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# Output Feedback Boundary Control by Backstepping and Application to Chemical Tubular Reactor

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***Abstract*— In this paper we design exponentially convergent observers for a class of parabolic partial integro-differential equations (P(I)DEs) with only boundary sensing available. The problem is posed as a problem of designing invertible coordinate transformation of the observer error system into exponentially stable target system. Observer gain (output injection function) is shown to satisfy a well-posed hyperbolic PDE that is closely related to the hyperbolic PDE governing backstepping control gain for the state-feedback problem. It is shown how observer gain can be obtained directly from the control gain. Backstepping controller and observer are then combined to obtain a solution for the boundary output-feedback problem. Collocated and anti-collocated positions of sensor and actuator are considered. Explicit solutions to the output-feedback problem are obtained for certain classes of PDEs. It is shown that the order of the compensator can be substantially lowered without affecting stability. Simulation study for the model of chemical tubular reactor is presented.**

## I. INTRODUCTION

In this paper we propose backstepping based infinite dimensional observers for a class of linear parabolic partial differential equations, and use them to develop output feedback controllers for stabilization by boundary control. The backstepping observer problem is dual to the backstepping control problem. While in the latter one is limited to employing control only at the boundary, in the former it is sensing that is restricted only to the boundary.

To solve this problem we draw inspiration from a recent paper of Krener and Wang [6] in which a finite dimensional backstepping observer is proposed for nonlinear ODEs. They discover and exploit a triangular structure dual to that for the backstepping controller design [7]. The complexities present due to nonlinearities in finite dimension make the Krener-Wang observer local. This limitation is not an issue in our problem, as the class of parabolic PDEs we consider is linear. Our observers, due to the infinite dimension, take a form in which they are almost unrecognizable as Krener-Wang observers, however their structure is exactly that of Krener and Wang, where duality with backstepping control is exploited.

Putting together our earlier state-feedback boundary stabilizers [11] and the present observers, yields output feedback compensators for a class of parabolic PDEs. The controller/observer pair uses gain kernels that are computed by solving a linear hyperbolic PDE, an object much easier, both conceptually and computationally, than operator Riccati equations arising in LQG approaches to boundary control. Moreover, we exploit duality and find a relationship between the two kernels, reducing the problem to solving only one hyperbolic PDE, which is then, with some variations, used both for state feedback in the controller, and for output injection in the observer.

While this is not the first solution to the problems of boundary observer design or output-feedback boundary control (see, e.g. [1], [3], [8], [10]), our approach has a distinguishing feature relative to the existing methods. For a number of physically relevant problems we are able to find the observer/controller kernels in closed form, i.e., as explicit functions of the spatial variable. This, in turn, allows to even find closed-loop solutions explicitly.

## II. PROBLEM STATEMENT

We consider the following class of parabolic PDEs:

$$\begin{aligned} u_t(x,t) &= \varepsilon u_{xx}(x,t) + b(x)u_x(x,t) + \lambda(x)\,u(x,t) \\ &\quad + g(x)\,u(0,t) + \int_0^x f(x,y)u(y,t)\,dy \end{aligned} \tag{1}$$

for $x \in (0,1)$, $t > 0$, with boundary conditions[1]

$$u_x(0,t) = qu(0,t)\,, \tag{2}$$

$$u(1,t) = U(t) \quad \text{or} \quad u_x(1,t) = U(t)\,, \tag{3}$$

and under the assumption

$$\varepsilon > 0, \quad q \in \mathbb{R}, \lambda, g \in C^1[0,1]\,, f \in C^1([0,1]\times[0,1]) \tag{4}$$

Without loss of generality we can set $b(x) \equiv 0$, since it can be eliminated from the equation with the transformation

$$u(x,t) \mapsto u(x,t)e^{-\frac{1}{2\varepsilon}\int_0^x b(\tau)\,d\tau} \tag{5}$$

and the appropriate changes of $q$, $\lambda(x)$, $g(x)$, and $f(x,y)$.

The PDE (1)–(2) is controlled at $x = 1$ (using either Dirichlet or Neumann actuation) by a boundary input $U(t)$ that can be any function of time or a feedback law.

The problem is to design an exponentially convergent observer for the plant with only boundary measurements available. Our ultimate objective is to use these observers for output-feedback stabilization by boundary control. We consider two possibilities: the anti-collocated case, when sensor and actuator are placed at the opposite ends, and the collocated case, when sensor and actuator are placed at the same end.

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[1] The case of Dirichlet boundary condition at the zero end can be handled by setting $q = +\infty$.

