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Model Reduction of Singular Systems via Covariance Approximation

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***Abstract*— Model reduction problem was investigated for singular systems. To solve the problem, the covariance for singular systems was defined. Then, a model reduction method based on covariance approximation was presented for obtaining a stable and impulse controllable models for singular systems. Thirdly, the error criterion was explicitly derived via a free parameter and the optimization procedure was presented in terms of gradient flow. Finally, illustrative examples were given to show the effectiveness of the proposed approach.**

I. INTRODUCTION

Singular systems have been investigated extensively due to their applications in modelling and control of electrical circuits, power systems and economics, etc. Some important characteristics of singular systems include combined dynamic and static solutions, impulsive behaviors and large dimensionality. Thus model reduction is vital for analysis and design of controller for such systems [6], [4] and it is the subject of current research.

The initial investigation of model reduction for singular systems was the chained aggregation method proposed in [7]. The authors there developed a generalized chained aggregation algorithm and gave an intuitive interpretation of the exact aggregation conditions for singular systems. The aim of the proposed method is to remove the unobservable subspace. Initial behavior of singular systems was also taken into consideration while performing model reduction. However, as pointed out in [11], the main drawback of this method is the high level of computational effort. Perev and Shafai [11] considered model reduction for singular system via balanced realization and gave a model reduction algorithm. Unfortunately, their method ignored the impulsive behavior which is of paramount importance to singular systems. The reduced order model may be a normal state space system, which has no impulsive behavior and does not track the original system response properly. Liu and Sreeram [5] proposed a new model reduction algorithm via Nehari's approximation algorithm and overcome the problem. The reduced-order model will be a really singular system and the approximation has been obtained as desired. For discrete singular systems, Zhang et al. [4] discussed the same problem based on $\mathcal{H}_2$ norm.

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Recently, Zhang et al. [3] discussed the $\mathcal{H}_\infty$ suboptimal model reduction problem for singular systems. In [3], it requires that the transfer function matrix of the error system is rational in order to guarantee that $\mathcal{H}_\infty$ norm exists. Further Wang, et al [17] give a sufficient and necessary condition of the existence of such systems. For singular systems without impulsive behavior, some model reduction approaches based on linear matrix inequalities are proposed in [8], [9] respectively for discrete and continuous systems.

In this paper, we will present a new error criterion via covariance approximation to investigate the model reduction problem for systems. Model reduction based on covariance has been investigated in [16] and many results have been obtained for normal linear systems. For singular systems, due to impulsive behavior, it is hard to define the covariance for the fact subsystems. In our recent paper [14], the covariance for singular systems was first defined and was used to investigate the regional stability for singular systems. Here we use the covariance defined in [14] to investigate the model reduction issue.

The organization of this paper is as following. In section 2, some preliminaries will be presented. In section 3, the model reduction problem is investigated for fast subsystems. In section 4, some main results about the model reduction are presented and an algorithm to reduce the original system will be given. In section 5, two numerical examples will be given to illustrate the effectiveness of the new proposed model reduction approaches. Conclusions will be given in section 6.

II. PRELIMINARY RESULTS

In this section, some basic results for singular systems will be presented for uses in the sequel.

Consider the following singular systems

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0-) = x_0, \\ y(t) &= Cx(t); \end{aligned} \tag{1}$$

where $x(t) \in \mathcal{R}^n$ is the state vector, $u(t) \in \mathcal{R}^q$ is the input vector and $y(t) \in \mathcal{R}^m$ is the output vector. $E \in \mathcal{R}^{n\times n}$, $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times q}$, $C \in \mathcal{R}^{m\times n}$ are constant matrices with E possibly singular. Assume that the matrix pair (E, A) is regular (i.e.,$|sE - A| \not\equiv 0$). In this paper, the realization quadruple (E, A, B, C) is used to represent system (1), which is assumed to be minimal. All the matrices in this paper are assumed to have appropriate dimensions.

