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A Factorization Approach to the Analysis of Asynchronous Interconnected Discrete-Time Systems with Arbitrary Clock Ratios

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***Abstract*— This paper presents a model for distributed feedback systems that operate at almost identical sampling rates. A Toeplitz model approach is used in order to capture the effect of small synchronization errors. In previous works, a similar model was used to formulate necessary and sufficient stability conditions, under the simplifying assumption that the frequency ratios of the subsystems involved have a special rational form. In this sequel this assumption is lifted, by proposing a factorization of the feedback matrix that holds for arbitrary clock frequency ratios. This factorization property constitutes the first necessary step towards the generalization of the previous stability results to the general case of arbitrary clock frequency ratios. Furthermore, this generalization may enable the derivation of simple stability and robustness criteria in the presence of time-varying uncertain synchronization errors.**

This paper presents a model for distributed feedback systems that operate at almost identical sampling rates. A Toeplitz model approach is used in order to capture the effect of small synchronization errors. In previous works, a similar model was used to formulate necessary and sufficient stability conditions, under the simplifying assumption that the frequency ratios of the subsystems involved have a special rational form. In this sequel this assumption is lifted, by proposing a factorization of the feedback matrix that holds for arbitrary clock frequency ratios. This factorization property constitutes the first necessary step towards the generalization of the previous stability results to the general case of arbitrary clock frequency ratios. Furthermore, this generalization may enable the derivation of simple stability and robustness criteria in the presence of time-varying uncertain synchronization errors.

I. INTRODUCTION

In classical discrete-time systems, it is often assumed that all system components have the same clock frequency, and are working synchronously [1], [2]. However, in distributed and networked systems, different computers always operate at different clock frequencies, even though they might only differ very slightly from each other. The problem of synchronization has therefore received a lot of attention but most work approaches the problem from a communication viewpoint [3], [4]. There have been a small number of results that consider the synchronization problem from a system theoretic viewpoint [5-9], but most results are based on the assumption of a rational frequency ratio. This paper takes a Toeplitz matrix approach to the synchronization problem between two communicating systems. The paper builds on previous results [7], [8] on Toeplitz matrix based models and lifts the unrealistic assumption of clock frequency ratios having a special rational form. This generalization of the Toeplitz model is a first step towards the derivation of stability conditions for distributed systems with uncertain non-identical clock frequencies.

This paper is structured as follows: section 2 introduces a model for the case of two interconnected discrete-time systems with arbitrary non-identical clock frequencies. Section 3 presents a timing model for the switching instants of the two systems. A Toeplitz matrix based approach to model the overall system dynamics is introduced. In section 4, a factorization property of the Toeplitz model is proposed. Finally some concluding remarks are given in section 5.

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II. MODEL FOR ASYNCHRONOUS DISCRETE-TIME FEEDBACK SYSTEMS

A. Notations

Figure 1 shows a closed loop with two LTI discrete-time systems that are working asynchronously. The system in the forward path has the clock period T_1, while the one in the feedback path operates with a clock period T_2.

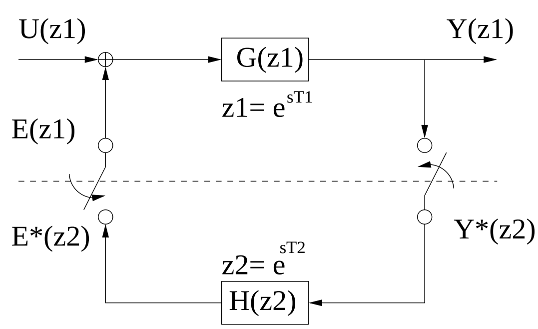

Fig. 1. Closed loop containing two asynchronously working discrete time systems

Due to the different clock periods it is necessary to introduce two variables z_1 and z_2 in the frequency domain:

$$z_i = e^{sT_i}, \qquad i = 1, 2 \tag{1}$$

(s: Laplace Transform variables). The transfer functions for the LTI system in the forward path and the feedback path are denoted by $G(z_1)$ and $H(z_2)$ respectively. $U(z_1)$ and $Y(z_1)$ stand for the Z transforms of the input and output sequences of the closed loop respectively. $Y^*(z_2)$ and $E^*(z_2)$ correspond to the input and output sequence transforms of

