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$\mathcal{H}_\infty$ Suboptimal Model Reduction for Singular Systems

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***Abstract*— Model reduction problem was investigated for singular systems. To solve the problem, the Silverman-Ho algorithm was given, and based on this algorithm, an optimal model reduction method for the fast subsystem was presented to obtain reduced-order stable models for singular systems. Sequentially, the $\mathcal{H}_\infty$ suboptimal model reduction algorithm was obtained for singular systems. The advantage of the presented algorithm is that the impulsive nature of singular system is preserved in the reduced-order models. Then some necessary and sufficient conditions for the existence of a stable reduced-order system were given and these conditions can be verified numerically. Finally, illustrative examples were given to show the effectiveness of the proposed approach.**

I. INTRODUCTION

In recent years, singular systems have been investigated extensively due to their applications in modelling and control of electrical circuits, power systems and economics, etc. Some important characteristics of singular systems include combined dynamic and static solutions, impulsive behaviors and large dimensionality. Thus model reduction is vital for analysis and design of controller for such systems [3], [5].

The initial investigation of model reduction for singular systems was the chained aggregation method in [6]. The authors there developed a generalized chained aggregation algorithm and gave an intuitive interpretation of the exact aggregation conditions for singular systems. The aim of the proposed method is to remove the unobservable subspace. Initial behavior of singular systems was also taken into consideration while performing model reduction. However, as pointed out in [7], the main drawback of this method is the high level of computational effort.

In 1994, Perev and Shafai [7] considered model reduction for singular system via balanced realizations and gave a model reduction algorithm. Unfortunately, their method ignored the impulsive behavior which is of paramount importance to singular systems. The reduced order model may become a normal state space system, which has no impulsive behavior and does not track the original system response properly. Liu and Sreeram [4] used the Nehari's approximation algorithm and overcome the problem. The reduced-order model is a really singular system and the approximation has been obtained as desired. For discrete singular systems, Zhang et al. [3] discussed the same problem with $\mathcal{H}_2$ norm. Recently, Zhang et al. [8] discussed the $\mathcal{H}_\infty$ suboptimal model reduction problem for singular systems and some sufficient conditions are obtained when the original singular system is not impulsive free. However, it requires that the transfer function matrix of the error system is rational in order to guarantee that $\mathcal{H}_\infty$ norm exists. Anyway, the existence problem for the $\mathcal{H}_\infty$ norm of the error system has not been solved there. Along this line of research, recently, the existence problem was investigated and a model reduction algorithm was proposed in [12]. When the original singular system is impulsive free, the model reduction problem for singular systems were investigated in [10], [11] respectively for the cases of continuous time and discrete time via linear matrix inequalities (LMI) approach.

In this paper, we will discuss the model reduction problem for singular systems based on the results in [12] and will present a new approach for the $\mathcal{H}_\infty$ suboptimal model reduction via solving the minimum rank problem for a matrix set. In order to preserve the impulsive nature of singular systems, we will use reduced-order fast subsystems to approximate the fast subsystems. Some necessary and sufficient conditions will be obtained for the existence of a stable reduced-order system. Further, an algorithm has been designed for the $\mathcal{H}_\infty$ suboptimal model reduction if the existence condition is met.

The organization of this paper is as following. In section 2, some preliminaries and the suboptimal model reduction problem will be presented. In section 3, the Silverman-Ho algorithm will be given. In section 4, the main results about the $\mathcal{H}_\infty$ suboptimal model reduction will be given and the proposed algorithm will be illustrated in section 5. Conclusions will be given in section 6.

II. PROBLEM FORMULATION

Consider the following singular systems

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0-) = x_0, \\ y(t) &= Cx(t); \end{aligned} \tag{1}$$

where $x(t)\in\mathcal{R}^n$ is the state vector, $u(t)\in\mathcal{R}^q$ is the input vector and $y(t)\in\mathcal{R}^m$ is the output vector. $E\in\mathcal{R}^{n\times n}$, $A\in\mathcal{R}^{n\times n}$, $B\in\mathcal{R}^{n\times q}$, $C\in\mathcal{R}^{m\times n}$ are constant matrices with E possibly singular. Assume that the matrix pair (E, A) is regular (i.e.,$|sE - A| \not\equiv 0$). In this paper, the realization quadruple (E, A, B, C) is used to represent the system (1). All the matrices in this paper are assumed to have appropriate dimensions.