From [2], it is known that there exist two square nonsingular matrices Q and P such that system (1) is transformed

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to the Weierstrass canonical form:

$$\begin{aligned}&\dot{x}_1(t)=A_1x_1(t)+B_1u(t),\quad x_1(0-)=x_{1,0},\\&y_1(t)=C_1x_1(t);\\&N\dot{x}_2(t)=x_2(t)+B_2u(t),\quad x_2(0-)=x_{2,0},\\&y_2(t)=C_2x_2(t);\end{aligned}\tag{2}$$

where $x_1(t)\in\mathcal{R}^{n_1}$, $x_2(t)\in\mathcal{R}^{n_2}$, $n_1+n_2=n$, $N\in\mathcal{R}^{n_2\times n_2}$ is nilpotent, and

$$\begin{aligned}&QEP=diag(I,N),\ \ QAP=diag(A_1,I),\\&CP=[C_1\ \ C_2],\ \ P^{-1}x(t)=[x_1^T(t),\ \ x_2^T(t)]^T,\\&QB=[B_1^T\ \ B_2^T]^T,\ \ y(t)=y_1(t)+y_2(t).\end{aligned}$$

System (1) is called system restricted equivalent(s.r.e) to system (2). The transfer function matrix $G(s)$ is invariant under s.r.e. transformation, i.e.,

$$\begin{aligned}G(s)&=C(sE-A)^{-1}B\\&=CP(sQEP-QAP)^{-1}QB\\&=C_1(sI-A_1)^{-1}B_1+C_2(sN-I)^{-1}B_2,\end{aligned}\tag{3}$$

and

$$\begin{aligned}&C_2(sN-I)^{-1}B_2\\&=-C_2B_2-sC_2NB_2-\cdots-s^{h-1}C_2N^{h-1}B_2,\end{aligned}$$

where h is the nilpotent index of N.

Impulsive controllability and impulsive observability are two important concepts for singular system introduced respectively by Cobb and Verghese in [1], [12]. Roughly speaking, they reflect the ability to remove the impulses in the state responses of a singular system with non-impulsive control. From [2], [1], one can obtain some criteria for impulsive controllability and impulsive observability.

For the slow subsystems (I,A_1,B_1,C_1), there are many possible methods to reduce their order. So in the reminder of this paper, we mainly discuss the model reduction for the fast subsystems (N,I,B_2,C_2), and use (N,I,B,C) to represent the discussed systems for convenience. In this paper, without loss of generality, we also assume that the nilpotent matrix N only contains Jordan blocks without zero blocks since it does not have impulse behavior otherwise.

$$N=diag(J_1,J_2,\cdots,J_k),\tag{4}$$

and

$$J_i=\begin{bmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&1\\0&0&0&\cdots&0\end{bmatrix}\in\mathcal{R}^{r_i\times r_i},$$

where $i=1,2,\cdots,k$ and $r_1(=h)\geq r_2\geq\cdots\geq r_k>0$, and the number h is called the nilpotent index of N. We use $ind(N)$ to denote the nilpotent index. In addition, all the nilpotent matrices in this paper are of the same form with N.

The discussed system (N,I,B,C) can be expressed as follows:

$$\begin{aligned}&N\dot{x}(t)=x(t)+Bu(t),\quad x(0-)=x_0,\\&y(t)=Cx(t).\end{aligned}\tag{5}$$

And its state response is

$$x(t)=-\sum_{i=1}^{h-1}\delta^{(i-1)}(t)N^ix_0-\sum_{i=0}^{h-1}N^iBu^{(i)}(t).$$

When the initial state $x_0=0$ and the input $u(t)$ is a zero mean white noise process with covariance $\delta(t)I$, where $\delta(t)$ is the Dirac impulse function, the jth state response at time t can be characterized as

$$x(j,t)=-\sum_{i=0}^{h-1}N^iBe_j\delta^{(i)}(t),$$

where $I=[\ e_1\ \ \ e_2\ \ \ \cdots\ \ \ e_q\ ]$, $e_i\in R^q$.