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system $H(z_2)$, respectively. Finally we assume that the ratio of the clock periods satisfies:

$$\frac{T_1}{T_2} = 1 - \frac{1}{q} \tag{2}$$

where q is no longer an integer as it was the case for previous works [8], [9]. This paper extends these previous results to arbitrary real parameters $q > 1$. Furthermore it is assumed that both clocks run with an offset at the origin of θ, which prevents them from producing coinciding samples at $t = 0$. Therefore the clocks in system 1 and 2 tick at time instants defined by:

$$t_1(k) = kT_1, \quad k \in \mathcal{Z} \tag{3}$$

$$t_2(k) = kT_2 + \theta, \quad k \in \mathcal{Z} \tag{4}$$

III. Toeplitz Approach to Synchronization

A. Matrix Representation for the Forward Path

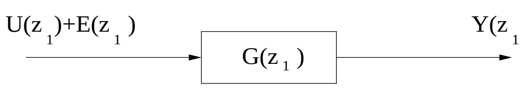

Fig. 2. Forward link: system 1

This section briefly presents the Toeplitz model approach introduced in [8]. It can be shown that the matrix representation of any causal discrete time-variant linear system can be obtained from its impulse response as follows:

$$G_n = [g_{(i-1,j-1)}]_{1 \le i,j \le n} \tag{5}$$

$\forall n > 0$, where $g(k, k_0), \quad g(k, k_0) = 0 \quad for \quad k < k_0$ represents the system response to a unit pulse at $k = k_0$. The input/output relationship between the signals $U(z_1) + E(z_1)$ and $Y(z_1)$ in Figure 2, can be fully determined by a matrix representation as in (5). In this case, matrix G_n is in Toeplitz form, due to the fact that the input/output relationship is linear, time-invariant:

$$G_n = \begin{pmatrix} g_0 & & & \\ g_1 & g_0 & O & \\ \vdots & \vdots & \ddots & \\ g_{n-1} & \cdots & g_1 & g_0 \end{pmatrix}, \tag{6}$$

where $\{g_i\}_{i \ge 0}$ are the coefficients of the impulse response of $G(z_1)$. G_n admits a simple factorization into two Toeplitz matrices with a few non-zero entries as follows:

$$\Rightarrow G_n = N_n^{(1)} (D_n^{(1)})^{-1} = (D_n^{(1)})^{-1} N_n^{(1)} \tag{7}$$

with

$$N_n^{(1)} = \begin{pmatrix} n_0^{(1)} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ n_1^{(1)} & n_0^{(1)} & \ddots & & & O & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ n_{m_1}^{(1)} & \ddots & \ddots & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & n_{m_1}^{(1)} & \cdots & n_1^{(1)} & n_0^{(1)} & 0 \\ 0 & \cdots & 0 & n_{m_1}^{(1)} & \cdots & n_1^{(1)} & n_0^{(1)} \end{pmatrix} \tag{8}$$

$$D_n^{(1)} = \begin{pmatrix} d_0^{(1)} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ d_1^{(1)} & d_0^{(1)} & \ddots & & & O & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ d_{n_1}^{(1)} & \ddots & \ddots & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & d_{n_1}^{(1)} & \cdots & d_1^{(1)} & d_0^{(1)} & 0 \\ 0 & \cdots & 0 & d_{n_1}^{(1)} & \cdots & d_1^{(1)} & d_0^{(1)} \end{pmatrix}, \tag{9}$$

where $n_i^{(1)}, i = 0, ..., m_1$ and $d_i^{(1)}, i = 0, ..., n_1$, are the coefficients of the transfer function $G(z^{-1})$:

$$G(z^{-1}) = \frac{n_0^{(1)} + ... + n_{m_1}^{(1)} z^{-m_1}}{d_0^{(1)} + ... + d_{n_1}^{(1)} z^{-n_1}} \tag{10}$$

B. Matrix Representation for the feedback path

In this section we derive the matrix operator that represents the linear relationship arising in the feedback link.

