Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Stability Analysis for Linear Systems under State Constraints

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***Abstract*— This paper revisits the problem of stability analysis for linear systems under state constraints. New and less conservative sufficient conditions are identified under which such systems are globally asymptotically stable. Based on these sufficient conditions, iterative LMI algorithms are proposed for testing global asymptotic stability of the system. In addition, these iterative LMI algorithms can be adapted for the design of globally stabilizing state feedback gains.**

I. Introduction and Problem Statement

In this paper, we will investigate stability analysis of two classes of linear systems under state constraints, which were recently studied in [4], [6], [7], [8], [10]. The first class of systems are defined as follows,

$$\dot{x} = h(Ax), \tag{1}$$

where $x \in \mathbf{D}^n = \{x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in \mathbf{R}^n : -1 \leq x_i \leq 1$ $i \in [1,n]\}$, $A = [a_{ij}] \in \mathbf{R}^{n\times n}$, and

$$h(Ax) = \begin{bmatrix} h_1\left(\sum_{j=1}^n a_{1j}x_j\right) \\ h_2\left(\sum_{j=1}^n a_{2j}x_j\right) \\ \vdots \\ h_n\left(\sum_{j=1}^n a_{nj}x_j\right) \end{bmatrix}.$$

Let $r_i = \sum_{j=1}^n a_{ij}x_j$, then, for each $i \in [1,n]$,

$$h_i(r_i) = \begin{cases} 0, & \text{if } |x_i| = 1 \text{ and } r_i x_i > 0, \\ r_i, & \text{otherwise.} \end{cases}$$

Such systems are defined on a closed hypercube as all state variables are constrained to the unit hypercube $\mathbf{D}^n$. For this reason, system (1) is sometimes referred to as a linear system subject to state saturation. Clearly, saturation occurs in the state x_i if $|x_i| = 1$ and $\left(\sum_{j=1}^n a_{ij}x_j\right) x_i > 0$.

The other class of systems are systems with partial state constraints and are described as

$$\begin{cases} \dot{x} = Ax + By, \\ \dot{y} = h(Cx + Ey), \end{cases} \tag{2}$$

where $x \in \mathbf{R}^{n-m}$ with $n \geq m$, $y \in \{(y_1, y_2, \cdots, y_m)^{\mathrm{T}} : -1 \leq y_i \leq 1,\ i \in [1,m]\}$, A, B, C and E are real matrices of appropriate dimensions, and

$$h(Cx+Ey) = \begin{bmatrix} h_1\left(\sum_{j=1}^{n-m} c_{1j}x_j + \sum_{k=1}^m e_{1k}y_k\right) \\ h_2\left(\sum_{j=1}^{n-m} c_{2j}x_j + \sum_{k=1}^m e_{2k}y_k\right) \\ \vdots \\ h_m\left(\sum_{j=1}^{n-m} c_{mj}x_j + \sum_{k=1}^m e_{mk}y_k\right) \end{bmatrix}. \tag{3}$$

Let $s_i = \left(\sum_{j=1}^{n-m} c_{ij}x_j + \sum_{k=1}^m e_{ik}y_k\right)$, then, for each $i \in [1,m]$,

$$h_i(s_i) = \begin{cases} 0, & |y_i| = 1 \text{ and } s_i y_i > 0 \\ s_i, & \text{otherwise.} \end{cases} \tag{4}$$

We note that the class of systems (2) reduces to the class (1) if $m = n$. These two classes of systems are encountered in a variety of applications, including signal processing, recurrent neural networks and control systems, and have been studied extensively (see, *e.g.*, [3], [4], [5], [6], [8], [12] and the references therein). In this paper, we revisit the problem of stability analysis for these two classes of systems. In particular, we are interested in conditions under which such systems are globally asymptotically stable at the origin. Here, by global asymptotic stability of the origin we mean that the origin is locally asymptotically stable within $\mathbf{D}^n$ (or $\mathbf{R}^{n-m} \times \mathbf{D}^m$), rather than the usual $\mathbf{R}^n$, being the domain of attraction.

Global asymptotic stability of these systems has been studied in [4], [8], [10]. For second order systems in the form of (1), necessary and sufficient conditions for global asymptotic stability were established in [4], [10]. For higher order systems in the form of either (1) or (2), various sufficient conditions for the global asymptotic stability were identified. Under the sufficient condition of [8], any system trajectory starting from inside $\mathbf{D}^n$ will never reach the boundary of $\mathbf{D}^n$, *i.e.*, the state never saturates. This saturation avoidance sufficient condition leads to a degree of conservatism. Using a Lyapunov function $V : \mathbf{D}^n \to \mathbf{R}$ that satisfies

$$\left[\frac{\partial V}{\partial x}(x)\right] h(Ax) \leq \left[\frac{\partial V}{\partial x}(x)\right] Ax, \tag{5}$$

[4] arrives at a sufficient condition that is less conservative than that of [8].