Definition 1: Given $N\in\mathcal{R}^{n\times n}$, $B\in\mathcal{R}^{n\times q}$, $C\in\mathcal{R}^{m\times n}$, N is nilpotent, then *the pseudo-nilpotent index* of (N,B,C) is denoted by $ind_p(N)$, and $ind_p(N)=h_p$, satisfying $CN^iB=0$, $i\geq h_p$, $CN^{h_p-1}B\neq 0$.

Obviously, there holds $ind_p(N)\leq ind(N)$. Then the transfer function of system (5) is

$$\begin{aligned}G(s)&=C(sN-I)^{-1}B\\&=-CB-sCNB-\cdots-s^{h_p-1}CN^{h_p-1}B.\end{aligned}$$

The impulse response matrix of system (5) is $g(t)=L^{-1}(G(s))=-\sum_{i=0}^{h_p-1}\delta^{(i)}(t)CN^iB$, and $L^{-1}(G(s))$ is the inverse Laplace transformation of $G(s)$. The next lemma is a property about Dirac function.

Lemma 2: [14] $t^n\delta^{(n)}(t)=(-1)^nn!\delta(t)$

With this lemma in mind, one can get

$$\begin{aligned}\tilde{x}(j,t)&=\int_{-\infty}^{\infty}\left(-\sum_{i=0}^{h-1}N^iB\frac{1}{(-1)^ii!}t^ie_j\delta^{(i)}(t)\right)dt\\&=-\sum_{i=0}^{h-1}N^iBe_j=-HBe_j,\end{aligned}\tag{6}$$

and

$$\begin{aligned}\hat{g}(t)&=-\sum_{i=0}^{h_p-1}\frac{1}{(-1)^ii!}t^i\delta^{(i)}(t)CN^iB\\&=-\sum_{i=0}^{h_p-1}CN^iB\delta(t)=-\Theta\delta(t),\end{aligned}\tag{7}$$

where $H=(I-N)^{-1}$, $\Theta=\sum_{i=0}^{h_p-1}CN^iB$, so we can use $\|\Theta\|_F$ to measure the magnitude of $\hat{g}(t)$.

It should be noted that $\tilde{x}(j,t)$ is a combination of $x(j,t)$ in order to get a compact form of $\hat{g}(t)$ in (7).

Definition 3: [14] The matrix

$$X=\sum_{j=1}^{q}\tilde{x}(j,t)\tilde{x}^T(j,t)=HBB^TH^T,$$

is called the steady-state covariance of system (5).

Then the steady-state output covariance of system (5) is

$$Y = CXC^T = CHBB^TH^TC^T = \Theta\Theta^T. \tag{8}$$

It should be noted that the state covariance and output covariance are used to investigate the impulsive controllability and impulsive observability in [14]. In this paper, we intend to use this index to investigate the model reduction issue for the fast subsystems. Actually, the defined covariance should reflect the capacity of impulsive behaviors of singular systems.

III. PROBLEM FORMULATION

It should be noted that the difficulty of model reduction for singular systems is to retain its impulsive nature. Without impulsive nature, the model reduction algorithms in [8], [9] are parallel to those for normal linear systems via LMIs. With impulsive behaviors considered, most of the results are based on system decompositions due to their difficulties. Even with system decomposition, it is not easy to characterize the capacity of the impulsive behavior. With the covariance defined in previous section as a possible alternative, we formulate our problem as below.

Problem Given the nth-order impulse-controllable and impulse-observable singular system (N, I, B, C) with Θ defined, find a rth-order system (N_r, I_r, B_r, C_r), $r < n$ with Θ_r such that the following conditions are simultaneously satisfied.

Condition 1 The new system is impulse-controllable;

Condition 2 N_r is also nilpotent, and $ind(N_r) \leq ind_p(N)$;

Condition 3 $\Theta = \Theta_r$;

Condition 4 For the fixed N_r, B_r,

$$\frac{\min\limits_{C_r} \sum_{i=0}^{h_p-1} \left\|CN^iB - C_rN_r^iB_r\right\|_F^2}{\sum_{i=0}^{h_p-1} \left\|CN^iB\right\|_F^2} \ll 1. \tag{9}$$

In this case, we can take the reduced system as the approximation of the original system.