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## III. BACKSTEPPING CONTROL DESIGN OVERVIEW

Though the main topic of this paper is observer design, it is crucial for further analysis to first recapitulate the backstepping approach to state-feedback boundary stabilization. This problem, for the class (1)–(2), was solved in [11] by finding a backstepping-style integral transformation

$$w(x,t) = u(x,t) - \int_0^x k(x,y)\,u(y,t)\,dy \tag{6}$$

that maps the system (1)–(2) into the system

$$\begin{aligned} w_t(x,t) &= \varepsilon w_{xx}(x,t) - cw(x,t), \quad x \in (0,1), & (7)\\ w_x(0,t) &= qw(0,t), & (8)\\ w(1,t) &= 0 \quad \text{or} \quad w_x(1,t) = 0, & (9) \end{aligned}$$

which is exponentially stable for $c \geq \varepsilon \bar{q}^2$ (respectively, $c \geq \varepsilon \bar{q}^2 + \varepsilon/2$) where $\bar{q} = \max\{0, -q\}$. Once the kernel $k(x,y)$ of the transformation (6) is found, the stabilizing boundary control at $x = 1$ can be obtained in the form

$$u(1,t) = \int_0^1 k_1(y)u(y,t)\,dy \tag{10}$$

for Dirichlet type of actuation, $k_1(y) = k(1,y)$, or

$$u_x(1,t) = k_1(1)u(1,t) + \int_0^1 k_2(y)u(y,t)\,dy \tag{11}$$

for Neumann type of actuation, $k_2(y) = k_x(1,y)$.

It was shown in [11] that the control gain kernel $k(x,y)$ satisfies the following hyperbolic PDE:

$$\begin{aligned} \varepsilon k_{xx}(x,y) - \varepsilon k_{yy}(x,y) &= (\lambda(y)+c)k(x,y) - f(x,y)\\ &\quad + \int_y^x k(x,\xi)f(\xi,y)\,d\xi, \qquad (12) \end{aligned}$$

for $(x,y) \in \mathcal{T} = \{x,y : 0 < y < x < 1\}$ with boundary conditions

$$\begin{aligned} \varepsilon k_y(x,0) &= \varepsilon q k(x,0) + g(x) - \int_0^x k(x,y)g(y)\,dy, & (13)\\ k(x,x) &= -\frac{1}{2\varepsilon}\int_0^x (\lambda(\xi)+c)\,d\xi, & (14) \end{aligned}$$

and the following theorem proved.

*Theorem 1:* The equation (12) with boundary conditions (13)–(14) has a unique $C^2(\mathcal{T})$ solution. For any $u_0 \in L_2(0,1)$ the system (1)–(2), (10) (or (11)) with the kernel $k_1(y) = k(1,y)$ (or $k_2(y) = k_x(1,y)$) has a unique classical solution $u(x,t) \in C^{2,1}((0,1)\times(0,\infty))$ and is exponentially stable at the origin, $u(x,t) \equiv 0$, in the $L_2(0,1)$ and $H_1(0,1)$ norms.

## IV. OBSERVER FOR ANTI-COLLOCATED OUTPUT FEEDBACK DESIGN

Suppose the only available measurement of our system is at $x = 0$, the opposite end to control actuation. We propose the following observer for the system (1)–(3) with Dirichlet actuation :

$$\begin{aligned} \hat{u}_t(x,t) &= \varepsilon \hat{u}_{xx}(x,t) + \lambda(x)\hat{u}(x,t) + \int_0^x f(x,y)\hat{u}(y,t)\,dy\\ &\quad + g(x)u(0,t) + p_1(x)[u(0,t) - \hat{u}(0,t)], & (15)\\ \hat{u}_x(0,t) &= qu(0,t) + p_{10}[u(0,t) - \hat{u}(0,t)], & (16)\\ \hat{u}(1,t) &= U(t). & (17) \end{aligned}$$

Here $p_1(x)$ and $p_{10}$ are output injection functions ($p_{10}$ is a constant) *to be designed*. Note, that we introduce output injection not only in the equation (15) but also at the boundary where measurement is available.

The observer error $\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t)$ satisfies the following PDE:

$$\begin{aligned} \tilde{u}_t(x,t) &= \varepsilon \tilde{u}_{xx}(x,t) + \lambda(x)\tilde{u}(x,t)\\ &\quad + \int_0^x f(x,y)\tilde{u}(y,t)\,dy - p_1(x)\tilde{u}(0,t), & (18)\\ \tilde{u}_x(0,t) &= -p_{10}\tilde{u}(0,t), & (19)\\ \tilde{u}(1,t) &= 0. & (20) \end{aligned}$$

Observer gains $p_1(x)$ and $p_{10}$ should be now chosen to stabilize the system (18)–(20). For linear finite dimensional systems the problem of finding the observer gains is dual to the problem of finding the control gains. This motivates us to try to solve the problem of stabilization of (18)–(20) by the same integral transformation approach as the (state feedback) boundary *control* problem reviewed in Section III. We look for a backstepping-like transformation

$$\tilde{u}(x,t) = \tilde{w}(x,t) - \int_0^x p(x,y)\,\tilde{w}(y,t)\,dy \tag{21}$$

that transforms system (18)–(20) into the exponentially stable (for $c \geq 0$) system

$$\begin{aligned} \tilde{w}_t(x,t) &= \varepsilon \tilde{w}_{xx}(x,t) - c\tilde{w}(x,t), \quad x \in (0,1), & (22)\\ \tilde{w}_x(0,t) &= 0, & (23)\\ \tilde{w}(1,t) &= 0. & (24) \end{aligned}$$