Motivated by the observation that the hypothesis (5) might be a source of conservatism, we will in this paper re-examine global asymptotic stability of such systems by exploring the special property of the function h. The sufficient conditions we thus arrive at are given in terms of matrix inequalities, which are shown to be less conservative than those of [8] and [4]. Based on these new sufficient conditions, iterative LMI algorithms are proposed for testing global asymptotic stability. In addition to the stability analysis, the proposed sufficient conditions and the iterative LMI algorithms can be readily adapted for designing globally stabilizing feedback gains for the following systems:

$$\dot{x} = h(Ax + Bu), \quad u = Fx, \tag{6}$$

where $x \in \mathbf{R}^n$ and $u \in \mathbf{R}^m$.

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This paper is organized as follows. The main result is presented in Section II. Numerical examples are given in Section III. A brief concluding remark is made in Section IV.

II. Stability Analysis

In this section, we will establish new sufficient conditions for global asymptotic stability for both the system (1) and (2). To this end, we first establish some technical lemmas.

Lemma 1: Consider a nonlinear system

$$\dot{x} = f(x), \quad x \in \Omega \subset \mathbf{R}^n,$$

with $f(0) = 0$. Assume that all trajectories remain inside Ω. If there exists a function $V : \Omega \to \mathbf{R}$ such that

$$\phi_1(\|x\|) \le V(x) \le \phi_2(\|x\|), \quad \forall x \in \Omega,$$

and

$$\dot{V}(x) \le -\phi_3(\|x\|), \quad \forall x \in \Omega,$$

for some class functions ϕ_1, ϕ_2 and ϕ_3, then the origin is globally asymptotically stable in the sense that the origin is locally asymptotically stable with Ω being the domain of attraction.

Proof. The main idea of the proof comes from [4]. Under these conditions, it follows from the standard Lyapunov theory that the system is locally asymptotically stable with a neighborhood of the origin $\mathbf{S} \subset \Omega$ contained in the domain of attraction. Let $d = \min \|x\|, x \in \partial\mathbf{S}$, where $\partial\mathbf{S}$ denotes the boundary of the set $\mathbf{S}$. Then, any trajectory starting from $\Omega \setminus \mathbf{S}$ will remain in Ω and enter $\mathbf{S}$ at some finite time t_0 and converge to the origin asymptotically. Otherwise, we must have $||x(t)|| > d$ for all $t \ge 0$, and

$$\begin{aligned} V(x(t)) = \quad & V(x(0)) + \int_0^t \dot{V}(x(\tau))d\tau \\ \le \quad & V(x(0)) - \int_0^t \phi_3(\|x(\tau)\|)d\tau \\ \le \quad & V(x(0)) - t\phi_3(d), \quad \forall t \ge 0, \end{aligned}$$

which is a contradiction to the fact that $V(x(t)) \ge 0$, $\forall t \ge 0$. $\square$

Recall that for a group of points, $u^1, u^2, \cdots, u$, their convex hull is defined as,

$$\mathrm{co}\left\{u^i : i \in [1, \quad]\right\} := \left\{\sum_{i=1} \alpha_i u^i : \sum_{i=1} \alpha_i = 1, \alpha_i \ge 0\right\}.$$

Lemma 2: [9] Let $u, u^1, u^2, \cdots, u \quad \in \mathbf{R}^{m_1}$, $v, v^1, v^2, \cdots, v \quad \in \mathbf{R}^{m_2}$, if $u \in \mathrm{co}\left\{u^i : i \in [1, \quad]\right\}$ and $v \in \mathrm{co}\left\{v^i : i \in [1, \quad]\right\}$, then

$$\begin{bmatrix} u \\ v \end{bmatrix} \in \mathrm{co}\left\{\begin{bmatrix} u^i \\ v^j \end{bmatrix} : i \in [1, \quad], j \in [1, \quad]\right\}.$$

Let $_n$ be the set of $n \times n$ diagonal matrices whose diagonal elements are either 1 or 0. $_n$ contains 2^n elements. Let D_i be an element of $_n$, denote $D_i^- = I - D_i$.