The first condition assures there exists a controller to eliminate the impulse of the reduced model; The second condition is to retain its impulsive nature; The third one, in fact, assures that the steady-state output covariance Y_r is equal to that of the original full-order system 5, that is $Y_r = Y$, and Y is defined in (8). The last condition is to make the state responses of the reduced system and the original system as close as possible when the initial state is zero.

Obviously, from [17], we know that the best case is that for all $i = 1, ..., h_p - 1$, there hold $CN^iB = C_rN_r^iB_r$, and in this case, the transfer function of the error system is rational then the $\mathcal{H}_\infty$ norm exists. In [17], a sufficient and necessary condition for the existence of such reduced-order system is given when $h_p > 1$. If the condition can be satisfied, the model reduction problem can be degenerated into a case for normal linear systems. In fact, there are many models not satisfying the existence condition. Therefore, it is necessary to consider the above suboptimal model reduction problem.

In this paper, two cases will be tackled. One is when $h_p = 1$ and the full-order system is output impulse-free; The other is when $h_p > 1$, and the original system does not satisfy the existence condition reported in [17].

IV. MAIN RESULTS

As stated in previous section, we present our results in two cases below.

A. $h_p = 1$

First, the following lemma is needed in the sequel.

Lemma 4: [15] For any initial state, system (5) is output impulse-free if and only if $CN = 0$.

Therefore, in this case, the transfer function of system (5) is $G(s) = C(sN - I)^{-1}B = -CB$. Noting that $CN{=}0$ in this case, one needs to find a rth-order system (N_r, I_r, B_r, C_r), $r < n = \dim N$ satisfying

(1) N_r is nilpotent and not zero;

(2) $C_rB_r = CB$;

(3) $C_rN_r = 0$;

(4) rank $\begin{bmatrix} N_r & 0 & 0 \\ I_r & N_r & B_r \end{bmatrix} = n_r + \text{rank}[N_r]$.

The following theorem gives the necessary and sufficient condition of the existence of such reduced-order system and the procedure of the proof is a constructive procedure to get N_r, B_r, and C_r.

Theorem 5: A reduced-order system satisfying the above conditions (1)-(4) exists if and only if rank$[CB] < n - 1$.

Proof (Necessity) Suppose that such a reduced-order system exists, then the inequality

$$\text{rank}[CB] = \text{rank}[C_rB_r] \leq r < n,$$

must hold. By contradiction, if rank$[CB] = n - 1$, that is rank$[C_rB_r] = n - 1 = r$. Then C_r will be full column rank and B_r is full row rank since $C_r \in \mathcal{R}^{m\times(n-1)}$, $B_r \in \mathcal{R}^{(n-1)\times q}$; Then from condition (3) there exists $N_r = 0$, which contradicts the first condition. So rank$[CB] < n-1$. (Sufficiency) Let $n_t = \text{rank}[CB] < n - 1$. Then take a full rank decomposition

$$CB = C_tB_t,$$

where $C_t \in \mathcal{R}^{m\times n_t}$, $B_t \in \mathcal{R}^{n_t\times q}$ are of full column and row rank, respectively. Choosing any nonzero vector $\beta_t = [\ \beta_{t1} \quad \beta_{t2} \quad \cdots \quad \beta_{tn_t}\ ]^T \in \mathcal{R}^{n_t}$, in which there exists no less than one zero element and letting $\alpha = C_t\beta_t$, $\beta = [\ \beta_t^T \quad -1\ ]^T$, then one can construct $N_r = [\ \alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_{n_t} \quad 0\ ]$, where

$$\alpha_i = \begin{cases} 0 & \beta_{ti} \neq 0, \\ \beta & \beta_{ti} = 0, \end{cases}$$

and $C_r = [\ C_t \quad \alpha\ ]$, $B_r = [\ B_t^T \quad 0\ ]^T$. Now it is easy to check that $N_r^2 = 0$, and the other three conditions are also satisfied for the system (N_r, I_r, B_r, C_r) with $r < n$. □