By substituting (21) into (18)–(20) and using (22)–(24) it can be shown that the kernel $p(x,y)$ must satisfy the following hyperbolic PDE:

$$\begin{aligned} \varepsilon p_{yy}(x,y) - \varepsilon p_{xx}(x,y) &= (\lambda(x)+c)p(x,y) - f(x,y)\\ &\quad + \int_y^x p(\xi,y)f(x,\xi)\,d\xi, \qquad (25) \end{aligned}$$

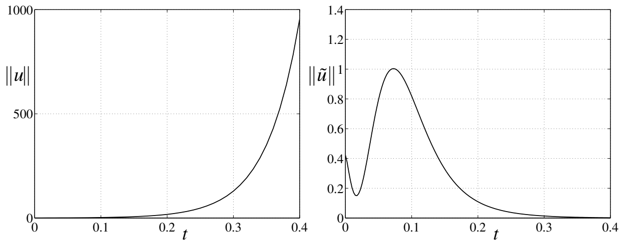
with the boundary conditions

$$\begin{aligned} \frac{d}{dx}p(x,x) &= \frac{1}{2\varepsilon}(\lambda(x)+c), & (26)\\ p(1,y) &= 0. & (27) \end{aligned}$$

In addition, the following conditions should be satisfied:

$$p_1(x) = \varepsilon p_y(x,0), \qquad p_{10} = p(0,0). \tag{28}$$

The problem is now to prove that PDE (25)–(27) is well-posed. Once the solution $p(x,y)$ to the problem (25)–(27) is found, the observer gains can be obtained from (28).




open-loop response  observer error

Fig. 1. Exponential convergence of the observer.

Let us make a change of variables $\breve{x} = 1 - y$, $\breve{y} = 1 - x$, $\breve{\lambda}(\breve{y}) = \lambda(x)$, $\breve{f}(\breve{x},\breve{y}) = f(x,y)$, $\breve{p}(\breve{x},\breve{y}) = p(x,y)$. In these new variables the problem (25)–(27) becomes

$$\begin{aligned} \varepsilon \breve{p}_{\breve{x}\breve{x}}(\breve{x},\breve{y}) - \varepsilon \breve{p}_{\breve{y}\breve{y}}(\breve{x},\breve{y}) &= (\breve{\lambda}(\breve{y}) + c)\breve{p}(\breve{x},\breve{y}) - \breve{f}(\breve{x},\breve{y}) \\ &\quad + \int_{\breve{y}}^{\breve{x}} \breve{p}(\breve{x},\xi)\breve{f}(\xi,\breve{y})\,d\xi, \quad (29) \\ \breve{p}(\breve{x},0) &= 0, \quad (30) \\ \breve{p}(\breve{x},\breve{x}) &= -\frac{1}{2\varepsilon}\int_0^{\breve{x}} (\breve{\lambda}(\xi) + c)\,d\xi. \quad (31) \end{aligned}$$

This is the same PDE as (12)–(14) with $q = \infty$, $g(x) = 0$ and $\lambda \to \breve{\lambda}$, $f \to \breve{f}$. Hence, using the results of Section III we immediately obtain the following result.

*Theorem 2:* The equation (25)–(27) has a unique $C^2(\mathcal{T})$ solution. For any $\tilde{u}_0(x) \in L_2(0,1)$ the system (18)–(20) with $p_1(x)$ and $p_{10}$ given by (28) has a unique classical solution $\tilde{u}(x,t) \in C^{2,1}((0,1)\times(0,\infty))$ and is exponentially stable at the origin, $\tilde{u}(x,t) \equiv 0$, in the $L_2(0,1)$ and $H_1(0,1)$ norms.

This result can be readily extended to the Neumann type of actuation as well.

The fact that the observer gain in transposed and switched variables satisfies the same class of PDEs as control gain is reminiscent of the duality property of state-feedback and observer design problems for linear finite-dimensional systems. The difference between the equations for observer and control gains is due to the fact that the observer error system does not contain terms with $g(x)$ and $q$ because $u(0,t)$ is measured.

The exponential convergence of the observer designed in this section is illustrated in Figure 1. Plant parameters were taken $g(x) \equiv 0$, $f(x) \equiv 0$, $q = \infty$, $\lambda = 5$. We can see that observer converges to the plant even though the plant is unstable.