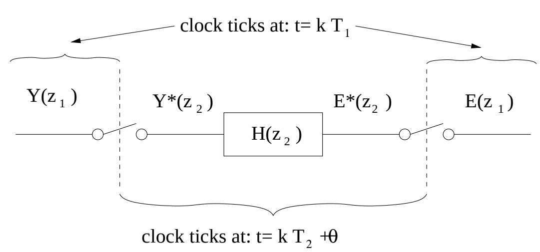

Fig. 3. Feedback link: serial connection of subsampling interface, system 2 and oversampling interface

H_n is defined as the matrix that relates the first n samples $Y_n = [y_0 ... y_{n-1}]^T$ of the sequence $\{y_k\}$ to the first k samples of the sequence U_k, $U_k = [u_0 ... u_{n-1}]^T$ as follows:

$$E_n = H_n Y_n \tag{11}$$

where,

$$H_n = \left(\begin{array}{cccc} h(0,0) & & O & \\ \vdots & \ddots & & \\ h(i-1,0) & & h(j-1,j-1) & \\ \vdots & & & \ddots \\ h(n-1,0) & \ldots & h(n-1,j) & \text{h(n-1,n-1)} \end{array} \right) \tag{12}$$

$$j > i \Rightarrow h(i,j) = 0, \quad i,j \in \mathcal{Z} \tag{13}$$

Due to the serial connection of the sub-sampling interface, system 2, and over-sampling interface, the overall feedback link is linear, but no longer time-invariant. This means that the input/output relationship can no longer be described by a transfer function or a time-invariant impulse response. As a replacement for the system description we can use a matrix operator that is obtained from the time-varying impulse response as in equation (5). (In retrospect, this motivates our introduction of a matrix operator for the forward path.) Therefore, we need to derive an explicit formulation for the feedback link time-varying impulse response $h(n,n_0)\mid_{n,n_0 \in \mathbb{Z}}$, which relates sequences $\{y_n\}_{n\in\mathbb{Z}}$ to $\{e_n\}_{n\in\mathbb{Z}}$ (see figure 1). This is made possible thanks to the following Theorem:

Theorem 1:

The time-varying discrete-time impulse response $h(n,n_0)|_{n,n_0>0}$, which relates sequences $\{y_n\}_{n>0}$ and $\{e_n\}_{n>0}$ through the feedback link in figure 1, satisfies:

$$h(i-1,j-1) = h_{k(i,j)}, \ \forall i,j \in \mathbb{Z} \tag{14}$$

where

$$k(i,j) = \begin{cases} -1, & \text{if } \lceil (j-1)\frac{T_1}{T_2} - \frac{\theta}{T_2} \rceil = \lceil j\frac{T_1}{T_2} - \frac{\theta}{T_2} \rceil \\ \left\lfloor (i-1)\frac{T_1}{T_2} - \frac{\theta}{T_2} \right\rfloor - \left\lceil (j-1)\frac{T_1}{T_2} - \frac{\theta}{T_2} \right\rceil, & \text{else} \end{cases}$$

and $\{h_n\}_{n>0}$ is the impulse response of the system in the feedback path $H(z_2)$, with the convention $h_n = 0, \ \forall n < 0$. $\square$

Proof:

The proof of this theorem consists in two steps. Each step distinguishes between two possible cases after a unit impulse is sent to the input of the feedback link at $t_0 = (j-1)T_1$ (i.e. $y(t) = \delta(t-t_0)$). The impulse either generates an output, or it is discarded by the system. The first step consists in identifying under which circumstances samples of the sequence $\{y_n\}_{n>0}$ are discarded by the feedback link. In other words, it identifies the values of j for which the samples $\{y_{j-1}\}$ do not contribute to the sequence $\{e_n\}_{n\in\mathbb{Z}}$. As a result the j^{th} columns of the H_n matrix will be all zero-valued. Let' s first assume that the impulse is discarded by the system. In that case, system 1 must have switched before system 2, erasing the impulse that was posted at t_0 (this is only possible because of the assumption $T_1 < T_2$). In other terms: the switching instants of system 1, t_0 and $t_0 + T_1$ will be most immediately followed by the same switching instant of system 2. The switching instant $t_2(k)$ of system 2 that most immediately follows an arbitrary instant t can be computed by:

$$\min\{t_2(k)|t_2(k) \geq t\} = \left\lceil \frac{t-\theta}{T_2} \right\rceil T_2 + \theta$$