From [2], it is known that there exist two square nonsingular matrices Q and P such that system (1) can be

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transformed into the Weierstrass canonical form:

$$\begin{aligned}
&\dot{x}_1(t) = A_1x_1(t) + B_1u(t), \quad x_1(0-) = x_{1,0},\\
&y_1(t) = C_1x_1(t);\\
&N\dot{x}_2(t) = x_2(t) + B_2u(t), \quad x_2(0-) = x_{2,0},\\
&y_2(t) = C_2x_2(t);
\end{aligned} \tag{2}$$

where $x_1(t)\in\mathcal{R}^{n_1}$, $x_2(t)\in\mathcal{R}^{n_2}$, $n_1 + n_2 = n$, $N\in\mathcal{R}^{n_2\times n_2}$ is nilpotent, and

$$\begin{aligned}
&QEP = diag(I, N), \quad QAP = diag(A_1, I),\\
&CP = [C_1 \;\; C_2], \quad P^{-1}x(t) = [x_1^T(t) \;\; x_2^T(t)]^T,\\
&QB = [B_1^T \;\; B_2^T]^T, \quad y(t) = y_1(t) + y_2(t).
\end{aligned}$$

System (1) is called system restricted equivalent(s.r.e) to system (2). The transfer function matrix $G(s)$ is invariant under the s.r.e. transformation, i.e.,

$$\begin{aligned}
G(s) &= C(sE - A)^{-1}B = CP(sQEP - QAP)^{-1}QB\\
&= C_1(sI - A_1)^{-1}B_1 + C_2(sN - I)^{-1}B_2,
\end{aligned} \tag{3}$$

and

$$\begin{aligned}
&C_2(sN - I)^{-1}B_2\\
&= -C_2B_2 - sC_2NB_2 - \cdots - s^{h-1}C_2N^{h-1}B_2,
\end{aligned}$$

where $C_2N^{h-1}B_2 \neq 0$.

The aim of this paper is to investigate the $\mathcal{H}_\infty$ suboptimal model reduction for singular systems. Suppose the reduced-order singular system is

$$\begin{aligned}
&E_r\dot{x}_r(t) = A_rx_r(t) + B_ru(t),\\
&y(t) = C_rx_r(t);
\end{aligned} \tag{4}$$

which is assumed to be regular. Then there are two matrices Q_r and P_r such that

$$\begin{aligned}
&\dot{x}_{1r}(t) = A_{1r}x_{1r}(t) + B_{1r}u(t), \quad x_{1r}(0-) = x_{1r,0},\\
&y_{1r}(t) = C_{1r}x_{1r}(t);\\
&N_r\dot{x}_{2r}(t) = x_{2r}(t) + B_{2r}u(t), \quad x_{2r}(0-) = x_{2r,0},\\
&y_{2r}(t) = C_{2r}x_{2r}(t);
\end{aligned} \tag{5}$$

where $x_{1r}(t) \in \mathcal{R}^{n_{1r}}$, $x_{2r}(t) \in \mathcal{R}^{n_{2r}}$, $n_{1r} + n_{2r} = n_r$, $N_r \in \mathcal{R}^{n_{2r}\times n_{2r}}$ is nilpotent, and

$$\begin{aligned}
&Q_rE_rP_r = diag(I, N_r), \quad Q_rA_rP_r = diag(A_{1r}, I),\\
&C_rP_r = [C_{1r} \;\; C_{2r}], \quad P_r^{-1}x_r(t) = [x_{1r}^T(t) \;\; x_{2r}^T(t)]^T,\\
&Q_rB_r = [B_{1r}^T \;\; B_{2r}^T]^T, \quad y(t) = y_{1r}(t) + y_{2r}(t).
\end{aligned}$$

The associated error system between the original system and the reduced-order system will be

$$\begin{aligned}
&E_e\dot{x}_e(t) = A_ex_e(t) + B_eu(t),\\
&y_e(t) = C_ex_e(t);
\end{aligned} \tag{6}$$

where $x_e^T(t) = [x^T(t) \;\; x_r^T(t)]^T$, $y_e \in \mathcal{R}^m$, and

$$\begin{aligned}
&E_e = diag(E, \;\; E_r), \quad A_e = diag(A, \;\; A_r),\\
&B_e^T = [B^T \;\; B_r^T]^T, \quad C_e = [C \;\; -C_r].
\end{aligned}$$