Definition 1: A matrix $M = [m_{ij}] \in \mathbf{R}^{n\times n}$ is said to be (row) diagonally dominant if

$$|m_{ii}| > \sum_{j=1, j\ne i}^{n} |m_{ij}|, \quad i \in [1, n].$$

Lemma 3: Let $G = [g_{ij}] \in \mathbf{R}^{n\times n}$ be (row) diagonally dominant and the diagonal be composed of negative elements (i.e., $g_{ii} < 0$ for all $i \in [1, n]$). Then,

$$h(Ax+K) \in \mathrm{co}\left\{D_i(Ax+K) + D_i^- Gx, i \in [1, 2^n]\right\}, \quad \forall x \in \mathbf{D}^n,$$

for any matrix $K \in \mathbf{R}^n$ independent of x.

Proof. Since G is (row) diagonally dominant and $g_{ii} < 0, i \in [1, n]$, for any $x \in \mathbf{D}^n$, $G_i x < 0$ if $x_i = 1$, or $G_i x > 0$ if $x_i = -1$. In the absence of state saturation, *i.e.*, $h(A_i x + K_i) = A_i x + K_i$, it is obvious that $h(A_i x + K_i) \in \mathrm{co}\{A_i x + K_i, G_i x\}$. In the event of state saturation, $h(A_i x + K_i)$=0, either when $x_i = 1$ and $A_i x + K_i > 0$, or when $x_i = -1$ and $A_i x + K_i < 0$. When $x_i = 1$ and $A_i x + K_i > 0$, $G_i x < 0$ and hence, $h(A_i x + K_i) = 0 \in \mathrm{co}\{A_i x + K_i, G_i x\}$. Similarly, when $x_i = -1$ and $A_i x + K_i < 0$, $G_i x > 0$ and hence $h(A_i x + K_i) = 0 \in \mathrm{co}\{A_i x + K_i, G_i x\}$. It then follows from Lemma 2 that

$$h(Ax+K) \in \mathrm{co}\left\{D_i(Ax+K) + D_i^- Gx, i \in [1, 2^n]\right\}, \quad \forall x \in \mathbf{D}^n.$$

$\square$

We are now ready to establish a new sufficient condition under which the system (1) is globally asymptotically stable at the origin.

Theorem 1: If there exist a symmetric positive definite matrix $P \in \mathbf{R}^{n\times n}$ and a $G \in \mathbf{R}^{n\times n}$ such that

$$(D_i A + D_i^- G)^{\mathrm{T}} P + P(D_i A + D_i^- G) < 0, \quad i \in [1, 2^n], \tag{7}$$

and G is (row) diagonally dominant with negative diagonal elements, then the system (1) is globally asymptotically stable at the origin.

Proof. Let $V(x) = x^{\mathrm{T}} P x$. We have

$$\dot{V}(x) = h(Ax)^{\mathrm{T}} P x + x^{\mathrm{T}} P h(Ax).$$

Since the matrix G is (row) diagonally dominant and the diagonal is composed of negative elements, by Lemma 3,

$$h(Ax) \in \mathrm{co}\left\{D_i Ax + D_i^- Gx, i \in [1, 2^n]\right\}, \quad \forall x \in \mathbf{D}^n.$$

It then follows that

$$\dot{V}(x) \le \max_{i\in[1,2^n]} x^{\mathrm{T}}((D_i A + D_i^- G)^{\mathrm{T}} P + P(D_i A + D_i^- G))x, \forall x \in \mathbf{D}^n.$$

Condition (7) then implies that

$$\begin{aligned} \dot{V} \le \quad & \max_{i\in[1,2^n]} x^{\mathrm{T}}((D_i A + D_i^- G)^{\mathrm{T}} P + P(D_i A + D_i^- G))x \\ \le \quad & -\delta x^{\mathrm{T}} x, \quad \forall x \in \mathbf{D}^n, \end{aligned}$$

for some $\delta > 0$. The results of the theorem then follow from Lemma 1. $\square$

Remark 1: In [8], it is established that the system (1) is globally asymptotically stable at the origin if A is (row) diagonally dominant and the diagonal is composed of negative elements. This condition on A implies that A is Hurwitz stable and hence there is a positive definite matrix P such that

$$A^{\mathrm{T}}P + PA < 0, \tag{8}$$

which can be written as (7) with $G = A$. Thus, the sufficient condition established in Theorem 1 is less conservative than the sufficient condition of [8].