Now, we use this equation to translate the fact that two consecutive switching instants of system 1, $t_1(j-1)$ and $t_1(j)$ will be immediately followed by the same switching instant of system 2:

$$\min\{t_2(k)|t_2(k) \geq t_1(j-1)\} = \min\{t_2(k)|t_2(k) \geq t_1(j)\}$$

$$\left\lceil (j-1)\frac{T_1}{T_2} - \frac{\theta}{T_2} \right\rceil + \theta = \left\lceil j\frac{T_1}{T_2} - \frac{\theta}{T_2} \right\rceil + \theta$$

Therefore system 1 will switch twice, at $t = (j-1)T_1$ and $t = jT_1$, before system, 2 if and only if:

$$\left\lceil (j-1)\frac{T_1}{T_2} - \frac{\theta}{T_2} \right\rceil = \left\lceil j\frac{T_1}{T_2} - \frac{\theta}{T_2} \right\rceil \tag{15}$$

In that case the response to a pulse of the feedback link at $t_0 = (j-1)T_1$ will be identically zero, and consequently the j^{th} column of H_n will only contain zeros. As a consequence, if (15) is satisfied for a certain value of $j \in \mathbb{N}$ then:

$$h(i-1,j-1) = 0, \ \forall i \tag{16}$$

Therefore, we artificially assign -1 to the value of the index function $k(i,j)$, to force the value of the time-varying transfer function to be zero, if condition (15) is satisfied. (by doing so we rely on the convention $h_k = 0, \ \forall k < 0$.) This completes the first step of the proof.

In a second step we derive the two-dimensional index function $k(i,j)$, which relates the time-invariant impulse response of system 2 $\{h_n\}|_{n\in\mathbb{Z}}$ to the overall feedback link time-varying impulse response through $h(i-1,j-1) = h_{k(i,j)}$. Let's now assume that the impulse is not discarded, but successfully produces an output. In that case, Figure 4 shows how the input of a unit pulse at $t_0 = (j-1)T_1$, causes the system in the feedback link to produce its first output after a variable amount of delay. This delay is simply caused by the synchronization errors between the clock of the two systems. The first sample will be produced at the switching instant of system 2 that most immediately follows the impulse. This is given by the following expression:

$$t_2'(j) \quad = \quad \left\lceil \frac{t_0-\theta}{T_2} \right\rceil T_2 + \theta$$

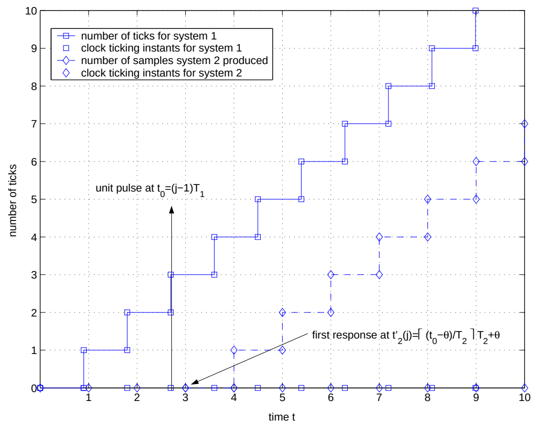

Fig. 4. Increments of the number of ticks for both systems after a unit pulse at $t_0 = (j-1)T_1$

$$= \left\lceil \frac{(j-1)T_1 - \theta}{T_2} \right\rceil T_2 + \theta \tag{17}$$