Let

$$Q_e = diag(Q, \;\; Q_r), \quad P_e = diag(P, \;\; P_r).$$

Then the $\mathcal{H}_\infty$ norm of the transfer function matrix $G_e(s)$ for the error system is

$$\begin{aligned}
&\|G_e(s)\|_\infty\\
&= \|C_eP_eP_e^{-1}(sE_e - A_e)^{-1}Q_e^{-1}Q_eB_e\|_\infty\\
&= \|C_1(sI - A_1)^{-1}B_1 - C_{1r}(sI - A_{1r})^{-1}B_{1r}\\
&\quad + C_2(sN - I)^{-1}B_2 - C_{2r}(sN_r - I)^{-1}B_{2r}\|_\infty.
\end{aligned} \tag{7}$$

Now the problem of the $\mathcal{H}_\infty$ suboptimal model reduction is to find a reduced-order singular system (E_r, A_r, B_r, C_r) with $\dim(E_r) < \dim(E)$ such that for a given positive number γ, the following holds:

$$\|G_e(s)\|_\infty < \gamma.$$

First, it is known from [7] that $||G_e(s)||_\infty$ is finite if and only if

$$C_2(sN-I)^{-1}B_2 - C_{2r}(sN_r-I)^{-1}B_{2r} = -C_2B_2 + C_{2r}B_{2r},$$

i.e.,

$$C_2N^iB_2 = C_{2r}N_r^iB_{2r}, \quad i = 1, 2, \cdots, h-1, \tag{8}$$

$$C_{2r}N_r^iB_{2r} = 0, \quad i \geq h. \tag{9}$$

In this case,

$$\begin{aligned}
&\|G_e(s)\|_\infty\\
&= \|C_1(sI - A_1)^{-1}B_1 - C_{1r}(sI - A_{1r})^{-1}B_{1r}\\
&\quad - C_2B_2 + C_{2r}B_{2r}\|_\infty.
\end{aligned}$$

Therefore, if (8) and (9) are satisfied, the $\mathcal{H}_\infty$ suboptimal model reduction problem can be solved via using the conventional approaches. As indicated by previous analysis, the main concern for the model reduction problem is to find suitable (N_r, B_{2r}, C_{2r}) such that equations (8) and (9) are satisfied simultaneously.

In addition, it is known that the transfer matrix for a system is determined only by the controllable and observable subsystem. Therefore, the core issue in this paper is to discuss the model reduction of the fast subsystems (N, I, B_2, C_2) which is controllable and observable, i.e., to find the fast subsystem $(N_r, I_r, B_{2r}, C_{2r})$ with $n_{2r} < n_2$ satisfying (8) and (9).

The approach adopted in [8] is to find N_r first, then one tries to solve (8) and (9) for obtaining B_{2r} and C_{2r}. The proposed approach has some significant disadvantages. In one hand, for a given N_r, (8) and (9) may not have solutions B_{2r},C_{2r}. On the other hand, even the solutions for these equations exist, it may be still very hard to solve them due to their nonlinear nature. In [12], a new approach was proposed with necessary and sufficient conditions obtained for the solving (8) and (9) and a model reduction algorithm was designed. However, those given conditions are not explicit since they are expressed by decomposed matrices.

In this paper, the following questions related to the model reduction problem will be addressed. The existence problem of $(N_r, I_r, B_{2r}, C_{2r})$, $n_{2r} < n_2$ satisfying (8) and (9) is given explicitly. Their solutions will be further investigated based on the results in [12]. Here we adopt a different approach based on the minimum rank for a matrix set instead of matrix decomposition. The lowest bound for the dimension of N_r is given and finally a model reduction algorithm will be presented.

In order to address all these problems, the following algorithm will be presented first and it will be used in the sequel.

III. SILVERMAN-HO ALGORITHM

We introduce the following lemma before presenting a useful algorithm.

Lemma 1: [1] For any polynomial matrix $P(s)$, there always exist matrices N, B, and C, with N nilpotent, such that $P(s) = C(sN - I)^{-1}B$.