## V. OBSERVER FOR COLLOCATED OUTPUT FEEDBACK DESIGN

Suppose now that the only available measurement is at the same end with actuation ($x = 1$). Of course, for the problem to make sense we should assume that the actuation is of Neumann type and $u(1)$ is available for measurement. We solve this problem with a restriction on the class (1)–(2) by setting $f(x,y) \equiv 0$, $g(x) \equiv 0$. This restriction is necessary because the boundary control problem is "lower-triangular" [7] whereas the boundary observer problem is "upper-triangular", thus the plant must be (tri-)diagonal.

Consider the following observer:

$$\begin{aligned} \hat{u}_t(x,t) &= \varepsilon \hat{u}_{xx}(x,t) + \lambda(x)\hat{u}(x,t) + p_1(x)[u(1) - \hat{u}(1)] \quad (32) \\ \hat{u}_x(0,t) &= q\hat{u}(0,t), \quad (33) \\ \hat{u}_x(1,t) &= -p_{10}[u(1,t) - \hat{u}(1,t)] + U(t). \quad (34) \end{aligned}$$

Here $p_1(x)$ and $p_{10}$ are output injection functions *to be designed*. The difference with the anti-collocated case (apart from injecting $u(1,t)$ instead of $u(0,t)$) is that gain $p_{10}$ is introduced in the other boundary condition.

The observer error $\tilde{u}(x)$ satisfies the equation

$$\begin{aligned} \tilde{u}_t(x,t) &= \varepsilon \tilde{u}_{xx}(x,t) + \lambda(x)\tilde{u}(x,t) - p_1(x)\tilde{u}(1,t), \quad (35) \\ \tilde{u}_x(0,t) &= q\tilde{u}(0,t), \quad (36) \\ \tilde{u}_x(1,t) &= p_{10}\tilde{u}(1,t). \quad (37) \end{aligned}$$

We are looking for the transformation

$$\tilde{u}(x,t) = \tilde{w}(x,t) - \int_x^1 p(x,y)\,\tilde{w}(y,t)\,dy \quad (38)$$

that maps (35)–(37) into the exponentially stable (for $c \geq \varepsilon\bar{q}^2 + \varepsilon/2$) target system

$$\begin{aligned} \tilde{w}_t(x,t) &= \varepsilon \tilde{w}_{xx}(x,t) - c\tilde{w}(x,t), \quad x \in (0,1), \quad (39) \\ \tilde{w}_x(0,t) &= q\tilde{w}(0,t), \quad (40) \\ \tilde{w}_x(1,t) &= 0. \quad (41) \end{aligned}$$

Note, that the transformation (38) is in upper-triangular form. By substituting (38) into (35)–(37) and using (39)–(41) it can be shown that the kernel $p(x,y)$ must satisfy the following hyperbolic PDE:

$$\varepsilon p_{yy}(x,y) - \varepsilon p_{xx}(x,y) = (\lambda(x) + c)p(x,y), \quad (42)$$

with the boundary conditions

$$\begin{aligned} p_x(0,y) &= qp(0,y), \quad (43) \\ p(x,x) &= -\frac{1}{2\varepsilon}\int_0^x (\lambda(\xi) + c)\,d\xi. \quad (44) \end{aligned}$$

In addition, the following conditions must be satisfied:

$$p_1(x) = -\varepsilon p_y(x,1), \qquad p_{10} = p(1,1). \quad (45)$$

Once the solution $p(x,y)$ to the problem (42)–(44) is found, the observer gains can be obtained from (45).

Similar to the anti-collocated case we introduce new variables $\breve{x} = y$, $\breve{y} = x$, $\breve{p}(\breve{x},\breve{y}) = p(x,y)$ in which (42)–(44) becomes

$$\begin{aligned} \varepsilon \breve{p}_{\breve{x}\breve{x}}(\breve{x},\breve{y}) - \varepsilon \breve{p}_{\breve{y}\breve{y}}(\breve{x},\breve{y}) &= (\lambda(\breve{y}) + c)\breve{p}(\breve{x},\breve{y}), \quad (46) \\ \breve{p}_{\breve{y}}(\breve{x},0) &= q\breve{p}(\breve{x},0), \quad (47) \\ \breve{p}(\breve{x},\breve{x}) &= -\frac{1}{2\varepsilon}\int_0^{\breve{x}} (\breve{\lambda}(\xi) + c)\,d\xi, \quad (48) \end{aligned}$$

This is exactly the same PDE as (12)–(14) for $k(\breve{x},\breve{y})$ and therefore using Theorem 1 we obtain the following result.

*Theorem 3:* The equation (42)–(44) has a unique $C^2(\mathcal{T})$ solution. For any $\tilde{u}_0(x) \in L_2(0,1)$ the system (35)–(37) with $p_1(x)$ and $p_{10}$ given by (45) has a unique classical solution $\tilde{u}(x,t) \in C^{2,1}((0,1)\times(0,\infty))$ and is exponentially stable at the origin, $\tilde{u}(x,t) \equiv 0$, in the $L_2(0,1)$ and $H_1(0,1)$ norms.