Remark 2: In [4], it is established that the system (1) is globally asymptotically stable at the origin if there exists a symmetric positive definite matrix P such that (8) is satisfied and

$$p_{ii} \geq \sum_{j\in[1,n],j\neq i}^{n} |p_{ij}|, \quad i \in [1,n]. \tag{9}$$

Existence of such a P implies the existence of a P that satisfies (8) and (9) with $\geq$ in (9) replaced with $>$. Indeed, if P satisfies (8) and (9), then, for a sufficiently small ε, $P+\varepsilon I$ will satisfies (8) and (9) with $>$ in (9). Let P be such that it satisfies (8) and

$$p_{ii} > \sum_{j\in[1,n],j\neq i} |p_{ij}|, \quad i \in [1,n]. \tag{10}$$

Define

$$\begin{aligned} a &= \max_{i\in[1,n]} \sum_{j\in[1,n]} |a_{ij}|, \\ b &= \min_{i\in[1,n]} \left(p_{ii} - \sum_{j\in[1,n],j\neq i} |p_{ij}| \right). \end{aligned}$$

Obviously, $a > 0$ and $b > 0$. Let $c > \frac{a}{b}$, then $A - cP$ is (row) diagonally dominant and the diagonal is composed of negative elements. Inequality (8) then implies

$$(D_iA+D_i^-(A-cP))^{\mathrm{T}}P+P(D_iA+D_i^-(A-cP)) < 0, i\in[1,2^n], \tag{11}$$

which is (7) with $G = A - cP$. We thus see that the sufficient condition of Theorem 1 is less conservative than the sufficient condition of [4].

In what follows, we will follow the idea of [2] to propose an iterative LMI algorithm for verifying the sufficient condition of Theorem 1. Let be the set of n-dimensional row vectors in which there is only one nonzero element which is 1. Denote $h_i, i \in [1,n]$, as an element of in which the ith element is 1. Let $_i$ be the set of n-dimensional column vectors in which the ith element is 1 and other elements are either 1 or -1. The elements of $_i$ are denoted as $y_{ij}, j \in [1,2^{n-1}]$. Then, the condition that G is (row) diagonally dominant and the diagonal is composed of negative elements can be expressed as the following LMIs,

$$h_iGy_{ij} < 0, \quad i \in [1,n], j \in [1,2^{n-1}]. \tag{12}$$

Algorithm 1: Global Asymptotic Stability of System (1)

Step 1. Select a $Q > 0$, and solve P from the following Lyapunov equation,

$$A^{\mathrm{T}}P + PA = -Q.$$

Set $k = 0$.

Step 2. Using P obtained previously, solve the following LMI optimization problem for G and α,

$$\begin{aligned} &\inf_{G} \alpha \\ \text{s.t.} \quad &(D_iA + D_i^-G)^{\mathrm{T}}P + P(D_iA + D_i^-G) < \alpha P, \\ &\qquad\qquad i \in [1,2^n], \\ &h_iGy_{ij} < 0, \quad i \in [1,n], j \in [1,2^{n-1}]. \end{aligned}$$

If $k = 0$ and $\alpha \leq 0$, go to Step 4. If $k > 0$, $\alpha \leq 0$ or $\alpha \not\leq \alpha_k$, go to Step 4. Otherwise, set $k = k+1, \alpha_k = \alpha$, go to the next step.

Step 3. Using G obtained in the previous step, solve the following LMI optimization problem for P and α,

$$\begin{aligned} &\inf_{P>0} \alpha \\ \text{s.t.} \quad &(D_iA + D_i^-G)^{\mathrm{T}}P + P(D_iA + D_i^-G) < \alpha P, \\ &\qquad\qquad i \in [1,2^n]. \end{aligned}$$

If $\alpha \leq 0$ or $\alpha \not\leq \alpha_k$, go to Step 4. Otherwise, let $k = k+1$, $\alpha_k = \alpha$, go to Step 2.

Step 4. If $\alpha \leq 0$, the system (1) is globally asymptotically stable at the origin. Otherwise, no conclusion can be drawn. A different Q may be selected and the algorithm may be repeated from Step 1.

Remark 3: We note that solutions to the LMI optimization problems in Algorithm 1 always exist and α_k is non-increasing. However, the number of constraints increases exponentially as n, the order of the system increases. The large number of constraints may cause numerical difficulties [1].

By viewing F as an additional variable, Algorithm 1 can also be readily adapted for the design of a globally stabilizing feedback law $u = Fx$ for the system (6). In particular, we have the following algorithm.

Algorithm 2: Design of Globally Stabilizing Feedback Gain F

Step 1. Select a $Q > 0$, and solve P from the following Lyapunov equation,

$$(A+BF)^{\mathrm{T}}P + P(A+BF) = -Q,$$

where F is chosen such that $A+BF$ is Hurwitz. Set $k = 0$.