Next, we will show the procedure to derive N, B, C for a given polynomial. For a given polynomial matrix,

$$P(s) = P_0 + P_1 s + \cdots + P_{h-1}s^{h-1},$$

where $P_i \in \mathcal{R}^{r\times m}$, $0 \le i \le h-1$. The above lemma assures the existence of B, C, and the nilpotent matrix N satisfying $P(s) = C(sN - I)^{-1}B$. Let

$$M_0 \triangleq \begin{bmatrix} -P_0 & -P_1 & \cdots & -P_{h-2} & -P_{h-1} \\ -P_1 & -P_2 & \cdots & -P_{h-1} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -P_{h-2} & -P_{h-1} & \cdots & \cdots & 0 \\ -P_{h-1} & 0 & \cdots & \cdots & 0 \end{bmatrix} \in \mathcal{R}^{hr\times hm}, \tag{10}$$

$$M_1 \triangleq \begin{bmatrix} -P_1 & -P_2 & \cdots & -P_{h-1} & 0 \\ -P_2 & -P_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -P_{h-1} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} \in \mathcal{R}^{hr\times hm}, \tag{11}$$

and $\tilde{n} \triangleq \text{rank}[M_0]$.

M_0 can be denoted as $M_0(P_0, P_1, \cdots, P_{h-1})$. Now one can decompose

$$M_0 = L_1 L_2,$$

where $L_1 \in \mathcal{R}^{hr\times\tilde{n}}$, $L_2 \in \mathcal{R}^{\tilde{n}\times hm}$ are of full column and row rank, respectively. Further, let $\tilde{B}$ and $\tilde{C}$, respectively, be the first m columns of L_2 and the first r rows of L_1. Then one can prove that

$$\tilde{N} = \left(L_1^T L_1\right)^{-1} L_1^T M_1 L_2^T \left(L_2 L_2^T\right)^{-1},$$

will be nilpotent and $(\tilde{N}, \tilde{B}, \tilde{C})$ will be a minimal realization for $P(s)$ [2].

This procedure will be very useful for us to design a model reduction algorithm for the fast subsystems.

IV. MAIN RESULTS

From previous analysis, it can be seen that the order of the minimal realization for $P(s)$ is determined by the rank of M_0. For a given system (N, I, B_2, C_2), let $P_i = C_2 N^i B_2, \quad i = 0, 1, \cdots, h-1$. Then the suboptimal model reduction problem is equivalent to finding a suitable $\tilde{P}_0$ to replace P_0, such that $n_2 = \text{rank}[M_0] > \text{rank}[\tilde{M}_0] = n_{2r}$, where $\tilde{M}_0$ corresponds to $\tilde{P}_0, P_1, \cdots, P_{h-1}$. So the existence issue of such $\tilde{P}_0$ will determine whether a given fast subsystem can be reduced or not. The following theorem will give a necessary and sufficient condition for the existence of such $\tilde{P}_0$. If such $\tilde{P}_0$ exists, the lowest order of the reduced system can also be found.

Without loss of generality, suppose that M_0 is partitioned as

$$M_0 = \begin{bmatrix} -P_0 & K_1 \\ K_2 & K_3 \end{bmatrix},$$

and

$$\left[\begin{array}{c|c} K_2 & K_3 \end{array}\right] = \left[\begin{array}{c|c} \gamma_1 & \beta_1 \\ \gamma_2 & \beta_2 \\ \vdots & \vdots \\ \gamma_{(h-1)m} & \beta_{(h-1)m} \end{array}\right].$$

In order to find suitable $\tilde{P}_0$ for the possible model reduction, one can decompose

$$\left[\begin{array}{c|c} -P_0 & K_1 \end{array}\right] = \left[\begin{array}{c|c} P_{01} & K_{11} \\ \hline P_{02} & K_{12} \end{array}\right] = \left[\begin{array}{c|c} \eta_{01} & \alpha_1 \\ \eta_{02} & \alpha_2 \\ \vdots & \vdots \\ \eta_{0(m-d)} & \alpha_{m-d} \\ \hline \eta_{0(m-d+1)} & \alpha_{m-d+1} \\ \vdots & \vdots \\ \eta_{0m} & \alpha_m \end{array}\right],$$

where K_{12} consists of the row vectors of K_1 with the fewest vector number satisfying

$$\text{rank}\begin{bmatrix} K_{12} \\ K_3 \end{bmatrix} = \text{rank}\begin{bmatrix} K_1 \\ K_3 \end{bmatrix}.$$

This is possible since one can choose the maximal independent vector set including K_3. Then one can show that

$$\text{rank}\begin{bmatrix} K_{12} \\ K_3 \end{bmatrix} = \text{rank}\left[K_3\right] + d,$$

i.e., any α_i $(1 \le i \le m-d)$ is a linear combination of $\alpha_{m-d+1}, \alpha_{m-d+2}, \cdots, \alpha_m$ and $\beta_1, \beta_2, \cdots, \beta_{(h-1)m}$. So one can obtain

$$\alpha_i = \sum_{j=m-d+1}^{m} a_{ij}\alpha_j + \sum_{k=1}^{(h-1)m} b_{ik}\beta_k, \quad \text{for all } 1 \le i \le m-d. \tag{12}$$

From the choice of the matrix K_{12}, one can derive that

$$\text{rank}\begin{bmatrix} P_{02} & K_{12} \\ K_2 & K_3 \end{bmatrix} = d + \text{rank}[K_2 \quad K_3]. \tag{13}$$

The above equation indicates that P_{02} has no effect on the rank for the matrix in left side of equation (13). Now the following theorem can be obtained.