The duality of the observer design to control design in the collocated case is even more evident than in the anti-collocated case. The kernel of the coordinate transformation (38) is equal to the kernel of the transformation (6) with switched variables: $p(x,y) = k(y,x)$. The observer gain is actually equal (up to a factor of $-\varepsilon$) to the control gain: $p_1(x) = -\varepsilon k_2(x)$, $p_{10} = k_1(1)$.

## VI. OUTPUT FEEDBACK CONTROL LAWS

Having at our disposal stabilizing controllers and exponentially convergent observers we can now combine the results of Sections III-V to obtain solutions to the output-feedback problem. First we formulate the result for the anti-collocated case.

*Theorem 4:* Let $k(x,y)$ be the solution of (12)–(14), $p_1(x)$, $p_{10}$ be the solutions of (25)–(28), and let the assumptions (4) and $c \geq \varepsilon \bar{q}^2$ hold. Then for any $u_0$, $\hat{u}_0 \in L_2(0,1)$ the system (1)–(2) with the controller

$$u(1,t) = \int_0^1 k_1(y)\hat{u}(y,t)\,dy \tag{49}$$

and the observer

$$\begin{aligned}
\hat{u}_t(x,t) &= \varepsilon \hat{u}_{xx}(x,t) + \lambda(x)\hat{u}(x,t) + \int_0^x f(x,y)\hat{u}(y,t)\,dy \\
&\quad + g(x)u(0,t) + p_1(x)[u(0,t) - \hat{u}(0,t)], \qquad (50)\\
\hat{u}_x(0,t) &= qu(0,t) + p_{10}[u(0,t) - \hat{u}(0,t)], \qquad (51)\\
\hat{u}(1,t) &= \int_0^1 k_1(y)\hat{u}(y,t)\,dy, \qquad (52)
\end{aligned}$$

have the unique classical solutions $u(x,t)$, $\hat{u}(x,t) \in C^{2,1}((0,1)\times(0,\infty))$ and are exponentially stable at the origin, $u(x,t) \equiv 0$, $\hat{u}(x,t) \equiv 0$, in the $L_2(0,1)$ and $H_1(0,1)$ norms.

*Proof:* The coordinate transformation

$$\hat{w}(x,t) = \hat{u}(x,t) - \int_0^x k(x,y)\hat{u}(y,t)\,dy \tag{53}$$

maps (50)–(52) into the system

$$\begin{aligned}
\hat{w}_t(x,t) &= \varepsilon \hat{w}_{xx}(x,t) - c\hat{w}(x,t) + \Big\{ p_1(x) + g(x) + \\
&\quad - \int_0^x k(x,y)(p_1(y) + g(y))\,dy \Big\} \tilde{w}(0,t)\,, \qquad (54)\\
\hat{w}_x(0,t) &= q\hat{w}(0,t) + (p_{10} + q)\tilde{w}(0,t)\,, \qquad (55)\\
\hat{w}(1,t) &= 0. \qquad (56)
\end{aligned}$$

The $\tilde{w}$-system (22)–(24) and the homogeneous part of the $\hat{w}$-system (54)–(56) (without $\tilde{w}(0,t)$, where $\tilde{w}(0,t)$ is driving the $\hat{w}$-system (54)–(55) through a $C^1$ function of $x$) are exponentially stable heat equations. The interconnection of the two heat equations ($\hat{w}$, $\tilde{w}$) is a cascade, and therefore the combined ($\hat{w}$, $\tilde{w}$) system is exponentially stable in $L^2$ and $H^1$. Hence, the system ($\hat{u}$, $\tilde{u}$) is also exponentially stable since it is related to ($\hat{w}$, $\tilde{w}$) by the invertible coordinate transformation (21), (53). This directly implies the closed-loop stability of ($u$, $\hat{u}$). ∎

A similar result holds for the collocated case.

*Theorem 5:* Let $k(x,y)$ be the solution of (12)–(14), $p_1(x)$ be the solution of (42)–(45) and let the assumptions (4) and $c \geq \varepsilon \bar{q}^2 + \varepsilon/2$ hold. Then for any $u_0$, $\hat{u}_0 \in L_2(0,1)$ the system (1)–(2) ($g(x) \equiv 0$, $f(x,y) \equiv 0$) with the controller

$$u_x(1,t) = k_1(1)u(1,t) + \int_0^1 k_2(y)\hat{u}(y,t)\,dy \tag{57}$$

and the observer

$$\begin{aligned}
\hat{u}_t(x,t) &= \varepsilon \hat{u}_{xx}(x,t) + \lambda(x)\hat{u}(x,t) + p_1(x)[u(1) - \hat{u}(1)]\,, \qquad (58)\\
\hat{u}_x(0,t) &= q\hat{u}(0,t)\,, \qquad (59)\\
\hat{u}_x(1,t) &= k_1(1)\hat{u}(1,t) + \int_0^1 k_2(y)\hat{u}(y,t)\,dy\,, \qquad (60)
\end{aligned}$$

have the unique classical solutions $u(x,t)$, $\hat{u}(x,t) \in C^{2,1}((0,1)\times(0,\infty))$ and are exponentially stable at the origin, $u(x,t) \equiv 0$, $\hat{u}(x,t) \equiv 0$, in the $L_2(0,1)$ and $H_1(0,1)$ norms.