Step 2. Using P obtained previously, solve the following LMI optimization problem for G, F and α,

$$\begin{aligned} &\inf_{G} \alpha \\ \text{s.t.} \quad &(D_i(A+BF) + D_i^-G)^{\mathrm{T}}P + \\ &P(D_i(A+BF) + D_i^-G) < \alpha P, \quad i \in [1,2^n], \\ &h_iGy_{ij} < 0, \quad i \in [1,n], j \in [1,2^{n-1}]. \end{aligned}$$

If $k = 0$ and $\alpha \leq 0$, go to Step 4. If $k > 0$, $\alpha \leq 0$ or $\alpha \not\leq \alpha_k$, go to Step 4. Otherwise, set $k = k+1, \alpha_k = \alpha$, go to the next step.

Step 3. Using G and F obtained in the previous step. Solve the following LMI optimization problem for P and α,

$$\begin{aligned} & \inf_{P>0} \alpha \\ \text{s.t.} \quad & (D_i(A+BF)+D_i^- G)^T P + \\ & P(D_i(A+BF)+D_i^- G) < \alpha P, \quad i \in [1,2^n]. \end{aligned}$$

If $\alpha \le 0$ or $\alpha \not\le \alpha_k$, go to Step 4. Otherwise, set $k = k+1$, $\alpha_k = \alpha$, go to Step 2.

Step 4. If $\alpha \le 0$, the system (6) is globally asymptotically stable at the origin. And the current F is the calculated feedback gain. Otherwise, no conclusion can be drawn. A different Q may be selected and the algorithm may be repeated from Step 1.

We next consider the second class of systems (2). Again, our interest here is to establish conditions under which the system is globally asymptotically stable at the origin. We have the following result.

Theorem 2: If there exist a symmetric positive definite matrix $P \in \mathbf{R}^{n\times n}$ and a $G \in \mathbf{R}^{m\times m}$ such that

$$\begin{bmatrix} A & B \\ D_iC & D_iE+D_i^- G \end{bmatrix}^{\mathrm{T}} P + P \begin{bmatrix} A & B \\ D_iC & D_iE+D_i^- G \end{bmatrix} < 0, \quad i \in [1,2^m], \tag{13}$$

where $D_i \in {}_m$ and G is (row) diagonally dominant and the diagonal is composed of negative elements, then the system (2) is globally asymptotically stable at the origin.

Proof. Let

$$V(x,y) = \begin{bmatrix} x^{\mathrm{T}} & y^{\mathrm{T}} \end{bmatrix} P \begin{bmatrix} x \\ y \end{bmatrix}.$$

Then,

$$\begin{aligned} \dot V(x,y) = & \begin{bmatrix} x^{\mathrm{T}} & y^{\mathrm{T}} \end{bmatrix} P \begin{bmatrix} Ax+By \\ h(Cx+Ey) \end{bmatrix} \\ & + \begin{bmatrix} (Ax+By)^{\mathrm{T}} & h(Cx+Ey)^{\mathrm{T}} \end{bmatrix} P \begin{bmatrix} x \\ y \end{bmatrix}. \end{aligned}$$

Recalling Lemma 3, we have

$$h(Cx+Ey) \in \mathrm{co}\left\{D_i(Cx+Ey)+D_i^- Gy\right\}, \quad i \in [1,2^m],$$

and hence,

$$\begin{aligned} \dot V(x,y) \le & \max_{i\in[1,2^m]} \begin{bmatrix} x^{\mathrm{T}} & y^{\mathrm{T}} \end{bmatrix} \left(P \begin{bmatrix} A & B \\ D_iC & D_iE+D_i^- G \end{bmatrix} \right. \\ & \left. + \begin{bmatrix} A & B \\ D_iC & D_iE+D_i^- G \end{bmatrix}^{\mathrm{T}} P \right) \begin{bmatrix} x \\ y \end{bmatrix}. \end{aligned}$$

By (13), we have

$$\dot V(x,y) \le -\delta \begin{bmatrix} x^{\mathrm{T}} & y^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

for some constant $\delta > 0$. It then follows from Lemma 1 that the system (2) is globally asymptotically stable at the origin. □

Remark 4: In [4], it is concluded in Theorem 3 that the system (2) is globally asymptotically stable at the origin if there exist symmetric positive matrices $P_1 \in \mathbf{R}^{(n-m)\times(n-m)}$, $P_2 \in \mathbf{R}^{m\times m}$ and $Q \in \mathbf{R}^{n\times n}$, with P_2 satisfying (9), such that

$$\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} A & B \\ C & E \end{bmatrix} + \begin{bmatrix} A & B \\ C & E \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} < -Q. \tag{14}$$