B. $h_p > 1$

Lemma 6: [13] Suppose that $A \in \mathcal{C}^{n\times n}$, $B \in \mathcal{C}^{p\times q}$, $H \in \mathcal{C}^{m\times q}$. Then the matrix equation

$$AXB = H, \tag{10}$$

is solvable, if and only if A^- and B^- satisfies the following condition

$$AA^- HB^- B = H. \tag{11}$$

If (10) is satisfied, the general solution is in the following form

$$X = A^- HB^- + Y - A^- AYBB^-,$$

where Y is any $n\times p$ matrix, A^- and B^- are the generalized inverse matrices.

Lemma 7: [5] Given system (5), and

$$B = \left[\begin{array}{ccccccc} \times & b_1^T & \times & b_2^T & \cdots & \times & b_k^T \end{array}\right]^T,$$

where "b_i" is the vector in B corresponding to the last row of J_i and "$\times$" are some matrices whose elements are not important for the analysis. Then system (5) is impulse-controllable if and only if

$$\text{rank}\left[\begin{array}{cccc} b_1^T & b_2^T & \cdots & b_p^T \end{array}\right] = p.$$

With these two lemmas, we can derive the following result.

Theorem 8: Given system (5), where $N \in \mathcal{R}^{n\times n}$, $I \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times q}$, $C \in \mathcal{R}^{m\times n}$, N_r is also given and nilpotent with d Jordan blocks, $r < n$. Then there exists impulse controllable reduced-order system (N_r, I_r, B_r, C_r), $N_r \in \mathcal{R}^{r\times r}$, $I_r \in \mathcal{R}^{r\times r}$, $B_r \in \mathcal{R}^{r\times q}$, $C_r \in \mathcal{R}^{m\times r}$, such that $\Theta = \Theta_r$, i.e.,

$$C_r(I_r - N_r)^{-1}B_r = C(I-N)^{-1}B.$$

Proof The proof is constructive. First, let $\Xi = C(I - N)^{-1}B$, the following steps are to get the matrices B_r and C_r with proper dimension such that $C_r(I_r - N_r)^{-1}B_r = \Xi$.

From Lemma 6, the necessary and sufficient condition of the existence of C_r is

$$\Xi\hat{B}_r^-\hat{B}_r = \Xi, \tag{12}$$

where $\hat{B}_r = (I_r - N_r)^{-1}B_r$, and $\hat{B}_r^-$ is a generalized inverse matrix of $\hat{B}_r$.

Suppose that $\hat{B}_r$ has the canonical form

$$\hat{B}_r = P\begin{bmatrix} I_{Br} & 0 \\ 0 & 0 \end{bmatrix} Q.$$

Then

$$\hat{B}_r^- = Q^{-1}\begin{bmatrix} I_{Br} & \times \\ \times & \times \end{bmatrix} P^{-1},$$

where "$\times$" are some matrices whose elements are not important for the analysis and P and Q are respectively $r\times r$ and $q\times q$ invertible matrices. Next, fix I_{Br} satisfying rank$(I_{Br}) \geq$ rank(Ξ), and rank$(I_{Br}) \geq d$. Then the equation (12) can be changed to the following:

$$\Xi Q^{-1}\begin{bmatrix} I_{Br} & 0 \\ \times & 0 \end{bmatrix} Q = \Xi,$$

i.e.,

$$\Xi Q^{-1}\begin{bmatrix} I_{Br} & 0 \\ \times & 0 \end{bmatrix} = \Xi Q^{-1}.$$

It can be deduced from Lemma 6 that Q satisfying the above equation must exist.