Theorem 2: Given (N, I, B_2, C_2), there exists a reduced-order, controllable and observable system $(N_r, I_r, B_{2r}, C_{2r})$ with its dimension $n_{2r} < n_2$, such that the $\mathcal{H}_\infty$ norm of the error system exists if and only if

$$n_2 > d + \text{rank}[K_2 \quad K_3].$$

From Theorem 2, one can see that d plays an important role in the existence issue for the reduced order systems. Next, this problem for the finite $\mathcal{H}_\infty$ norm of the error system will be investigated in different point of view and we will give an explicit formula for d. First, the following two lemmas will be presented.

Lemma 3: [13] Let $A \in \mathcal{R}^{m\times n}$, $B \in \mathcal{R}^{m\times p}$, $C \in \mathcal{R}^{q\times n}$ be constant, $F \in \mathcal{R}^{p\times q}$ be variable. Then

$$\begin{aligned} &\min_F \ \text{rank} \begin{bmatrix} A & B \\ C & F \end{bmatrix} \\ &= \text{rank}[A \quad B] + \text{rank} \begin{bmatrix} A \\ C \end{bmatrix} - \text{rank}\,[A]\,. \end{aligned}$$

Lemma 4: [13] Let $r_{\min} = \min_K \ \text{rank}\,[A + BKC]$, $r_{\max} = \max_K \ \text{rank}\,[A + BKC]$. Then for any r_0 within $r_{\min} \le r_0 \le r_{\max}$, there exists $K \in \mathcal{R}^{p\times q}$, such that

$$r_0 = \text{rank}\,[A + BKC]\,.$$

From the above two lemmas, one can easily get the following lemma and theorem. The following theorem gives another necessary and sufficient condition for the existence of $\tilde{P}_0$ such that the rank of $M_0(\tilde{P}_0, P_1, \cdots, P_{h-1})$ is reduced. If it exists, the minimum order of the reduced system can also be found.

Lemma 5: Let $A \in \mathcal{R}^{m\times n}$, $B \in \mathcal{R}^{m\times p}$, $C \in \mathcal{R}^{q\times n}$ be constant, $F \in \mathcal{R}^{p\times q}$ be variable,

$$r_{\min} = \min_F \ \text{rank} \begin{bmatrix} A & B \\ C & F \end{bmatrix},$$

$$r_{\max} = \max_F \ \text{rank} \begin{bmatrix} A & B \\ C & F \end{bmatrix},$$

then for any r_0 satisfying $r_{\min} \le r_0 \le r_{\max}$, there exists $F \in \mathcal{R}^{p\times q}$, such that

$$r_0 = \text{rank} \begin{bmatrix} A & B \\ C & F \end{bmatrix}.$$

Now Theorem 2 can be interpret in another way.

Theorem 6: Given (N, I, B_2, C_2). There exists a reduced-order, controllable and observable system $(N_r, I_r, B_{2r}, C_{2r})$, with $n_{2r} < n_2$, such that the $\mathcal{H}_\infty$ norm of the error system is finite if and only if

$$n_2 > n_{2r} \ge \text{rank} \begin{bmatrix} K_1 \\ K_3 \end{bmatrix} + \text{rank}[K_2 \quad K_3] - \text{rank}\,[K_3]\,.$$

The proof for this theorem is obvious from Theorem 2 and Lemma 3. Here the necessary and sufficient condition is given by the parameters K_1, K_2 and K_3.

Corollary 7: The number d in Theorem 2 is

$$d = \text{rank} \begin{bmatrix} K_1 \\ K_3 \end{bmatrix} - \text{rank}[K_3].$$

Though the necessary and sufficient conditions for the existence for the finite $\mathcal{H}_\infty$ norm of the error system are given in previous theorems, it is still hard to construct an effective algorithm for obtaining the lower order fast system. Next, we will present a constructive procedure for the model reduction problem. In other words, with given n_{2r} satisfying the requirements, one needs to design a procedure for finding $\tilde{P}_0$.