As explained in Remark 2, it is without loss of generality to assume that P_2 satisfies (10). Let

$$\begin{aligned} a &= \max_{i\in[1,n]} \sum_{j\in[1,n]} |e_{ij}|, \\ b &= \min_{i\in[1,n]} \left(p_{2ii} - \sum_{j\in[1,n], j\ne i} |p_{2ij}| \right). \end{aligned}$$

Then, choose $c > \frac{a}{b}$ to be large enough such that $E - cP_2$ is (row) diagonally dominant and the diagonal is composed of negative elements, and

$$Q \ge \begin{bmatrix} \frac{1}{c}C^{\mathrm{T}}C & 0 \\ 0 & \frac{1}{c}E^{\mathrm{T}}E \end{bmatrix}. \tag{15}$$

Because for any $(x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}} \in \mathbf{R}^n \setminus \{0\}$, the following inequality is valid,

$$\begin{aligned} & -\frac{1}{c} y^{\mathrm{T}}E^{\mathrm{T}}Ey - cy^{\mathrm{T}}P_2D_i^- P_2 y \\ & -\left(\frac{1}{\sqrt{c}}Cx + \sqrt{c}D_i^- P_2 y\right)^{\mathrm{T}} \left(\frac{1}{\sqrt{c}}Cx + \sqrt{c}D_i^- P_2 y\right) < 0, \\ & \qquad i \in [1,2^m], \end{aligned} \tag{16}$$

which implies

$$\begin{aligned} & -\begin{bmatrix} \frac{1}{c}C^{\mathrm{T}}C & 0 \\ 0 & \frac{1}{c}E^{\mathrm{T}}E \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -D_i^- C & -D_i^- cP_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \\ & + \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -D_i^- C & -D_i^- cP_2 \end{bmatrix} \le 0, i \in [1,2^m]. \end{aligned} \tag{17}$$

Therefore, by (15) and (17), the inequality (14) implies

$$\begin{aligned} & \begin{bmatrix} A & B \\ D_iC & E - D_i^- cP_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \\ & + \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} A & B \\ D_iC & E - D_i^- cP_2 \end{bmatrix} < 0, i \in [1,2^m], \end{aligned} \tag{18}$$

which is (13) with $G = E - cP_2$ and $P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$. Therefore, we can see that Theorem 3 of [4] is more conservative than Theorem 2.

We next propose an iterative LMI algorithm for the verification of the conditions of Theorem 2.

Algorithm 3: Global Asymptotic Stability of System (2)

Step 1. Select a $Q > 0$, and solve the following Lyapunov equation for $P > 0$,

$$\begin{bmatrix} A & B \\ C & E \end{bmatrix}^{\mathrm{T}} P + P \begin{bmatrix} A & B \\ C & E \end{bmatrix} = -Q.$$

Set $k = 0$.

Step 2. Using P obtained previously, solve the following optimization problem for G and α,

$$\begin{aligned} &\inf_{G} \alpha \\ \text{s.t.} \quad &\begin{bmatrix} A & B \\ D_iC & D_iE + D_i^- G \end{bmatrix}^{\mathrm{T}} P + \\ &P \begin{bmatrix} A & B \\ D_iC & D_iE + D_i^- G \end{bmatrix} < \alpha P, \quad i \in [1, 2^m]. \end{aligned} \tag{12}$$

If $k = 0$ and $\alpha \leq 0$, go to Step 4. If $k > 0$, $\alpha \leq 0$ or $\alpha \not\leq \alpha_k$, go to Step 4. Otherwise, set $k = k+1$, $\alpha_k = \alpha$, go to the next step.

Step 3. Let $G = G_k$. Solve the following optimization problem for P and α,

$$\begin{aligned} &\inf_{P>0} \alpha \\ \text{s.t.} \quad &\begin{bmatrix} A & B \\ D_iC & D_iE + D_i^- G \end{bmatrix}^{\mathrm{T}} P + \\ &P \begin{bmatrix} A & B \\ D_iC & D_iE + D_i^- G \end{bmatrix} < \alpha P, \quad i \in [1, 2^m]. \end{aligned}$$

If $\alpha \leq 0$ or $\alpha \not\leq \alpha_k$, go to Step 4. Otherwise, let $k = k+1$, $\alpha_k = \alpha$, go to Step 2.

Step 4. If $\alpha \leq 0$, the system (2) is globally asymptotically stable at the origin. Otherwise, No conclusion can be drawn. A different Q may be selected and the algorithm may be repeated from Step 1.