Consequently, during the time interval $t'_2(j) \leq t < t'_2(j) + T_2$ the signal sample value which will be posted and available for reading on the feedback output $e^*(t'_2(j))$ will be h_0:

$$e^*(t) = h_0,\ t'_2(j) \leq t < t'_2(j) + T_2,$$

the first sample value in the impulse response of system 2. Since after each period of T_2 the index of the samples in the impulse response is incremented, we can generalize the expression for the output signal at any time instant $t > t'_2(j)$ as follows:

$$\begin{aligned} e^*(t) &= h_{k(t)} \\ k(t) &= \left\lfloor \frac{t - t'_2(j)}{T_2} \right\rfloor \end{aligned} \tag{18}$$

where $k(t)$ is the continuous-time integer valued index function plotted in Figure 4 (as the dashed line stair-case function). Finally, the coefficients of the j^{th} row in matrix H_n are obtained after sampling $e^*(t)$ with sampling period T_1 and storing the first n sample values in a column-vector. Sampling equation (18) at $(i-1)T_1,\ i \geq 0$ yields by identification:

$$e^*((i-1)T_1) = h_{k((i-1)T_1)} = h_{k(i,j)}, \tag{19}$$

and since this should be true for any impulse response $\{h_k\}_{k \geq 0}$, we have:

$$k(i,j) = k((i-1)T_1), \tag{20}$$

With (20), (18) and (17) we have:

$$\begin{aligned} k(i,j) &= \left\lfloor \frac{(i-1)T_1 - t'_2(j)}{T_2} \right\rfloor \\ &= \left\lfloor (i-1)\frac{T_1}{T_2} - \left\lceil (j-1)\frac{T_1}{T_2} - \frac{\theta}{T_2} \right\rceil - \frac{\theta}{T_2} \right\rfloor \\ &= \left\lfloor (i-1)\frac{T_1}{T_2} - \frac{\theta}{T_2} \right\rfloor - \left\lceil (j-1)\frac{T_1}{T_2} - \frac{\theta}{T_2} \right\rceil \end{aligned} \tag{21}$$

$\forall (i,j)$, such that

$$(i,j) \notin \left\{ (i,j) \middle| \left\lceil (j-1)\frac{T_1}{T_2} - \frac{\theta}{T_2} \right\rceil = \left\lceil j\frac{T_1}{T_2} - \frac{\theta}{T_2} \right\rceil \right\}$$

In the following sections we will need the complete index function $k'(i,j)$ (without zeroed columns) which is defined as follows:

$$k'(i,j) = \left\lfloor (i-1)\frac{T_1}{T_2} - \frac{\theta}{T_2} \right\rfloor - \left\lceil (j-1)\frac{T_1}{T_2} - \frac{\theta}{T_2} \right\rceil, \tag{22}$$

such that:

$$k(i,j) = \begin{cases} -1 & \text{if } \lceil (j-1)\frac{T_1}{T_2} - \frac{\theta}{T_2} \rceil = \lceil j\frac{T_1}{T_2} - \frac{\theta}{T_2} \rceil \\ \text{k'(i,j)} & \text{otherwise,} \end{cases} \tag{23}$$

This completes the second step of the proof of theorem 1.

IV. Factorization of the Toeplitz Model

In section 3.1 we have seen that the Toeplitz matrix operator for the forward path admits a factorization with sparse matrix factors. In the following we will show that such a factorization also exists for the feedback path. Unlike in the case of the forward path, the matrix factors here no longer have a Toeplitz structure. Note that expression (23) does not require the assumption that the ratio $\frac{T_1}{T_2}$ be a rational number. Therefore matrix H_n can be computed from expression (5), with arbitrary values of $\frac{T_1}{T_2}$ (of q), and only requires the knowledge of $\frac{T_1}{T_2}$ and $\frac{\theta}{T_2}$.