According to Theorem 6, one only needs to discuss two cases respectively.

Case 1. $n_{2r} > d + \text{rank}[K_2 \quad K_3]$

From (12), one can replace η_{0i} with

$$\tilde{\eta}_{0i} = \sum_{j=m-d+1}^{m} a_{ij}\eta_{0j} + \sum_{k=1}^{(h-1)m} b_{ik}\gamma_k, \quad \text{for all } 1 \le i \le m-d, \tag{14}$$

and obtain a new $\tilde{P}_{01}$. The associated $\tilde{M}_0$ will satisfy $\text{rank}[\tilde{M}_0] = d + \text{rank}[K_2 \quad K_3]$

With the above $\tilde{M}_0$, one can see that

$$\begin{aligned} M_0 &= \tilde{M}_0 + M_0 - \tilde{M}_0 \\ &= \tilde{M}_0 + \begin{bmatrix} P_{01} - \tilde{P}_{01} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \tilde{M}_0 + \sum_{i=1}^{m-d} p_i, \end{aligned}$$

where

$$p_i = \begin{bmatrix} \tilde{p}_i & 0 \\ 0 & 0 \end{bmatrix},$$

in which the i-th row of the matrix $\tilde{p}_i$ is $\eta_{0i} - \tilde{\eta}_{0i}$, with all the other rows being zeros. Note that $\text{rank}[\tilde{M}_0 + p_i]$ is one more than $\text{rank}[\tilde{M}_0]$ at most. Therefore, there exists r such that $\text{rank}[\tilde{M}_0 + \sum_{i=1}^{r} p_i] = n_{2r}$, $r < m - d$ due to

$$\text{rank}[M_0] = n_2,$$

and

$$\text{rank}[\tilde{M}_0] = d + \text{rank}[K_2 \quad K_3] < n_{2r} < n_2.$$

In order to reduce the computation cost, one can first compute

$$\text{rank}\left[\tilde{M}_0 + \sum_{i=0}^{p} p_i\right] = n_{2r}^{(1)},$$

where $p = n_{2r} - d - \text{rank}[K_2 \quad K_3]$. If $n_{2r}^{(1)} < n_{2r}$, one can further compute

$$\text{rank}\left[\tilde{M}_0 + \sum_{i=0}^{p+n_{2r}-n_{2r}^{(1)}} p_i\right] = n_{2r}^{(2)};$$

Otherwise one can deduce that $r = p$. If $n_{2r}^{(2)} < n_{2r}$, one computes

$$\text{rank}\left[\tilde{M}_0 + \sum_{i=0}^{p+n_{2r}-n_{2r}^{(1)}+n_{2r}-n_{2r}^{(2)}} p_i\right] = n_{2r}^{(3)};$$

Otherwise, $r = p + n_{2r} - n_{2r}^{(1)}$, and so on. After finite steps, r must be able to be reached such that

$$\mathrm{rank}[\tilde{M}_0 + \sum_{i=0}^{r} p_i] = n_{2r}.$$

Actually, It can be seen that the maximal number of steps is

$$\left\lfloor \frac{m-d}{p} \right\rfloor + m - d - p \left\lfloor \frac{m-d}{p} \right\rfloor,$$

where $\left\lfloor \frac{m-d}{p} \right\rfloor$ is the largest integer less than $\frac{m-d}{p}$.

Case 2. $n_{2r} = d + \mathrm{rank}[K_2 \quad K_3]$

In case 1, P_{02} keeps unchanged, but from the choice of K_{12}, P_{02} doesn't affect the rank of

$$\begin{bmatrix} P_{02} & K_{12} \\ K_2 & K_3 \end{bmatrix}.$$

Thus, P_{02} can be taken as free variables. In this case, one can replace η_{0i} in P_{01} with

$$\eta_{0i}^{*} = \sum_{j=m-d+1}^{m} a_{ij}\eta_{0j} + \sum_{k=1}^{(h-1)m} b_{ik}\gamma_k, \quad \text{for all } 1 \le i \le m-d. \tag{15}$$