*Proof:* Using the fact that $u(1,t)$ is measured and $p_{10} = k(1,1)$ we can avoid boundary injection term in (60) by using $u(1,t)$ instead of $\hat{u}(1,t)$ in the controller (57). With this modification the error system (35)–(37) remains the same. The rest of the proof is similar to the proof of Theorem 4. ∎

## VII. EXPLICIT CONSTRUCTION

For some classes of systems our approach can give explicit solutions for the boundary output-feedback stabilization problem which is not the case with existing methods. In this section we present two important cases.

### A. Explicit solution for constant $\lambda(x) \equiv \lambda_0$

Consider the unstable heat equation with anti-collocated boundary actuation and sensing ($u(0,t)$ measured, $u(1,t)$ controlled)

$$\begin{aligned}
u_t(x,t) &= \varepsilon u_{xx}(x,t) + \lambda_0 u(x,t)\,, \qquad (61)\\
u_x(0,t) &= 0\,, \qquad (62)
\end{aligned}$$

The open-loop system (61)–(62) (with $u(1,t) = 0$) is unstable with arbitrarily many unstable eigenvalues.

*Theorem 6:* The controller (49) with the observer (50)–(52) where

$$k_1(x) = -a\frac{I_1(\sqrt{a(1-x^2)})}{\sqrt{a(1-x^2)}}, \quad k_1(1) = -\frac{a}{2}\,, \tag{63}$$

$$p_1(x) = \frac{a(1-x)}{x(2-x)} I_2(\sqrt{ax(2-x)}), \quad p_{10} = -\frac{a}{2}\,, \tag{64}$$

exponentially stabilizes the zero solution of (61)–(62).

*Proof:* The gain kernel for the state-feedback problem has been found in [11] by solving (12)–(14) analytically:

$$k(x,y) = -ax\frac{I_1(\sqrt{a(x^2-y^2)})}{\sqrt{a(x^2-y^2)}}\,, \tag{65}$$

where $a = (\lambda_0 + c)/\varepsilon$ and $I_1$ is the modified Bessel function of the first order. The solution to (29)–(31) with $\lambda(x) \equiv \lambda_0$, $q = \infty$, $g(x) \equiv 0$, and $f(x,y) \equiv 0$ is [11]

$$\check{p}(\check{x},\check{y}) = -a\check{y}\frac{I_1(\sqrt{a(\check{x}^2-\check{y}^2)})}{\sqrt{a(\check{x}^2-\check{y}^2)}}\,. \tag{66}$$

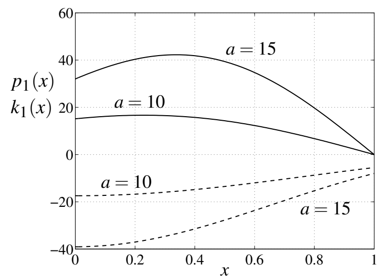



Fig. 2. Controller (dashed) and observer (solid) gains for the anti-collocated case.

This gives the observer gains (64). The stability of the closed-loop system is ensured by Theorem 4. ■

In Figure 2 the control and the observer gains for the different values of the parameter $a$ are shown.

The closed-loop system was simulated with $\varepsilon = 1$, $\lambda_0 = 10$, $c = 5$, $u(x,0) = 2e^{-2x}\sin(\pi x)$. With this choice of the parameters the open-loop system has two unstable eigenvalues. The plant and the observer are discretized using a finite difference method. Since designs exist where, in principle, the order of the observer can be as low as the number of unstable eigenvalues, we design the low order compensator by taking a coarse grid (keeping the fine discretization of the plant, 100 points in our case). The controller is able to stabilize the system with just a 6th order compensator. In Figure 3 the pole-zero map and Bode plots of the low order compensator are shown.