In addition, suppose that B_r and $\hat{B}_r$ can be partitioned as follows

$$B_r = \left[\begin{array}{ccccccc} \times & b_{r1}^T & \times & b_{r2}^T & \cdots & \times & b_{rd}^T \end{array}\right]^T,$$
$$\hat{B}_r = \left[\begin{array}{ccccccc} \times & \hat{b}_{r1}^T & \times & \hat{b}_{r2}^T & \cdots & \times & \hat{b}_{rd}^T \end{array}\right]^T.$$

After some manipulations, one can obtain that

$$\text{rank}\left[\begin{array}{cccc} b_{r1}^T & b_{r2}^T & \cdots & b_{rd}^T \end{array}\right] = d,$$

if and only if

$$\text{rank}\left[\begin{array}{cccc} \hat{b}_{r1}^T & \hat{b}_{r2}^T & \cdots & \hat{b}_{rd}^T \end{array}\right] = d. \tag{13}$$

Now one can get matrix P such that (13) holds, which assures that the new system is impulse controllable. Finally, from Lemma 6, we can obtain

$$C_r = \Xi\hat{B}_r^- + Y - Y\hat{B}_r\hat{B}_r^-, \tag{14}$$

where Y is any constant $m\times r$ matrix. □

Now let us consider the state covariance optimization problem. For this purpose, let

$$\begin{aligned}
\gamma &= \sum_{i=0}^{h_p-1}\left\|CN^iB - C_rN_r^iB_r\right\|_F^2 \\
&= \sum_{i=0}^{h_p-1} Tr\left(\left(CN^iB - C_rN_r^iB_r\right)\left(CN^iB - C_rN_r^iB_r\right)^T\right) \\
&= \sum_{i=0}^{h_p-1} Tr\left(\left[\begin{array}{cc} C & -C_r \end{array}\right]\begin{bmatrix} N^i & 0 \\ 0 & N_r^i \end{bmatrix}\begin{bmatrix} B \\ B_r \end{bmatrix}\right. \\
&\qquad \left.\times\left[\begin{array}{cc} B^T & B_r^T \end{array}\right]\begin{bmatrix} \left(N^i\right)^T & 0 \\ 0 & \left(N_r^i\right)^T \end{bmatrix}\begin{bmatrix} C^T \\ -C_r^T \end{bmatrix}\right) \\
&= Tr\left(\left[\begin{array}{cc} C & -C_r \end{array}\right] P \begin{bmatrix} C^T \\ -C_r^T \end{bmatrix}\right),
\end{aligned}$$

where

$$\begin{aligned}
P = &\sum_{i=0}^{h_p-1}\begin{bmatrix} N^i & 0 \\ 0 & N_r^i \end{bmatrix}\begin{bmatrix} B \\ B_r \end{bmatrix} \\
&\times\left[\begin{array}{cc} B^T & B_r^T \end{array}\right]\begin{bmatrix} \left(N^i\right)^T & 0 \\ 0 & \left(N_r^i\right)^T \end{bmatrix}.
\end{aligned}$$

Let

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix},$$

where

$$P_1 = \sum_{i=0}^{h_p-1} \left(N^i B\right)\left(N^i B\right)^T,$$

$$P_2 = \sum_{i=0}^{h_p-1} \left(N^i B\right)\left(N_r^i B_r\right)^T,$$

$$P_3 = \sum_{i=0}^{h_p-1} \left(N_r^i B_r\right)\left(N_r^i B_r\right)^T.$$

If condition 3 is satisfied and B_r is obtained as in Theorem 8, then P_1, P_2 and P_3 are derived easily. So C_r can be expressed as in (14) where Y is variable. Therefore,

$$\begin{aligned}
\gamma &= Tr\left(CP_1C^T - 2CP_2C_r^T + C_rP_3C_r^T\right) \\
&= Tr\Bigg(CP_1Cr^T - 2CP_2\left(\Xi\hat{B}_r^-\right)^T \\
&\quad + \Xi\hat{B}_r^- P_3\left(\Xi\hat{B}_r^-\right)^T \\
&\quad + \left(\Xi\hat{B}_r^- P_3 - 2CP_2\right)\left(I_r - \hat{B}_r\hat{B}_r^-\right)^T Y^T \\
&\quad + Y\left(I_r - \hat{B}_r\hat{B}_r^-\right)P_3\left(\Xi\hat{B}_r^-\right)^T \\
&\quad + Y\left(I_r - \hat{B}_r\hat{B}_r^-\right)P_3\left(I_r - \hat{B}_r\hat{B}_r^-\right)^T Y^T\Bigg),
\end{aligned} \tag{15}$$