In (15), η_{0j} is regarded as free parameters to be determined in the following optimization process, $m - d + 1 \le j \le m$. This indicates that the updated matrix P_{01}^{*} is the function with variable P_{02}. In order to differentiate with the original P_{02}, here the notation Q is used to replace P_{02} and denote P_{01}^{*} as $P_{01}^{*}(Q)$. Then

$$-P_0^{*} = \begin{bmatrix} P_{01}^{*}(Q) \\ Q \end{bmatrix},$$

where Q is a free parameter to be determined. Now, one can see that the suboptimal model reduction problem is equivalent to finding A_{r1}, B_{r1}, C_{r1}, and Q such that

$$\begin{aligned} &\|C_1(sI - A_1)^{-1}B_1 - C_{1r}(sI - A_{1r})^{-1}B_{1r} \\ &-C_2B_2 - P_0^{*}(Q)\|_{\infty} < \gamma, \end{aligned} \tag{16}$$

for a given positive number γ. As discussed previously, P_0^{*} can be obtained as a function of the matrix Q if the fast system can be reduced and the conventional approach in [9] can be used to solve this unconstrained optimization problem.

With previous analysis, one can obtain the following conclusion.

Corollary 8: The lowest order of the reduced-order system is $d + \mathrm{rank}[K_2 \quad K_3]$.

Now it is time to present a procedure for constructing a lower order fast subsystems.

Algorithm

Step 1. To decompose the original system and obtain the fast subsystem. If there exists controllable and observable fast part, denote it as (N, I, B_2, C_2), else stop.

Step 2. To compute $P_i = -C_2N^iB_2$, and obtain M_0.

Step 3. To testify whether the fast system can be reduced according to Theorem 2 or Theorem 6. If yes, continue, else stop.

Step 4. To find out $P_{01}$and P_{02}. If

$$n_{2r} > d + \mathrm{rank}[K_2 \quad K_3],$$

one can obtain P_0^{*} as in case 1 in the previous section, and go to Step 6. Otherwise, one can obtain P_0^{*} as a function of free variable Q in case 2.

Step 5. To solve the unconstrained optimization problem (16) and find A_{1r}, B_{1r}, C_{1r}, Q_1 and P_0^{*}.

Step 6. To obtain the minimal realization $(N_r, I_{2r}, B_{2r}, C_{2r})$ via the Silverman-Ho algorithm for

$$P^{*}(s) = P_0^{*} + P_1s + \cdots + P_{h-1}s^{h-1}.$$

Step 7. The following system will be the reduced-order system for the original system;

$$\begin{aligned} &\dot{x}_{1r}(t) = A_{1r}x_{1r}(t) + B_{1r}u(t), \quad x_{1r}(0-) = x_{1r,0}, \\ &y_{1r}(t) = C_{1r}x_{1r}(t); \\ &N_r\dot{x}_{2r}(t) = x_{2r}(t) + B_{2r}u(t), \quad x_{2r}(0-) = x_{2r,0}, \\ &y_{2r}(t) = C_{2r}x_{2r}(t). \end{aligned}$$

V. ILLUSTRATIVE EXAMPLE

In this section, we will present an example to show the effectiveness of the proposed algorithm. Also we will do some comparisons with the results in [8]. Consider system (N, I, B, C) with

$$N = \begin{bmatrix} 0.2532 & -0.0273 & 0.1175 & -0.0267 \\ 0.0207 & -0.3704 & 0.0195 & 0.1147 \\ -0.0304 & -0.8768 & 0.0096 & 0.2827 \\ -0.8642 & 0.0412 & -0.3995 & 0.1075 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.5996 & -0.7491 \\ 0.7287 & -0.5638 \\ -0.2961 & 0.2501 \\ -0.1478 & -0.2418 \end{bmatrix},$$

$$C = \begin{bmatrix} -14.0707 & -2.2387 & -0.1650 & -0.9886 \\ -4.7730 & 2.8568 & -0.0778 & -0.2263 \\ -12.7743 & 1.1680 & 1.2586 & 0.9315 \end{bmatrix}.$$

It can be verified that N is nilpotent and this system is a minimal realization. Now, one can compute

$$-P_0 = CB = \begin{bmatrix} 7 & 12 \\ 5 & 2 \\ 8 & 9 \end{bmatrix}, -P_1 = CNB = \begin{bmatrix} 3 & 1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix},$$

$$-P_2 = CN^2B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 3 & 0 \end{bmatrix},$$

and

$$K_1 = [\ -P_1 \ \ -P_2 \ ], \quad K_2 = \begin{bmatrix} -P_1 \\ -P_2 \end{bmatrix}, \quad K_3 = \begin{bmatrix} -P_2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then, one can obtain that

$$\begin{aligned} &d = 1, \quad \mathrm{rank}[K_2 \quad K_3] = 2, \\ &d + \mathrm{rank}[K_2 \quad K_3] = 3 < 4, \end{aligned}$$

so this system can be reduced according to Theorem 2. Let $n_{2r} = 3 = d + \text{rank}[K_2 \quad K_3]$, one can use the approach in case 2 to obtain a lower order model.