### B. Explicit solution for a family of nonconstant $\lambda(x)$

The other case in which the output-feedback problem can be solved explicitly is the heat equation with a non-constant coefficient:

$$u_t(x,t) = \varepsilon u_{xx}(x,t) + \lambda_{\alpha\beta}(x)u(x,t)\,, \quad x \in (0,1)\,, \tag{67}$$
$$u(0,t) = 0\,, \tag{68}$$

where

$$\lambda_{\alpha\beta}(x) = \frac{2\varepsilon\alpha^2}{\cosh^2(\alpha x - \beta)}\,. \tag{69}$$

Equations of the form (67)–(68) often describe the heat/mass transfer systems with heat generation or volumetric chemical reactions. The coefficient $\lambda_{\alpha\beta}(x)$ parameterizes a family of "one-peak" functions. The free parameters $\alpha$ and $\beta$ are chosen so that the maximum of $\lambda_{\alpha\beta}(x)$ is $2\alpha^2$ and is achieved at $x = \beta/\alpha$. Examples of $\lambda_{\alpha\beta}(x)$ for different values of $\alpha$ and $\beta$ are shown in Figure 4. The open-loop system (67)–(68) (with $u(1,t) = 0$) is unstable for all three cases shown in Figure 4.

Since the plant is in the diagonal form (there is no terms with $g(x)$ and $f(x,y)$), we choose to collocate the sensor and the actuator at $x = 1$.

*Theorem 7:* The controller (57) with the observer (58)–(60) where

$$k_2(x) = -\alpha^2\tanh(\beta)e^{(1-x)\alpha\tanh\beta}(\tanh\beta - \tanh(\beta - \alpha x)) \tag{70}$$
$$p_1(x) = -\varepsilon k_2(x), \quad p_{10} = -\alpha(\tanh(\beta) - \tanh(\beta - \alpha)) \tag{71}$$

exponentially stabilizes the zero solution of (67)–(68).

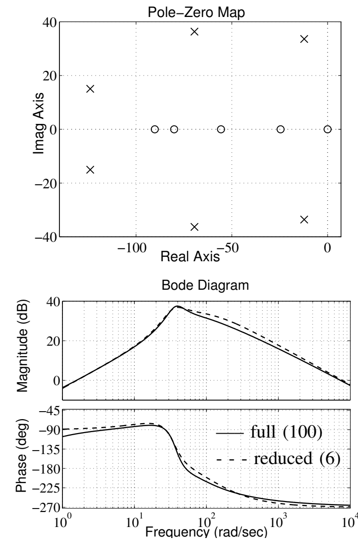



Fig. 3. Pole-zero map and Bode plot of the compensator for $\lambda(x) \equiv \lambda_0$.

*Proof:* The stabilizing kernel (70) for (67)–(68) was obtained in [11]. The stability of the closed-loop system is ensured by Theorem 5. ■

In Figure 5 the observer gains corresponding to $\lambda_{\alpha\beta}(x)$ from Figure 4 are shown.

### C. Explicit solution for $\lambda(x) = \lambda_0 + \lambda_{\alpha\beta}(x)$

In Sections VII-A and VII-B we considered two interesting examples of solving an output-feedback problem explicitly, in a closed form. One can actually combine these two solutions to get a solution for a heat equation with $\lambda(x) = \lambda_0 + \lambda_{\alpha\beta}(x)$. It can be done in two steps. First, transform the error system into the target system (39)–(41) with $c = -\lambda_0$. It will give a PDE for $p(x,y)$ with $\lambda(x) = \lambda_{\alpha\beta}(x)$ whose solution we know. The target system will not be stable, but it will have constant coefficients. Second, stabilize this target system with $p_1(x)$ corresponding to a constant $\lambda_0$. The resulting gain will be expressed in quadratures in terms of gains for $\lambda_0$ and $\lambda_{\alpha\beta}(x)$. Denote by $p^{\alpha\beta}(x,y)$ and $p^a(x,y)$ the observer gains for the heat equation with $\lambda(x) = \lambda_{\alpha\beta}(x)$ and $\lambda(x) = \lambda_0$ ($a = (\lambda_0 + c)/\varepsilon$), respectively. Then following the procedure described above we obtain the observer gain for the heat equation with $\lambda(x) = \lambda_0 + \lambda_{\alpha\beta}(x)$:

$$p_1(x) = p_1^a(x) + p_1^{\alpha\beta}(x) + \varepsilon p_{10}^a p^{\alpha\beta}(x,1) - \int_x^1 p^{\alpha\beta}(x,\xi)p_1^a(\xi)\,d\xi, \tag{72}$$

and $p_{10} = p_{10}^a + p_{10}^{\alpha\beta}$. For example for $\beta = 0$ one can get the closed-form solution

$$p_1(x) = \frac{\varepsilon a^2}{\varphi(x)^2}I_2(\varphi(x)) + \varepsilon a\alpha\tanh(\alpha x)\frac{I_1(\varphi(x))}{\varphi(x)}\,. \tag{73}$$

## VIII. CHEMICAL TUBULAR REACTOR EXAMPLE

In this section we present the simulation results for a linearization of an adiabatic chemical tubular reactor. For the case when Peclet numbers for heat and mass transfer

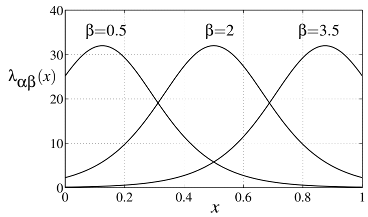



Fig. 4. "One-peak" $\lambda_{\alpha\beta}(x)$ for $\alpha = 4$, $\varepsilon = 1$.