$$\begin{aligned}
\frac{\partial\gamma}{\partial y_{ij}} &= Tr\Bigg(\left(\Xi\hat{B}_r^- P_3 - 2CP_2\right)\left(I_r - \hat{B}_r\hat{B}_r^-\right)^T \frac{\partial Y^T}{\partial y_{ij}} \\
&\quad + \frac{\partial Y}{\partial y_{ij}}\left(I_r - \hat{B}_r\hat{B}_r^-\right) \times P_3\left(\Xi\hat{B}_r^-\right)^T \\
&\quad + 2\frac{\partial Y}{\partial y_{ij}}\left(I_r - \hat{B}_r\hat{B}_r^-\right)P_3\left(I_r - \hat{B}_r\hat{B}_r^-\right)^T Y^T\Bigg),
\end{aligned} \tag{16}$$

where $\frac{\partial Y}{\partial y_{ij}} = \xi_i\eta_j^T, i = 1, 2, \cdots, m, j = 1, 2, \cdots, r, Y = (y_{ij})_{m\times r}$, $\xi_k(k = 1, 2, \cdots, m)$ and $\eta_j\ (j = 1, 2, \cdots, r)$ are the standard basis vectors of R^m and R^r respectively.

Since we have obtained the expression for γ and its partial derivatives with respect to the parameter C_r, a gradient-based method [18] can be used to obtain the optimal parameter C_r.

With all these results, given r and N_r, satisfying $\text{rank}\,(\Theta) \le r < n$, $ind(N_r) \le h_p$, we present the following model reduction algorithm.

Algorithm:

Step1. As in the proof of Theorem 8, obtain the matrix B_r;

Step2. Choose an initial value of the parameter Y;

Step3. Obtain the optimal parameter Y by solving the unconstrained optimization problem $\min_Y \gamma$;

(a) Calculate the function γ from (15);

(b) Compute the derivative of γ with respect to Y given by (16);

(c) Obtain the optimal parameter Y using a gradient-based method; if $\min_Y \gamma = 0$, then go to step 5, else continue;

Step4. Calculate the norm $\sum_{i=0}^{h_p-1} \left\|CN^iB\right\|_F^2$;

Step5. Verify whether the inequality (9) holds, if yes, one gets an approximation of the original system.

Remark 9: The range of the reduced order r is $\text{rank}\,(\Theta) \le r < n$ and one can choose any one N_r satisfying condition 2.

Remark 10: The corresponding results for discrete-time case can be derived similarly.

V. NUMERICAL EXAMPLES

In this section, we will use two examples to illustrate the effectiveness of the model reduction approaches proposed in this paper.

Example 1. Consider system (N, I, B, C)

$$N = \begin{bmatrix} 0&1&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{bmatrix},\ B = \begin{bmatrix} 1&1 \\ 2&1 \\ 3&1 \\ 1&2 \\ 2&2 \end{bmatrix},\ C = \begin{bmatrix} 0&1&0&0&3 \\ 0&2&0&0&1 \\ 0&1&0&0&1 \end{bmatrix}.$$

Its transfer function is

$$G(s) = -CB = \begin{bmatrix} 8&7 \\ 6&4 \\ 4&3 \end{bmatrix}.$$

It can be easily checked that in this example, the full-order system is output impulsive-free and $h_p = 1$.