Theorem 2:

If $\{h_k\}_{k \in (\mathcal{Z})}$ has the Z-transform

$$H(z^{-1}) = \frac{n_0 + ... + n_m z^{-m}}{d_0 + ... + d_m z^{-m}} \tag{24}$$

then:

$$H_n D_n = N_n \tag{25}$$

where,

$$D_n = \begin{pmatrix} d_{k'(2,1)} & \dots & d_{k'(2,j)} & \dots & d_{k'(2,n)} \\ \vdots & & \vdots & & \vdots \\ d_{k'(i,1)} & \dots & d_{k'(i,j)} & \dots & d_{k'(i,n)} \\ \vdots & & \vdots & & \vdots \\ d_{k'(n+1,1)} & \dots & d_{k'(n+1,j)} & \dots & d_{k'(n+1,n)} \end{pmatrix} \quad (26)$$

and

$$N_n = \begin{pmatrix} n_{k'(1,1)} & \dots & n_{k'(1,j)} & \dots & n_{k'(1,n)} \\ \vdots & & \vdots & & \vdots \\ n_{k'(i,1)} & \dots & n_{k'(i,j)} & \dots & n_{k'(i,n)} \\ \vdots & & \vdots & & \vdots \\ n_{k'(n,1)} & \dots & n_{k'(n,j)} & \dots & n_{k'(n,n)} \end{pmatrix} \quad (27)$$

with the convention $n_k = d_k = 0 \ \forall k \not\in \{0, 1, ..., m\}\Box$.

Proof:

In a first step, we show that matrices D_n and N_n are sparse. In fact, D_n and N_n only have non zero values on a diagonal band defined by:

$$0 \leq k'(i,j) \leq m \quad (28)$$

From (22), we have:

$$\begin{aligned} k'(i,j) &\geq \{(i-1)\frac{T_1}{T_2} - \frac{\theta}{T_2} - 1\} - \{(j-1)\frac{T_1}{T_2} - \frac{\theta}{T_2} + 1\} \\ &\geq (i-j)\frac{T_1}{T_2} - 2 \end{aligned} \quad (29)$$

Therefore if the inequality:

$$(i-j)\frac{T_1}{T_2} > m+2, \quad (30)$$

is satisfied, after substitution into inequality (29) we obtain:

$$k'(i,j) > m. \quad (31)$$

Therefore in the in a diagonal band defined by

$$0 \leq i - j \leq \frac{T_2}{T_1}(m+2), \quad (32)$$

we have

$$-1 \leq k'(i,j) \leq m, \quad (33)$$

and since $n_k = d_k = 0 \ \forall k \not\in \{0, 1, ..., m\}$, any matrix entry outside that diagonal band must be zero. Finally, on order the demonstration of theorem 2, we need to prove that the matrix product (25) yields. This is equivalent to prove the following equality for $i = 1, ..., n$ and $j = 1, ..., n$:

$$n_{k'(i,j)} = [h_{k(i,1)}...h_{k(i,n)}][d_{k'(2,j)}...h_{k'(n+1,j)}]^T. \quad (34)$$

Note that the zero coefficients of the row vector $[h_{k(i,1)}...h_{k(i,n)}]$ which correspond to the zero columns in matrix H_n, are multiplied with repeated samples $d_{k'(l,j)}$ satisfying $d_{k'(l,j)} = d_{k'(l,j)}$. As a consequence, after removal of all null products in (35), and re-indexation of the coefficients we find that the row column product in (35) equals the following convolutional product:

$$[h_{k(i,1)}...h_{k(i,n)}][d_{k'(2,j)}...h_{k'(n+1,j)}]^T = \sum_{k=0}^{m} h_{k'(i,j)-k} d_k, \quad (35)$$

and,

$$\sum_{k=0}^{m} h_{k'(i,j)-k} d_k = n_{k'(i,j)}, \quad (36)$$

as can be shown by taking the inverse Z-transform of (24), which immediately yields the desired result (35).

