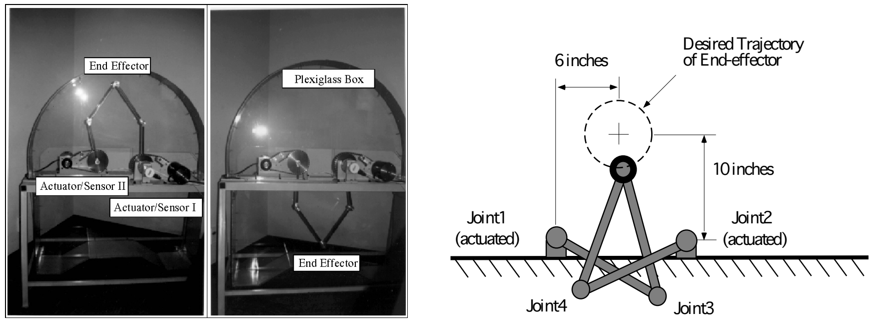
References

[1] K. E. Brenan, S. L. Campbell, and L. R. Petzold, *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*. Philadelphia, PA: SIAM, 1996.

[2] F. Ghorbel, O. Chételat, R. Gunawardana, and R. Longchamp, "Modeling and set point control of closed-chain mechanisms: Theory and experiment," *IEEE Trans. Contr. Syst. Technol.*, vol. 8, no. 5, pp. 801 – 815, September 2000.

[3] R. Gunawardana, F. Ghorbel, and Z. Wang, "Reduced model based tracking control of robots with closed kinematic chains: analysis and implementation," *IEEE Trans. Contr. Syst. Technol.*, accepted for publication.

[4] J. Dabney, F. Ghorbel, and Z. Wang, "Modeling closed kinematic chains via singular perturbations," in *Proceeding of the American Control Conference*, Anchorage, AK, May 2002, pp. 4104 – 4110.



(a) Up and Down Configurations of RPDR (b) Tracking a Circle

Fig. 4. Pictures of RPDR and Trajectory Tracking

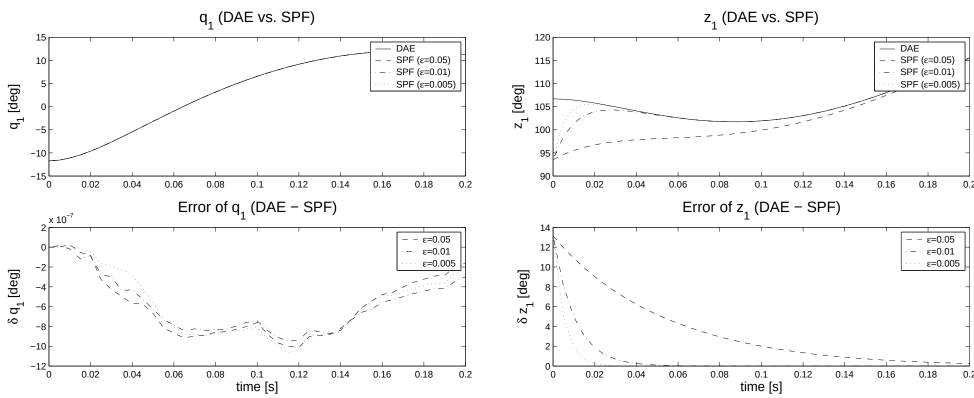


Fig. 5. Transient Behavior of the SPF Model with $\epsilon = 0.05, 0.01, 0.005$

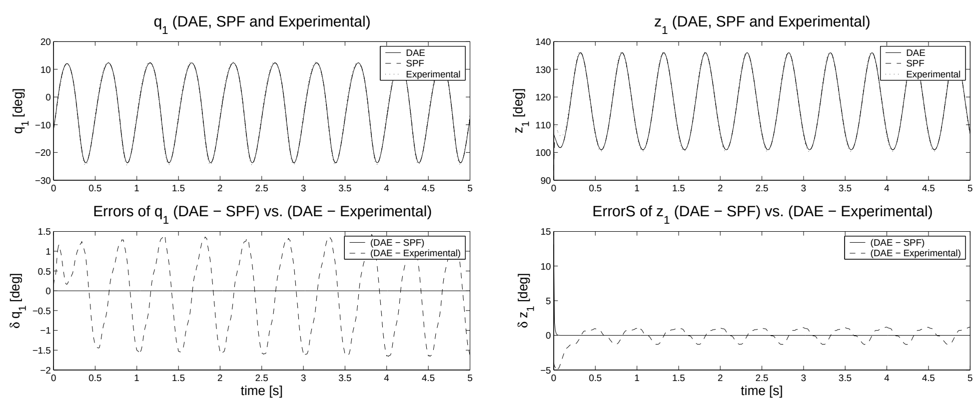


Fig. 6. Trajectory Tracking of the RPDR— DAE Model, SPF Model and Experimental Data

[5] B. W. Gordon and S. Liu, "A singular perturbation approach for modeling differential-algebraic systems," *Journal of Dynamic Systems, Measurement and Control*, vol. 120, no. 4, pp. 541 – 545, 1998.

[6] P. Kokotovic, H. K. Khalil, and J. O'Reilly, *Singular Perturbation Methods in Control: Analysis and Design*, 1st ed. Philadelphia, PA: SIAM, 1999.

[7] H. K. Khalil, *Nonlinear Systems*, 2nd ed. Upper Saddle River, NJ: Prentice Hall, 1996.

[8] A. Kumar and P. Daoutidis, *Control of nonlinear differential algebraic equation systems*. London, UK: Chapman & Hall/CRC, 1999.

[9] Z. Wang and F. H. Ghorbel, "On the error characterization of a singular perturbation model for closed kinematic chains," Robotics Lab, Department of Mechanical Engineering and Material Science, Rice University, Tech. Rep., 2003.

[10] P. Wenger and D. Chablat, "Workspace and assembly modes in fully-parallel manipulators: descriptive study," *Advances in Robot Kinematics: Analysis and Control, J. Lenarcic and M.L. Husty (eds)*, pp. 117 – 126, 1998.

[11] R.E.O'Malley, "On nonlinear singularly perturbed initial value problems," *SIAM Review*, vol. 30, no. 2, pp. 193 – 212, June 1988.

[12] M. Knorrenschild, "Differential/algebraic equations as stiff odinary differential equations," *SIAM Journal of Numerical Analysis*, vol. 29, no. 6, pp. 1694 – 1715, Dec. 1992.

[13] J. J. E. Slotine and W. Li, "On the adaptive control of robot manipulators," *Int. J. Robotics Res.*, vol. 6, no. 3, 1987.

[14] B. Eryilmaz and B. H. Wilson, "Improved tracking control of hydraulic systems," *Journal of Dynamic Systems, Measurement, and Control*, vol. 123, pp. 457 – 462, 2001.