III. Numerical Examples

Example 1. Consider the following system

$$\dot{x} = h(Ax) = h\left(\begin{bmatrix} 1 & 25 & 0 \\ -0.1 & -1.1 & 0 \\ 0 & -2 & -1 \end{bmatrix} x \right). \tag{19}$$

The matrix A is not (row) diagonally dominant, hence no conclusion can be drawn based on the condition of [8]. It can also be verified that the condition (8)-(9) (from [4]) can't be satisfied either. However (7) of Theorem 1 is satisfied with

$$P = \begin{bmatrix} 9.42 & 99.2 & 0.04 \\ 99.2 & 2255 & 0.6 \\ 0.04 & 0.6 & 0.5 \end{bmatrix},$$
$$G = \begin{bmatrix} -1.8 & -0.3 & 0 \\ -128 & -2908.1 & -0.8 \\ -0 & -78.4 & -79.9 \end{bmatrix}.$$

Thus, by Theorem 1, the system is globally asymptotically stable. A trajectory of (19) is shown in Fig 1. Fig 2 shows that Theorem 1 relaxes the hypothesis (5) (from [4]).

Example 2. Consider the following system

$$\dot{x} = h(Ax + Bu) = h\left(\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} Fx \right). \tag{20}$$

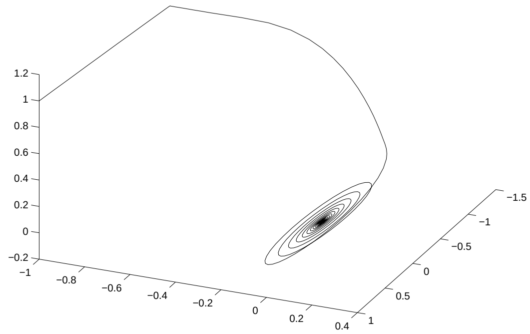

Fig. 1. A trajectory of the system (19).

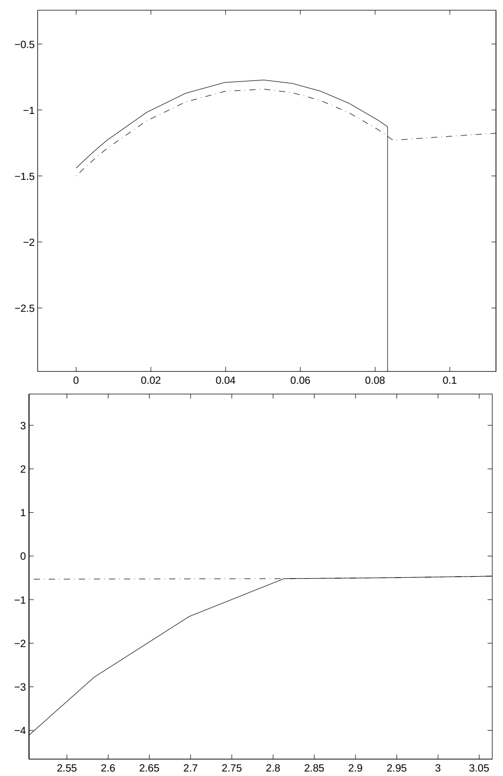

Fig. 2. $\left[\frac{\partial V}{\partial x}(x(t))\right] h(Ax(t))$(solid line) vs $\left[\frac{\partial V}{\partial x}(x(t))\right] Ax(t)$(dash-dotted line).

This system is open loop unstable as the matrix A is unstable. Using Algorithm 2, we obtain a globally stabilizing feedback gain F as

$$F = \begin{bmatrix} -3.5304 & -1.9255 \end{bmatrix},$$

with

$$P = \begin{bmatrix} 11.7205 & -2.6335 \\ -2.6335 & 1.1867 \end{bmatrix},$$
$$G = \begin{bmatrix} -2.6060 & 0.1635 \\ -3.0105 & -4.4588 \end{bmatrix}.$$

A trajectory of the closed-loop system (20) is shown in Fig. 3.