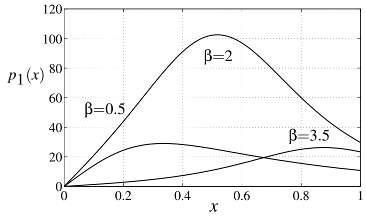



Fig. 5. Observer gain (71) for $\alpha = 4$, $\varepsilon = 1$.

are equal (Lewis number of unity) the two equations for the temperature and concentration can be reduced to one equation [4]

$$
\begin{aligned}
v_t &= Pe^{-1} v_{\xi\xi} - v_\xi + Da\,(b-v) e^{\frac{v}{1+\mu v}}, \quad \xi \in (0,1)\,, && (74)\\
v_\xi(0) &= Pe\, v(0)\,, && (75)\\
v_\xi(1) &= 0\,, && (76)
\end{aligned}
$$

where $Pe$ is Peclet number, $Da$ is the Damköhler number, $\mu$ is the dimensionless activation energy, and $b$ is the dimensionless adiabatic temperature rise.

For a particular choice of system parameters ($Pe = 3$, $Da = 0.05$, $\mu = 0.05$, and $b = 10$) system (74)–(76) has three equilibrium profiles (Figure 6) [5]. As shown in [5], the middle profile is unstable while the outer two profiles are stable. Linearization of the system (74)–(76) around the unstable equilibrium profile $\bar{v}(\xi)$ gives [2]

$$
\begin{aligned}
v_t &= Pe^{-1} v_{\xi\xi} - v_\xi + Da\,F(\bar{v}(\xi))v, \quad \xi \in (0,1)\,, && (77)\\
v_\xi(0) &= Pe\, v(0)\,, && (78)\\
v_\xi(1) &= 0\,, && (79)
\end{aligned}
$$

where $v$ now stands for the deviation variable from the steady state $\bar{v}(\xi)$ and $F$ is a spatially dependent coefficient defined as

$$
F(\bar{v}) = \left( \frac{b-\bar{v}}{(1+\mu\bar{v})^2} - 1 \right) e^{\frac{\bar{v}}{1+\mu\bar{v}}} \qquad (80)
$$

In a real application control would be implemented through small variations of $T_{in}$ and $C_{in}$ at the 0-end [4]. To put the system in our standard form with control at 1-end and without $v_\xi$-term, we introduce variable change $x = 1-\xi$, $u(x,t) = v(1-x,t)\exp(Pe\,x/2)$ and obtain the following system

$$
\begin{aligned}
u_t &= Pe^{-1} u_{xx} + \lambda_c(x) u\,, && (81)\\
u_x(0) &= (Pe/2)u(0)\,, && (82)\\
u_x(1) &= -(Pe/2)u(1) + u_x^c(1)\,, && (83)
\end{aligned}
$$

where $\lambda_c(x)$ is defined as

$$
\lambda_c(x) = (Pe-2)/4 + Da\,F(\bar{v}(1-x))\,, \qquad (84)
$$

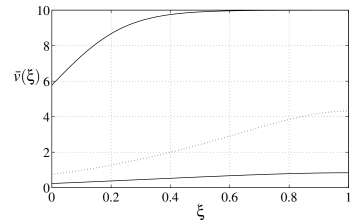



Fig. 6. Steady-state profiles for the adiabatic chemical tubular reactor with $Pe = 3$, $Da = 0.05$, $\mu = 0.05$, and $b = 10$.

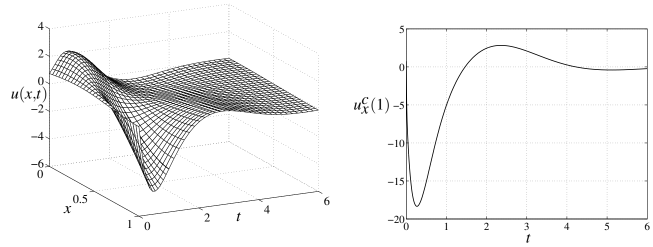



Fig. 7. Closed-loop response and the control effort for the chemical tubular reactor example.

and $u_x^c(1)$ stands for the control law to be designed. We consider the collocated case, i.e. both the sensor and the actuator are placed at 1-end.

The observer gain has been found by solving (46)–(48) numerically. In simulations the observer was discretized using a finite difference method on a coarse grid with order 6. The order of the plant (81)–(83) was taken 100. The closed-loop response of the system and the control effort for $u(x,0) = 0.25\sin(3\pi x/2) + 0.75$ and $c = 1$ are shown in Figure 7. We can see that the low-order compensator can successfully stabilize the system.

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