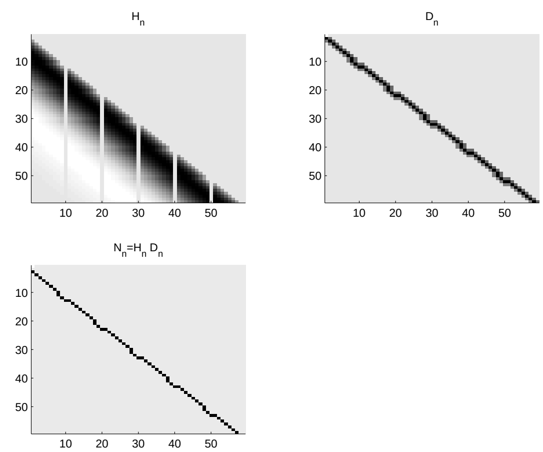

Fig. 5. Picture of Matrices H_n, D_n and N_n; zero is coded as white, and dark gray represents large magnitude values.

Figure (5) shows the pictures of non-zero entries of matrices H_n, D_n and N_n , for $q = \pi^2$, $\frac{\theta}{T_2} = 0.5$, $n = 50$ and $H(z^{-1}) = \frac{z^{-1}}{1-1.87z^{-1}+0.81z^{-2}}$

Remarks:

• Coinciding switching events lead to non-zeros coefficients on the main diagonal of H_n. This is a consequence of expression (22). Indeed, if $t_1(j-1)$ coincides with a switching instant of system 2 then $k'(j,j) = 0$.

• If q is an integer all matrices H_n, D_n and N_n have a block Toeplitz structure, with block dimension $(q \times q)$. This follows from (22) after observing that in that case:

$$k'(i,j) = k'(i+q, j+q)$$

• If q is not an integer, the diagonal blocks alternate in dimension between $\lfloor q \rfloor \times \lfloor q \rfloor$ or $\lceil q \rceil \times \lceil q \rceil$.

• If q is rational, the sequence of block dimensions $\lfloor q \rfloor$, $\lceil q \rceil$ is periodic.

• If q is an arbitrary real number, it can be shown that q asymptotically equals the average dimension of the diagonal blocks, if n becomes arbitrary large.

V. Conclusion

A Toeplitz approach model, which was introduced earlier in [8], has been further investigated. The factorization property, which was obtained in that same paper for a special case, has been generalized to the case of arbitrary frequency ratios. The extension of this factorization property constitutes the first step towards the derivation of simple and efficient stability criteria, for the case of uncertain and/or time-varying clock frequency ratios.

References

[1] O. Katsuhiko, "Discrete-Time Control Systems", Prentice Hall, 1995.

[2] A. Antoniou, "Digital Filters: Analysis, Design, and Applications", McGraw-Hill, New York, 1993

[3] D. Mills, "Internet time synchronization, the Network time protocol", *IEEE trans. on Communication*, vol. 39, pp. 1485-1493, October 1991

[4] J. Elson and D. Estrin, "Time synchronization for wireless sensor networks", *IPDPS'01*, 2001.

[5] A. F. Kleptsyn, V.S. Kozyakin, M. A. Krasnosel'skii, ams N. A.Kuznetsov, "Effect of small synchronization errors on stability of complex systems II", *Avtomatika i Telemekhanika.* Translated in: *Automation and Remote Control*,pp 309-314, 1984.

[6] A. F. Kleptsyn, V.S. Kozyakin, M. A. Krasnosel'skii, ams N. A.Kuznetsov, "Effect of small synchronization errors on stability of complex systems III", *Avtomatika i Telemekhanika.* Translated in: *Automation and Remote Control*,pp 1014-1018, 1984.

[7] Bauer, P.H., Lorand, C., Premaratne, K., "Stablity of Interconnected Discrete-Time Systems with Synchronization Errors", *Proceedings of the 2003 International Symposium on* ,pp. 568 -571, Volume: 4 , May 25 - 28, Bankok, Thailand, 2003.

[8] C. Lorand, P. H. Bauer, "A Model for Closed-Loop Discrete-time Systems with small Frequency Clock Drifts" , *Proceedings of the American Control Conference*, pp. 3317-3322, Denver, Colorado June 4-6, 2003.

[9] C. Lorand, P. H. Bauer, "Stablity analysis of Asynchronous Closed-Loop Discrete-time Systems" , *Proceedings of the American Control Conference*,pp. 3323-3328, Denver, Colorado June 4-6, 2003.