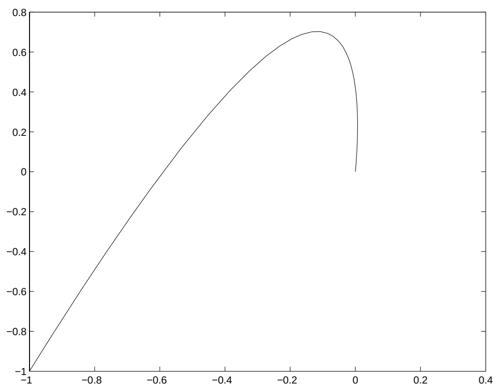

Fig. 3. A trajectory of the system (20)

Example 3. Consider the following system

$$\left\{\begin{array}{l} \dot{x} = -x + \left[\begin{array}{cc} 0.5 & 0.5 \end{array}\right] y, \\ \dot{y} = h\left(\left[\begin{array}{c} -0.2 \\ 0 \end{array}\right] x + \left[\begin{array}{cc} 1 & 14 \\ -0.2 & -2.3 \end{array}\right] y \right). \end{array}\right. \qquad (21)$$

It can be verified that the system (21) does not satisfy the condition of [4] and hence no conclusion on the global asymptotic stability can be drawn based on the results of [4]. However, by using Algorithm 3, it can be verified that the condition of Theorem 2 is satisfied with

$$\begin{array}{l} P = \left[\begin{array}{ccc} 0.5643 & -0.3214 & -2.5945 \\ -0.3214 & 4.0051 & 21.7223 \\ -2.5945 & 21.7223 & 131.8759 \end{array}\right], \\ G = \left[\begin{array}{cc} -2.1126 & -0.1906 \\ -0.3535 & -3.3845 \end{array}\right]. \end{array} \qquad (22)$$

We thus conclude that the system (21) is globally asymptotically stable at the origin. A trajectory is shown in Fig. 4.

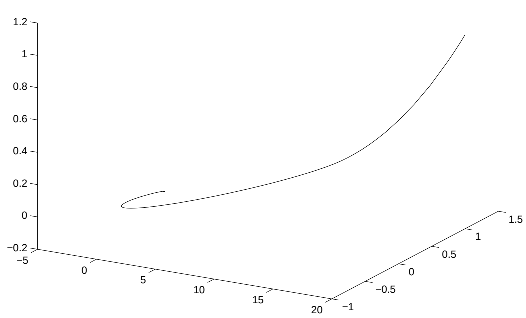

Fig. 4. A trajectory of the system (21)

IV. Conclusions

In this paper, we established simple conditions under which linear systems defined on a closed hypercube and linear systems with partial state saturation are globally asymptotically stable at the origin. These conditions were shown to be less conservative than the existing conditions. Based on these conditions, iterative LMI algorithms are proposed for verifying global asymptotic stability of these systems. Numerical examples were used to show the effectiveness of the proposed algorithms.

References

[1] S. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory*, SIAM Studies in Appl. Mathematics, Philadelphia, 1994.

[2] Y.-Y. Cao, J. Lam and Y.-X. Sun, “Static output feedback stabilization: an ILMI approach,” *Automatica*. Vol. 34, No. 12, pp. 1641-1645, 1998.

[3] J.J. Hopfield, “ ‘Neural’ computation of decisions in optimization problems,” *Biol. Cybern.*, Vol. 52, pp. 141-152, 1985.

[4] L. Hou and A.N. Michel, “Asymptotic Stability of Systems with Saturation Constraints,” *Proc. of the 35th IEEE Conference on Decision and Control*, pp. 2624-2629, 1996.

[5] J.-H. Li, A.N. Michel and W. Porod, “Analysis and synthesis of a class of neural networks: linear systems operating on a closed hypercube,” *IEEE Transactions on Circuits and Systems*, Vol. 36, pp. 1405-1422, 1989.

[6] D. Liu and A.N. Michel, “Asymptotic stability of systems operating on a closed hypercube,” *Systems & Control Letters*, Vol. 19, pp. 281-285, 1992.

[7] D. Liu and A.N. Michel, “Stability analysis of systems with partial state saturation nonlinearities,” *Proceedings of the 33rd IEEE Conference on Decision and Control*, pp. 1311-1316, 1994.

[8] D. Liu and A.N. Michel, *Dynamical Systems with Saturation Nonlinearities: Analysis and Design*, Springer, London, 1994.

[9] T. Hu and Z. Lin, *Control Systems with Actuator Saturation: Analysis and Design*, Birkhäuser, Boston, 2001.

[10] R. Mantri, A. Saberi and V. Venkatasubramanian, “Stability analysis of continuous time plannar systems with state saturation nonlinearity,” *Proc. of IEEE ISCAS’96*, pp. 60-63, Atlanta, GA, 1996.

[11] R.K. Miller and A.N. Michel, *Ordinary Differential Equations*, Academic Press, New York, 1982.

[12] D.W. Tank and J.J. Hopfield, “Simple ‘neural’ optimization networks: an A/D converter, signal decision circuit, and a linear programming circuit,” *IEEE Transactions on Circuits and Systems*, Vol. 33, pp. 533-541, 1986.