Let $Q = [\ q_1 \quad q_2\ ]$ be free variable, where q_1 and q_2 are to be determined. Then one can change $[\ 8 \quad 9\ ]$ to

$$3[\ q_1 \quad q_2\ ] + 2[\ 0 \quad 1\ ] = [\ 3q_1 \quad 3q_2 + 2\ ],$$

and change $[\ 7 \quad 12\ ]$ to

$$[\ q_1 \quad q_2\ ] + 3[\ 0 \quad 1\ ] = [\ q_1 \quad q_2 + 3\ ].$$

Then one will derive that

$$-P_0^* = \begin{bmatrix} q_1 & q_2 + 3 \\ q_1 & q_2 \\ 3q_1 & 3q_2 + 2 \end{bmatrix}.$$

By using the Matlab function fminunc(), one can achieve the optimal solution $[\ q_1 \quad q_2\ ] = [\ 3.1241 \quad 3.0072\ ]$ with the minimal $\mathcal{H}_\infty$ norm of the error system

$$\|G_e(s)\|_\infty^2 = \|CB - C_r B_r\|_\infty^2 = 56.9381.$$

Implementing the Silverman-Ho algorithm, the parameters of the reduced order fast system are obtained as the following:

$$B_r = \begin{bmatrix} -0.6101 & -0.7252 \\ 0.6357 & -0.3916 \\ -0.4729 & 0.4091 \end{bmatrix},$$

$$C_r = \begin{bmatrix} -7.2234 & -2.1531 & -0.1818 \\ -4.4328 & 0.9827 & 0.4336 \\ -15.1587 & 0.8577 & 0.8906 \end{bmatrix},$$

$$N_r = \begin{bmatrix} 0.2860 & 0.0505 & 0.1324 \\ 0.3177 & -0.5067 & 0.2912 \\ 0.0582 & -0.7463 & 0.2207 \end{bmatrix}.$$

It should be noted that

$$N_r^3 = (1.0e-15) \times \begin{bmatrix} 0.0945 & -0.1249 & 0.0763 \\ -0.0928 & 0.1284 & -0.0781 \\ -0.2134 & 0.3053 & -0.1735 \end{bmatrix} \approx 0,$$

which indicates that N_r is nilpotent. Then the system (N_r, I_r, B_r, C_r) can be taken as the approximation of the original system.

In order to compare the effectiveness of the proposed algorithm, one can obtain the reduced order system with the algorithm in [8] as below

$$N_r = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, C_r = \begin{bmatrix} 1 & 2 & 2 \\ 1 & -1 & 0 \\ 3 & -1 & 2 \end{bmatrix}, B_r = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

The corresponding $\mathcal{H}_\infty$ norm of the error system is

$$\|G_e(s)\|_\infty^2 = \|CB - C_r B_r\|_\infty^2 = 138.7694 > 56.9381.$$

This verifies that the proposed algorithm improves the $\mathcal{H}_\infty$ norm significantly.

VI. CONCLUSIONS

In this paper, we developed a new procedure of $\mathcal{H}_\infty$ suboptimal reduction algorithm for singular systems. Some necessary and sufficient conditions are obtained which can guarantee the existence of a reduced-order system with finite $\mathcal{H}_\infty$ norm of the error system. The contribution of this paper can be concluded as following.

First, the existence of the reduced order system with finite $\mathcal{H}_\infty$ norm of the error system was investigated thoroughly. Some necessary and sufficient conditions are obtained.

Second, a design procedure is designed for obtaining the reduced order systems. The core contribution is that a free parameter is identified in the optimization process. This free parameter can reduce the $\mathcal{H}_\infty$ norm of the error system significantly as evidenced in the illustrative example.

Finally, the results in this paper can be extended to the case for discrete singular systems without much difficulty.

Compared to the results in [10], [11], one can see that the disadvantage of this paper is that the system decomposition is used in this paper in stead of the original parameters. This is due to the difference that the singular systems with impulsive behavior are treated in this paper and the results in [10], [11] only deal with singular systems without impulsive dynamics.

VII. ACKNOWLEDGMENTS

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