Using the method in Theorem 5, one can get

$$N_r = \begin{bmatrix} 0&2&0 \\ 0&0&0 \\ 0&-1&0 \end{bmatrix},\ C_r = \begin{bmatrix} 1&3&2 \\ 2&1&4 \\ 1&1&2 \end{bmatrix},\ B_r = \begin{bmatrix} 2&1 \\ 2&2 \\ 0&0 \end{bmatrix}.$$

Example 2. Consider the system (N, I, B, C) of the example in [17], where

$$N = \begin{bmatrix} 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{bmatrix},\ B = \begin{bmatrix} 1&1 \\ 0&2 \\ 1&0 \\ 2&1 \\ 0&1 \end{bmatrix},\ C = \begin{bmatrix} 1&3&0&3&2 \\ 1&0&2&1&0 \\ 3&2&3&1&1 \end{bmatrix}.$$

It can be easily checked that in this example, $h_p > 1$.

In [17], this system is reduced to a 4th-order system, and it can not be reduced further based on $\mathcal{H}_\infty$ norm. Now, using the algorithm in this paper, we reduce this system to a 3rd-order system. Choosing

$$N_r = \begin{bmatrix} 0&1&0 \\ 0&0&1 \\ 0&0&0 \end{bmatrix},$$

and applying the algorithm in section 4, one can obtain the following

$$\hat{B}_r = \begin{bmatrix} 3&1 \\ -2&2.5 \\ 2&1 \end{bmatrix},\ Y = \begin{bmatrix} -2.8291&0.0573&4.5475 \\ -0.7435&0.8062&1.9140 \\ -2.3027&-0.1766&4.6734 \end{bmatrix},$$

$$C_r = \begin{bmatrix} 0.8600 & 3.4084 & 7.6185 \\ 0.5378 & 0.6483 & 2.8415 \\ 1.2043 & 2.8863 & 7.5799 \end{bmatrix}, \ B_r = \begin{bmatrix} 5 & -1.5 \\ -4 & 1.5 \\ 2 & 1 \end{bmatrix},$$

$$\gamma = 20.1481, \quad \sum_{i=0}^{h_p-1} \left\| CN^iB \right\|_F^2 = 474,$$

and

$$\frac{\min\limits_{C_r} \sum_{i=0}^{h_p-1} \left\| CN^iB - C_rN_r^iB_r \right\|_F^2}{\sum_{i=0}^{h_p-1} \| CN^iB \|_F^2} = 0.0425 \ll 1.$$

System (N_r, I_r, B_r, C_r) can be taken as the approximation the original system. In order to view the differences between the state responses of these two systems, the following figures give the output responses of the original and reduced systems with the sinusoidal input (the X axis label unit is Radian(Rad)). It can be seen from these figures that the reduced order system can approximate the original system quite well.

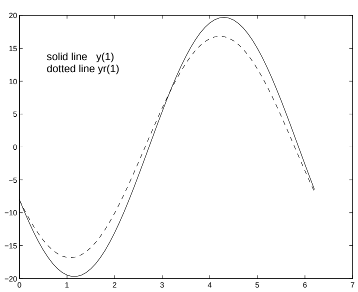

Fig. 1. The 1st output responses for reduced system and original system

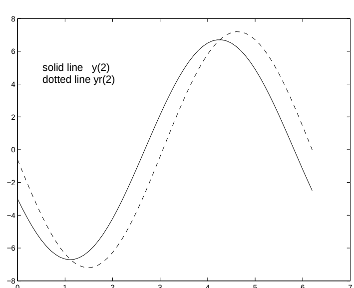

Fig. 2. The 2nd output responses for reduced system and original system

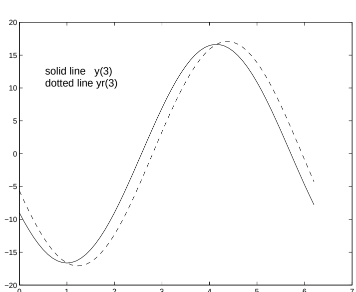

Fig. 3. The 3rd output responses for reduced system and original system

VI. CONCLUSIONS

In this paper, we developed a new model reduction algorithm for singular systems via covariance approximation. This is the first time that the covariance defined for singular systems is used for model reduction issue. From the results of this paper, it can be seen that the proposed covariance can reflect the capacity of impulsive behavior for singular systems. This will motivates us for further research along this direction.

VII. ACKNOWLEDGMENTS

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